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THE STUDY OF EQUILIBRIA IN ABSTRACT ECONOMIES AND GENERALIZED GAMES

Saeideh Pirbavafa[1*] and S. Mansour Vaezpour[2]

The purpose of this paper is to present a new best proximity point existence theorem for set-valued maps that are almost w-upper semicontinuous. We also give examples that support the usability of our results. Next, we establish new existence results of equilibria for free abstract economies in which constraint correspondences may not be upper semicontinuous. To expose the existence result of equilibria, we use our best proximity point theorem and prove the new existence theorem of an equilibrium pair in free abstract economies with two constraints set-valued. Finally, we state some illustrative examples to ensure our main theorems.

Keywords: Almost w-upper semicontinuous map, Best proximity point, Equilibrium pair, F-majorized set-valued map

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1. **Introduction**

Mathematical economics applies mathematical methods to formulate economic theories and analyze economic problems. Existence theorems for equilibrium points are central to studying many economic models. In 1951, Nash [14] established the existence of equilibrium in finite games using Brouwer's fixed point theorem, and Debreu (1952) [15] introduced the notion of an abstract economy, deriving Walrasian equilibrium. Arrow and Debreu (1954) [1] later developed a competitive economy model and proved an equilibrium existence theorem for such abstract economies. Beyond models based on utility functions, Shafer and Sonnenschein (1975) [17] extended Debreu's equilibrium theorem [3]. Borglin and Keiding [2] introduced the concept of KF-majorization and obtained a weaker equilibrium existence theorem, slightly extending results of Gale and Mas-Colell [8]. Yannelis et al. (1983) [19] later introduced the class of L-majorized set-valued maps and derived an equilibrium existence theorem for abstract economies with infinitely many commodities and countably many agents.

[1*]Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran, e-mail: bavafa@aut.ac.ir (Corresponding Author)

[2]Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran, e-mail: vaez@aut.ac.ir

Given the importance of equilibrium existence theorems in abstract economies, many researchers have studied models with two constraints. Ding and Tan (1992) [5] proved an equilibrium existence theorem for a one-person game $\Gamma = (X; A, B; P)$ under upper semicontinuity, and Tan and Yuan [18] used a new maximal element theorem to establish equilibrium results for non-compact generalized games with L_C-correspondences. Ding (1998) [4] introduced the class of upper semicontinuous U-majorized set-valued maps and derived equilibrium results for qualitative and generalized games with infinitely many agents and non-compact strategy sets. The classical theorem of [1] has since been extended in various directions (see [9], [7], [11]). More recently, Hervs-Beloso and Patriche (2014) [10] introduced several new classes of set-valued maps, w-upper semicontinuous, almost w-upper semicontinuous, and those with the e-USS property, and established equilibrium existence for abstract economies with two constraints using a new fixed point theorem.

The first time, the notion of best approximant point was introduced by Ky Fan [6] which extends the Tychonoff fixed point theorem. In 2006, Kim and Lee [13] combined the optimal case of Fan's best approximation theorem and equilibrium existence theorems to provide a new version of the equilibrium concept in generalized games. They considered an economic situation and defined an equilibrium pair concept in free n-person games. Recently, Pirbavafa and Vaezpour in [16], as an application of best proximity point theorems, obtained new existence theorems of an equilibrium pair in free abstract economies.

In this study, we will first prove an interesting existence theorem of best proximity pairs for almost weakly upper semicontinuous set-valued maps. Also, we will state examples that are suitable for our results. Moreover, we will give applications of our best proximity point theorem to the study of the equilibrium existence results in free abstract economies which the preference set-valued map is F-majorized. Finally, we will present some illustrative examples to find an equilibrium pair in free generalized games.

2. **Preliminaries**

Let X be a subset of a vector space E, 2^X denotes the power set of X and $co\ X$ denotes the convex hull of X. If X is a subset of a topological space E, $int\ X$ denotes the interior of X in E and $cl\ X$ denotes the closure of X in E. Suppose that X and Y are two topological spaces and $T : X \to 2^Y$ is a set-valued map. Then $dom\ T$ is the domain of set-valued map T and is defined by $dom\ T = \{x \in X : T(x) \neq \emptyset\}$. The map T is said to be upper semicontinuous on X, if for each $x \in X$ and each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood U of x in X such that $T(x') \subseteq V$ for each $x' \in U$. The set-valued map $\overline{T} : X \to 2^Y$ is defined by

$$\overline{T}(x) = \{y \in Y : \ (x, y) \in cl_{X\times Y} graph\ T\},$$

where, $cl_{X\times Y} graph\ T$ is called the adherence of the graph of T.

Assume that I is a finite index set and for every $i \in I$, X_i and Y_i be non-empty subsets of a normed linear space E with a metric $d(x, y)$ induced by the norm. Then we use the following notations as in [12], for each $i, j \in I$:

$d(X_i, Y_j) := \inf \{d(x, y) \mid x \in X_i, y \in Y_j\}$;

${X_i}^o := \{x \in X_i \mid \text{for each } j \in I, \exists\, y_j \in Y_j \text{ such that } d(x, y_j) = d(X_i, Y_j)\}$;

${Y_i}^o := \{y \in Y_i \mid \text{for each } j \in I, \exists\, x_j \in X_j \text{ such that } d(x_j, y) = d(X_j, Y_i)\}$.

Also, $X^o := \prod_{i \in I} {X_i}^o$ and $Y^o := \prod_{i \in I} {Y_i}^o$. Let $T_i : X = \prod_{i \in I} X_i \to 2^{Y_i}$ be a set-valued map. Then the pair $(\hat{x_i}, T_i(\hat{x})) \in X_i \times Y_i$ is called a best proximity pair for a set-valued map T_i if $d(\hat{x_i}, T_i(\hat{x})) = d(X_i, Y_i)$ for each $i \in I$.

Let X be a topological space, Y be a non-empty subset of a topological vector space E and D be a subset of Y.

Definition 2.1. *The set-valued map $T : X \to 2^Y$ is called to be w-upper semicontinuous (weakly upper semicontinuous) with respect to the given set D if, there exists a basis β of open symmetric neighborhoods of* 0 *in E such that, for each $V \in \beta$, the set-valued map $T^V : X \to 2^Y$ is upper semicontinuous, where $T^V(x) := (T(x) + V) \cap D$ for each $x \in X$.*

Definition 2.2. *The set-valued map $T : X \to 2^Y$ is said to be almost w-upper semicontinuous (almost weakly upper semicontinuous) with respect to the set D if, there exists a basis β of open symmetric neighborhoods of* 0 *in E such that, for each $V \in \beta$, the set-valued map $\overline{T^V}$ is non-empty valued and upper semicontinuous.*

Remark 2.1. *In Definitions 2.1 and 2.2, when $D = Y$, then we recall that T is w-upper semicontinuous and almost w-upper semicontinuous, respectively.*

Definition 2.3. [20] *Let X be a topological space and A be a subset of X. The set A is called to be compactly open in X, whenever $A \cap C$ is open in C, for each non-empty compact subset C of X.*

Definition 2.4. [20] *Let X be a topological space and Y be a non-empty subset of a topological vector space E. Let $P : X \to 2^Y$ be a set-valued mapping. Then*

(1) *(ϕ_x, ψ_x, N_x) is a F-majorant of P at x if $\phi_x, \psi_x : X \to 2^Y$ are two mappings and N_x is an open neighborhood of x in X such that*
  - (*a*) *for each $z \in N_x$, $z \notin co\ \phi_x(z) \subseteq Y$ and $\{z \in N_x : P(z) \neq \emptyset\} \subseteq \{z \in N_x : \psi_x(z) \neq \emptyset\}$,*
  - (*b*) *for each $z \in X$, $\psi_x(z) \subseteq \phi_x(z)$ and*
  - (*c*) *for each $y \in Y$, ${\psi_x}^{-1}(y)$ is compactly open in X;*

(2) *P is said to be F-majorized if*
  - (*a*) *for each $x \in X$ with $P(x) \neq \emptyset$, there exists an F-majorant (ϕ_x, ψ_x, N_x) of P at x and*
  - (*b*) *for each $A \in \mathcal{F}(domP)$, $domP \cap (\cap_{x \in A} N_x) \subseteq \{z \in \cap_{x \in A} N_x : \cap_{x \in A} \psi_x(z) \neq \emptyset\}$, (where $\mathcal{F}(domP)$ is the family of all non-empty finite subsets of $domP$).*

Note that, if $P, Q : X \to 2^Y$ are two set-valued maps such that $Q(x) \subseteq P(x)$ for each $x \in X$ and P is F-majorized, then Q is F-majorized.

We recall the following Theorem which is useful to obtain our results.

Theorem 2.1. [20] *Let I be a set of agents and for each $i \in I$, let Y_i be a non-empty compact and convex subset of a topological vector space E_i and $\mathcal{G} = (Y_i, \Phi_i)_{i\in I}$ be a qualitative game. If for each $i \in I$,*

(a) *$\Phi_i : Y = \prod_{j\in I} Y_j \to 2^{Y_i}$ is F-majorized in Y;*

(b) *$\bigcup_{i\in I}\{y \in Y : \Phi_i(y) \neq \emptyset\} = \bigcup_{i\in I} int_Y\{y \in Y : \Phi_i(y) \neq \emptyset\}$.*

Then there exists an equilibrium point for $\mathcal{G}$, i.e., there exists $\hat{y} \in Y$ such that for each $i \in I$, $\Phi_i(\hat{y}) = \emptyset$. In this case, it is said that Φ_i has a maximal element.

3. Main Results

At first, we obtain a new existence theorem of best proximity pairs for almost w-upper semicontinuous set-valued maps of which needed in the equilibrium existence theorem. Our outputs generalize and improve some recent results in the literature.

Theorem 3.1. *Let I be an index set and for each $i \in I$, X_i and Y_i be non-empty compact and convex subsets of a normed linear space E and D_i be a non-empty compact and convex subset of Y_i. Suppose that $S_i, T_i : X := \prod_{i\in I} X_i \to 2^{Y_i}$ are two set-valued maps such that for each $i \in I$:*

(i) *for each $x \in X$, $\overline{S_i}(x) \subseteq T_i(x)$;*

(ii) *S_i is almost weakly upper semicontinuous with respect to D_i such that for each open symmetric neighborhood V_i of 0 in E and for each $x \in X$, $\overline{S_i^{V_i}}(x)$ is a convex subset of D_i and $\overline{S_i^{V_i}}(x) \cap {D_i}^o \neq \emptyset$.*

Then, for each $i \in I$, the set-valued map T_i has a best proximity point with respect to the given set D_i, i.e., there exists $x^* \in X = \prod_{i\in I} X_i$ such that for each $i \in I$, $d({x_i}^*, T_i(x^*)) = d(X_i, D_i)$.

Proof. For each $i \in I$, since S_i is almost weakly upper semicontinuous so there exists a basis β_i of open symmetric neighborhoods of 0 in E such that, for each $V_i \in \beta_i$, the set-valued map $\overline{S_i^{V_i}}$ is upper semicontinuous, for which the set-valued map $S_i^{V_i} : X \to 2^{D_i}$ is defined by $S_i^{V_i}(x) = (S_i(x) + V_i) \cap D_i$. It is easy to check that, for each $i \in I$ and for each $x \in X$, $\overline{S_i^{V_i}}(x)$ is a closed subset of D_i. For each system of neighborhoods $V = \prod_{i\in I} V_i \in \prod_{i\in I} \beta_i = \beta$, we define $S^V : X \to 2^D$ by

$$S^V(x) = \prod_{i\in I} \overline{S_i^{V_i}}(x),$$

which in this case, $D = \prod_{i\in I} D_i$. The set-valued map S^V is upper semicontinuous with closed convex values, also $S^V(x) \cap D^o \neq \emptyset$. Therefore, according to Theorem 2 in [12], there exists $x_V^* = \prod_{i\in I} x_{V_i}^* \in X$ such that

$d(x_V^*, S^V(x_V^*)) = d(X, D)$. And hence, there is an $y_V^* = \prod_{i\in I} y_{V_i}^* \in \overline{S^V}(x_V^*)$ such that $d(x_V^*, y_V^*) = d(X, D)$, i.e., for each $i \in I$, there exists $y_{V_i}^* \in \overline{S_i^{V_i}}(x_V^*)$ such that $d(x_{V_i}^*, y_{V_i}^*) = d(X_i, D_i)$.
For each $V = \prod_{i\in I} V_i \in \beta$, we consider the set

$$\mathcal{D}_V = \cap_{i\in I}\{(x, y) \in X \times D \mid y_i \in \overline{S_i^{V_i}}(x) \text{ and } d(x_i, y_i) = d(X_i, D_i)\}.$$

$\mathcal{D}_V \neq \emptyset$, since $(x_V^*, y_V^*) \in \mathcal{D}_V$. In order to show that $\mathcal{D}_V$ is a closed subset of $X \times D$, let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}_V$ which converges to $(x, y) \in X \times D$. Therefore, for each $i \in I$, $y_n^i \in \overline{S_i^{V_i}}(x_n)$ and $d(x_n^i, y_n^i) = d(X_i, D_i)$. We conclude that $(x_n, y_n^i) \in cl_{X\times D_i} graph(S_i^{V_i})$ and so $(x, y_i) \in cl_{X\times D_i} graph(S_i^{V_i})$. Therefore $y_i \in \overline{S_i^{V_i}}(x)$. On the other hand,

$$d(X_i, D_i) \leq \|x_i - y_i\| \leq \|x_i - x_n^i\| + \|x_n^i - y_n^i\| + \|y_n^i - y_i\|,$$

and since $\|x_i - x_n^i\| \to 0$, $\|y_n^i - y_i\| \to 0$ and $\|x_n^i - y_n^i\| = d(X_i, D_i)$, this implies that $\|x_i - y_i\| = d(X_i, Y_i)$. Thus, $\mathcal{D}_V$ is closed subset of $X \times D$.
Now we will prove $\{\mathcal{D}_V\}_{V\in\beta}$ has finite intersection property. Let $\{V^1, V^2, \ldots, V^n\}$ be any finite subset of β. For each $k = 1, 2, \ldots, n$, let $V^k = \prod_{i\in I} V_i{}^k$, where $V_i{}^k \in \beta_i$ for each $i \in I$. So, $V = \prod_{i\in I}(\cap_{k=1}^n V_i{}^k) \in \prod_{i\in I} \beta_i = \beta$. Obviously, $\mathcal{D}_V \subseteq \cap_{k=1}^n \mathcal{D}_{V^k}$. Note that $\cap_{k=1}^n \mathcal{D}_{V^k}$ is non-empty since $\mathcal{D}_V$ is non-empty. Therefore, the family $\{\mathcal{D}_V\}_{V\in\beta}$ has the finite intersection property. And since $X \times D$ is compact, so $\cap_{V\in\beta}\mathcal{D}_V \neq \emptyset$.
Now, take $(x^*, y^*) \in \cap\{\mathcal{D}_V : V \in \beta\}$, then for each $i \in I$ and for each $V_i \in \beta_i$,

$$y_i{}^* \in \overline{S_i^{V_i}}(x^*) \text{ and } d(x_i{}^*, y_i{}^*) = d(X_i, D_i).$$

According to [10, Lemma 2.2], we have that $y_i{}^* \in \overline{S_i}(x^*)$, for each $i \in I$. Therefore, $y_i{}^* \in T_i(x^*)$ for all $i \in I$ and $d(x_i{}^*, y_i{}^*) = d(X_i, D_i)$. □

If the set-valued map $S_i = T_i$ for each $i \in I$ in Theorem 3.1, we have the following existence result.

Corollary 3.1. *Let I be an index set and for each $i \in I$, let X_i and Y_i be non-empty compact and convex subsets of a normed linear space E, D_i be a non-empty compact and convex subset of Y_i, and for each $i \in I$, $T_i : X := \prod_{i\in I} X_i \to 2^{Y_i}$ be an almost weakly upper semicontinuous set-valued map with respect to D_i such that for each open symmetric neighborhood V_i of 0 in E and for each $x \in X$, $\overline{T_i^{V_i}}(x)$ is a convex subset of D_i and $\overline{T_i^{V_i}}(x) \cap D_i{}^o \neq \emptyset$. Then, for each $i \in I$, the set-valued map $\overline{T_i}$ has a best proximity point with respect to D_i.*

As an immediate consequence of Theorem 3.1, when $|I| = 1$ we have the following result.

Corollary 3.2. *Let X and Y be non-empty compact and convex subsets of a normed linear space E, D be a non-empty compact and convex subset of Y,* *and $S, T : X \to 2^Y$ be two set-valued maps that the following conditions are satisfied:*

(i) *for each $x \in X$, $\overline{S}(x) \subseteq T(x)$;*

(ii) *S is almost weakly upper semicontinuous with respect to D such that for each open symmetric neighborhood V of 0 in E and for each $x \in X$, $\overline{S^V}(x)$ is a convex subset of D and $\overline{S^V}(x) \cap D^o \neq \emptyset$.*

Then, the set-valued map T has a best proximity point with respect to D.

If $X_i = Y_i$ for each $i \in I$, by applying Theorem 3.1 we can also obtain the following corollary which corresponds to Theorem 3.1 in [10].

Corollary 3.3. *Let I be an index set and for each $i \in I$, let X_i be a non-empty compact and convex subset of a normed linear space E, D_i be a non-empty compact and convex subset of X_i, and $S_i, T_i : X := \prod_{i \in I} X_i \to 2^{X_i}$ be two set-valued maps that the following conditions are satisfied for each $i \in I$:*

(i) *for each $x \in X$, $\overline{S_i}(x) \subseteq T_i(x)$;*

(ii) *S_i is almost weakly upper semicontinuous with respect to D_i such that for each open symmetric neighborhood V_i of 0 in E and for each $x \in X$, $\overline{S_i^{V_i}}(x)$ is a convex subset of D_i.*

Then for each $i \in I$, the set-valued map T_i has a fixed point in D, i.e., there exists $x^* \in D = \prod_{i \in I} D_i$ such that ${x_i}^* \in T_i(x^*)$ for each $i \in I$.

When $|I| = 1$, by applying Theorem 3.1, we can obtain an best proximity pair for the following example:

Example 3.1. *Let $X = [0, 1]$ and $Y = [2, 5]$. Suppose the set-valued maps $S, T : X \to 2^Y$ defined as follows:*

$$S(x) = \begin{cases} (2, 4 - x), & \text{if } x \in (0, 1], \\ \{2\}, & \text{if } x = 0. \end{cases}$$

And, for each $x \in [0, 1]$, $T(x) = [2, 5 - x)$. It is clear that S is not upper semicontinuous. For $\varepsilon > 0$, let $V = (-\varepsilon, \varepsilon)$ be an open symmetric neighborhood of 0 in $\mathbb{R}$. We now consider $S^V(x) = (S(x) + V) \cap Y$ for each $x \in [0, 1]$. Then, it results that:

(1) *for $0 < \varepsilon < 1$,*

$$S^V(x) = \begin{cases} [2, 4 - x + \varepsilon), & \text{if } 0 < x \leq 1, \\ [2, 2 + \varepsilon), & \text{if } x = 0, \end{cases}$$

(2) *for $1 \leq \varepsilon \leq 2$,*

$$S^V(x) = \begin{cases} [2, 5], & \text{if } 0 < x < -1 + \varepsilon, \\ [2, 4 - x + \varepsilon), & \text{if } -1 + \varepsilon \leq x \leq 1, \\ [2, 2 + \varepsilon), & \text{if } x = 0, \end{cases}$$

(3) *for* $2 < \varepsilon \leq 3$,

$$S^V(x) = \begin{cases} [2,5], & \text{if } 0 < x \leq 1, \\ [2, 2+\varepsilon), & \text{if } x = 0, \end{cases}$$

(4) *and for* $\varepsilon > 3$,

$$S^V(x) = [2,5], \text{ if } 0 \leq x \leq 1.$$

Therefore for $0 < \varepsilon < 1$,

$$\overline{S^V}(x) = [2, 4 - x + \varepsilon] \text{ if } 0 \leq x \leq 1,$$

for $1 \leq \varepsilon \leq 2$,

$$\overline{S^V}(x) = \begin{cases} [2,5], & \text{if } 0 \leq x < -1+\varepsilon, \\ [2, 4 - x + \varepsilon], & \text{if } -1+\varepsilon \leq x \leq 1, \end{cases}$$

and for $\varepsilon > 2$,

$$\overline{S^V}(x) = [2,5] \text{ if } 0 \leq x \leq 1.$$

For each $V = (-\varepsilon, \varepsilon)$ *with* $\varepsilon > 0$, $\overline{S^V}$ *is non-empty valued and upper upper semicontinuous, thus the set-valued map* S *is almost* w*-upper semicontinuous. Also,* $\overline{S^V}(x) \cap Y^o \neq \emptyset$ *and* $\overline{S^V}(x)$ *is convex subset of* Y*, for each* $x \in [0,1]$. *Note that* $[2, 4-x] = \overline{S}(x) \subseteq T(x) = [2, 5-x)$ *for each* $x \in [0,1]$. *Thus, all the hypotheses of Theorem 3.1 are satisfied and hence there exists* $(x^*, y^*) = (1,2) \in X \times Y$ *such that* $y^* \in T(x^*)$ *and* $d(x^*, y^*) = d(X, Y) = 1$.

In the sequel, our aim is the study of equilibrium existence for free generalized games with two constraints which the preference set-valued map is F-majorized. Next, we are going to give an example of an economic situation which has not an equilibrium point with one constraint set-valued map but it has an equilibrium point with two constraint set-valued maps.

Let I be a non-empty set of locations or agents. For each $i \in I$, let X_i and Y_i be non-empty subsets of a normed linear space E, D_i be a non-empty subset of Y_i and $A_i, B_i : X = \prod_{j \in I} X_j \to 2^{Y_i}$ be the constraint set-valued maps, and $P_i : Y = \prod_{j \in I} Y_j \to 2^{Y_i}$ be the preference set-valued map.

Definition 3.1. *A free generalized game (free abstract economy) with two constraints is defined as a family of ordered tuples* $\Gamma = (X_i, Y_i; A_i, B_i; P_i)_{i \in I}$. *An equilibrium pair for* Γ *with respect to the set* D_i *is defined as a pair of points* $(x^*, y^*) = ((x_i{}^*), (y_i{}^*))_{i \in I} \in X \times Y$ *such that for each* $i \in I$, $y_i{}^* \in \overline{B_i}(x^*)$ *with* $d(x_i{}^*, y_i{}^*) = d(X_i, D_i)$, *and* $A_i(x^*) \cap P_i(y^*) = \emptyset$.

In Definition 3.1, when $X_i = Y_i = D_i$ for each $i \in I$, then it can be reduced to the definitions of equilibrium theory in economics due to Ding [4] or Tan et al. [18].

Moreover, if for each $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$ and B_i has closed graph, then Definition 3.1 can be reduced to the definition of equilibrium pairs for free abstract economies as was introduced by Kim et al. [13].

In order to comprehend the concept of the equilibrium from an economic viewpoint, we consider an economic situation and derive an equilibrium point for an abstract economy $\Gamma = (X; A; P)$.

Example 3.2. *Consider a firm that has a single output. First, the firm chooses its capacity, x_1, then it chooses the output level, x_2. Its capacity cannot be longer than one. Hence, $x = (x_1, x_2) \in X = [0, 1]^2$.*
The constraint correspondence for this firm denoted by $A : X \to 2^X$ can be defined as:

$$A(\hat{x}_1, \hat{x}_2) := \{(x_1, x_2) : \; x_2 \leq \hat{x}_1\}.$$

The objective of the firm is to maximize its profit. Assume that the marginal cost of production, denoted by c, and the cost of one unit of capacity, denoted by r, are constants. Let p be the price of one unit of outputs ($p > r + c$). The profit of the firm can be derived as $\pi = p \cdot x_2 - c \cdot x_2 - r \cdot x_1$. Hence, the preference correspondence $P : X \to 2^X$ can be defined as:

$$P(\hat{x}_1, \hat{x}_2) = \{(x_1, x_2) \in X : \; x_1 < (\frac{p-c}{r})x_2 - (\frac{p-c}{r})\hat{x}_2 + \hat{x}_1\}.$$

It can be shown that the unique equilibrium for this example is $(1, 1)$.

Theorem 3.2. *Let X_i and Y_i be non-empty compact and convex subsets of a normed linear space E, D_i be a non-empty compact and convex subset of Y_i and $\Gamma = (X_i, Y_i; A_i, B_i; P_i)$ be a free abstract economy with two constraints. For each $i \in I = \{1, 2, \ldots, n\}$, consider the following conditions:*

(i) for each $x \in X$, $A_i(x)$ is non-empty and $\overline{A_i}(x) \subseteq \overline{B_i}(x)$;
(ii) A_i is almost weakly upper semicontinuous with respect to D_i such that for each open symmetric neighborhood V_i of 0 in E and for each $x \in X$, $\overline{A_i^{V_i}}(x)$ is a convex subset of D_i and $\overline{A_i^{V_i}}(x) \cap {D_i}^o \neq \emptyset$;
(iii) $P_i(y)$ is non-empty for each $y = (y_i)_{i \in I} \in Y$ with $y \in Y \setminus E_{i_x}$, whenever $E_{i_x} = \{z \in D \mid z_i \in \overline{B_i}(x)$ and $d(x_i, z_i) = d(X_i, D_i)\}$ is non-empty;
(iv) for each $i \in I$, P_i is F-majorized;
(v) the set $\{y \in Y : A_i(x) \cap P_i(y) \neq \emptyset\}$ is open in Y, whenever E_{i_x} is non-empty.

Then there exists an equilibrium pair $(x^*, y^*) = (({x_i}^*), ({y_i}^*))_{i \in I} \in X \times Y$ for Γ, i.e., for each $i \in I = \{1, 2, \cdots, n\}$,

$${y_i}^* \in \overline{B_i}(x^*) \;\; \text{such that} \;\; d({x_i}^*, {y_i}^*) = d(X_i, D_i) \;\; \text{and} \;\; A_i(x^*) \cap P_i(y^*) = \emptyset.$$

Proof. Since $\overline{B_i}$ satisfies all of the assumptions of Theorem 3.1, we can obtain a best proximity pair $\{x^*\} \times \overline{B_i}(x^*) \in X \times Y_i$ for the set-valued map $\overline{B_i}$ with respect to D_i, i.e., $d({x_i}^*, \overline{B_i}(x^*)) = d(X_i, D_i)$.
We now consider the set $E_{i_{x^*}} = \{y \in D \mid y_i \in \overline{B_i}(x^*)$ and $d({x_i}^*, y_i) = d(X_i, D_i)\}$. Then, it is easy to see that $E_{i_{x^*}}$ is a non-empty closed subset of D. It is sufficient to show that there exists a point $y^* \in E_{i_{x^*}}$ such that

$A_i(x^*) \cap P_i(y^*) = \emptyset$ for each $i \in I$.
For each $i \in I$ and each $y \in Y$, we define $\Phi_i : Y \to 2^{Y_i}$ by

$$\Phi_i(y) = \begin{cases} A_i(x^*) \cap P_i(y), & y \in E_{i_{x^*}}, \\ P_i(y), & y \in Y \setminus E_{i_{x^*}}. \end{cases}$$

We prove that the mapping Φ_i has a maximal element. For this purpose, we shall show that the qualitative game $\mathcal{G} = (Y_i, \Phi_i)_{i \in I}$ satisfies all hypotheses of Theorem 2.1. For each $i \in I$, we have that the set

$$\begin{aligned} \{y \in Y : \Phi_i(y) \neq \emptyset\} &= \{y \in E_{i_{x^*}} : \Phi_i(y) \neq \emptyset\} \cup \{y \in Y \setminus E_{i_{x^*}} : \Phi_i(y) \neq \emptyset\} \\ &= \{y \in E_{i_{x^*}} : A_i(x^*) \cap P_i(y) \neq \emptyset\} \cup \{y \in Y \setminus E_{i_{x^*}} : P_i(y) \neq \emptyset\} \\ &= (E_{i_{x^*}} \cap \{y \in Y : A_i(x^*) \cap P_i(y) \neq \emptyset\}) \cup (Y \setminus E_{i_{x^*}}) \\ &= \{y \in Y : A_i(x^*) \cap P_i(y) \neq \emptyset\} \cup Y \setminus E_{i_{x^*}} \end{aligned}$$

is open in Y and hence condition (b) of Theorem 2.1 is satisfied. Eventually, we check that Φ_i is F-majorized.
Since P_i is F-majorized, so for each $y \in Y$ with $P_i(y) \neq \emptyset$ there exists an F-majorant (ϕ_y, ψ_y, N_y) of P_i at y and for each finite subset A of $domP_i$, we have $domP_i \cap (\cap_{y \in A} N_y) \subseteq \{z \in \cap_{y \in A} N_y : \cap_{y \in A} \psi_y(z) \neq \emptyset\}$. We consider the following two cases:
Case 1. if $y \in E_{i_{x^*}}$, then $\Phi_i(y) = A_i(x^*) \cap P_i(y)$. Now, if $\Phi_i(y) = \emptyset$, then Φ_i is automatically F-majorized and if $\Phi_i(y) \neq \emptyset$, then $P_i(y) \neq \emptyset$ and since P_i is F-majorized and $\Phi_i(y) \subseteq P_i(y)$, so Φ_i is also F-majorized in Y.
Case 2. if $y \in Y \setminus E_{i_{x^*}}$, then $\Phi_i(y) = P_i(y)$. Since $P_i(y) \neq \emptyset$ and P_i is F-majorized, so Φ_i is also F-majorized in Y.
Therefore, Φ_i is F-majorized in Y for each $i \in I$. Hence, the whole hypotheses of Theorem 2.1 are satisfied so that there exists a point $y^* = ({y_i}^*)_{i \in I} \in Y$ such that $\Phi_i(y^*) = \emptyset$ for each $i \in I$. By (iii) and the definition of Φ_i, this implies that $y^* \in E_{i_{x^*}}$ such that $A_i(x^*) \cap P_i(y^*) = \emptyset$ for each $i \in I$. This completes the proof. □

Applying Theorem 3.2, we can obtain an equilibrium pair for the following one-person game:

Example 3.3. *Let $\Gamma = (X, Y; A, B; P)$ be a free one-person game with two constraint set-valued maps such that $E = \mathbb{R}$, $X = [0, 1]$ and $Y = [3, 6]$. Let constraint set-valued maps $A, B : X \to 2^Y$ and the preference set-valued map $P : Y \to 2^Y$ defined by*

$$A(x) = \begin{cases} (3, 5 - x), & \text{if } 0 < x \leq 1, \\ \{3\}, & \text{if } x = 0, \end{cases}$$

and,

$$B(x) = \begin{cases} (3, 6 - x), & \text{if } 0 < x \leq 1, \\ \{3\}, & \text{if } x = 0, \end{cases}$$

and also,

$$P(y) = \begin{cases} (3, y^{\frac{1}{2}} + 2], & \text{if } y \in (3, 6], \\ \emptyset, & \text{if } y = 3. \end{cases}$$

Note that the set-valued maps $\overline{A}, \overline{B} : X \to 2^Y$ satisfy $[3, 5-x] = \overline{A}(x) \subseteq \overline{B}(x) = [3, 6-x]$ for each $x \in X$. It is straightforward to see that $Y^0 = \{3\}$, $E_x = \{3\}$ and $d(X, Y) = 2$. Then, the assumptions (i) and (ii) are satisfied. Also, the set $\{y \in Y : A(x) \cap P(y) \neq \emptyset\}$ is open in Y and $P(y)$ is nonempty for each $y \in Y \setminus E_x$. Thus the assumptions (iii) and (v) are valid.
Now, for each $y \in Y$ with $P(y) \neq \emptyset$, let $N_y = [2, 5]$ be an open neighborhood of y in Y and define the set-valued maps $\phi_y, \psi_y : Y \to 2^Y$ by

$$\phi_y(z) = \psi_y(z) = \begin{cases} \{3\}, & \text{if } z \in (3, 6], \\ \emptyset, & \text{if } z = 3. \end{cases}$$

Then, for each $z \in Y$ we have

$$\psi_y^{-1}(z) = \begin{cases} \emptyset, & \text{if } z \in (3, 6], \\ Y \setminus \{3\}, & \text{if } z = 3. \end{cases}$$

So each $\psi_y^{-1}(z)$ is compactly open in Y, and hence P is F-majorized in Y. Thus, all the hypotheses of Theorem 3.2 are satisfied so that we can obtain an equilibrium pair $(x^*, y^*) = (1, 3) \in X \times Y$ such that $y^* \in \overline{B}(x^*)$ with $d(x^*, \overline{B}(x^*)) = d(X, Y) = 2$ and $A(x^*) \cap P(y^*) = \emptyset$.
Note that $(1, 3)$ is an equilibrium pair of the free abstract economy $\Gamma = ([0, 1], [3, 6]; A, B; P)$ (in the sense of our definition for equilibria, Definition 3.1), but $(1, 3)$ is not an equilibrium pair of the free abstract economy $\Gamma = ([0, 1], [3, 6]; A; P)$ (in the sense of Kim and Lee's definition in [13]*) as $3 \notin A(1)$ even though $d(1, 3) = d(X, Y) = 2$ and $A(1) \cap P(3) = \emptyset$.*

When $X_i = Y_i$ for each $i \in I$ in Theorem 3.2, the best proximity point set of the set-valued map $\overline{B_i}$ in D is the fixed point set of $\overline{B}$ in D, i.e., $E_{i_x} := F_D(\overline{B_i}) := \{x \in D \mid x_i \in \overline{B_i}(x)\}$.
Therefore, we can easily derive the following equilibrium existence result for the abstract economies with two constraints. As a consequence of Theorem 3.2, if in the above corollary we take $A_i = B_i$ for all $i \in I$ and A_i has closed graph, then we deduce the following result.

Corollary 3.4. *Let $\Gamma = (X_i; A_i; P_i)$ be a generalized game with a constraint set-valued map such that for each $i \in I$, X_i is a non-empty compact and convex subset of a normed linear space E, D_i is a non-empty compact and convex subset of X_i. For each $i \in I$, Suppose that*

- (i) *A_i is almost w-upper semicontinuous with respect to D_i such that for each open symmetric neighborhood V_i of 0 in E and for each $x \in X$, $\overline{A_i^{V_i}}(x)$ is a convex subset of D_i;*
- (ii) *$P_i(x)$ is non-empty for each $x = (x_i)_{i \in I} \in X$ with $x \in X \setminus F_D(A_i)$, whenever $F_D(A_i)$ is non-empty;*
- (iii) *for each $i \in I$, P_i is F-majorized;*
- (iv) *the set $\{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X, whenever $F_D(A_i)$ is non-empty;*
- (v) *A_i has closed graph.*

Then there exists an equilibrium point $x^* = ({x_i}^*)_{i\in I} \in X$ *for* Γ, *i.e., for each* $i \in I$, ${x_i}^* \in A_i(x^*)$ *and* $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Let us illustrate the above corollary with the following example.

Example 3.4. *Let* $\Gamma = (X; A; P)$ *be a one-person game with a constraint set-valued map such that* $E = \mathbb{R}$ *and* $X = [0, 1]$. *Let the constraint set-valued map* $A : X \to 2^X$ *and the preference set-valued map* $P : X \to 2^X$ *defined by*

$$A(x) = [0, 1 - x] \text{ for all } 0 \le x \le 1,$$

and $P(x) = \begin{cases} \emptyset, & \text{if } 0 \le x \le \frac{1}{2}, \\ (\frac{1}{2}, 1], & \text{if } \frac{1}{2} < x \le 1. \end{cases}$ *. For* $\varepsilon > 0$, *let* $V = (-\varepsilon, \varepsilon)$ *be an open symmetric neighborhood of* 0 *in* $\mathbb{R}$. *We now consider* $A^V(x) = (A(x) + V) \cap X$ *for each* $x \in [0, 1]$. *Then, it results that:*

(1) *for* $0 < \varepsilon \le 1$,

$$A^V(x) = \begin{cases} [0, 1], & \text{if } 0 \le x < \varepsilon, \\ [0, 1 - x + \varepsilon), & \text{if } \varepsilon \le x \le 1. \end{cases}$$

(2) *for* $\varepsilon > 1$,

$$A^V(x) = [0, 1], \text{ for all } 0 \le x \le 1.$$

Therefore, for $0 < \varepsilon \le 1$,

$$\overline{A^V}(x) = \begin{cases} [0, 1], & \text{if } 0 \le x < \varepsilon, \\ [0, 1 - x + \varepsilon], & \text{if } \varepsilon \le x \le 1, \end{cases}$$

and for $\varepsilon > 1$,

$$\overline{A^V}(x) = [0, 1], \text{ for all } 0 \le x \le 1.$$

Note that for each $V = (-\varepsilon, \varepsilon)$ *with* $\varepsilon > 0$, $\overline{A^V}$ *is convex valued. Also, it is upper semicontinuous and non-empty valued, then* A *is almost w-upper semi-continuous.*

Now, for each $x \in X$ *with* $P(x) \ne \emptyset$, *let* $N_x = [0, 1]$ *be an open neighborhood of* x *in* X *and we define the set-valued maps* $\phi_x, \psi_x : X \to 2^X$ *by*

$$\phi_x(z) = \psi_x(z) = \begin{cases} \emptyset, & \text{if } 0 \le z \le \frac{1}{2}, \\ [0, \frac{1}{2}], & \text{if } \frac{1}{2} < z \le 1. \end{cases}$$

It is clear that each $\psi_x^{-1}(z)$ *is compactly open in* X, *and hence* P *is* F*-majorized on* X. *Also, the set* $\{x \in X : A(x) \cap P(x) \ne \emptyset\}$ *is open in* X. *Therefore, all the conditions of Corollary 3.4 hold and since the fixed point set of* A *is* $[0, \frac{1}{2}]$, *so the all points of interval* $[0, \frac{1}{2}]$ *are equilibrium points of one-person game* Γ.

4. **Conclusions**

This paper establishes a new best proximity point theorem for set-valued maps that are nearly w-upper semicontinuous and illustrates it with examples. Using this result, we derive equilibrium existence theorems for free abstract economies with possibly non-upper semicontinuous constraints and prove the existence of equilibrium pairs for economies with two set-valued constraints. Examples are provided to demonstrate the applicability of the main results.

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