# MILD SOLUTIONS FOR NEUTRAL CONFORMABLE FRACTIONAL ORDER FUNCTIONAL EVOLUTION EQUATIONS USING MEIR-KEELER TYPE FIXED POINT THEOREM

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*Our mission is to demonstrate the existence, uniqueness, attractiveness, and controllability of mild solutions to neutral conformable fractional-order functional evolution equations, specifically of order between 1 and 2. These intriguing equations encompass finite delay, all while adhering to local conditions within a separable Banach space. By invoking Meir-Keeler's fixed-point Theorem and enhancing it with measures of noncompactness, we establish the existence of these solutions. To highlight the potency of our approach, we present a captivating example.*

**Keywords:** Neutral functional differential equation, mild solution, finite delay, infinite delay, fixed point, condensing operator, measure of noncompactness, conformable fractional.

**MSC2020:** 47H 10, 54H 25.

## 1. Introduction

This paper establishes criteria for the existence, uniqueness, attractivity, and controllability of mild solutions to conformable fractional-order neutral functional evolution equations with finite delay. The analysis is conducted within the framework of a separable Banach space, where the completeness of the space and local conditions are leveraged to prove the existence of a unique mild solution, forming the cornerstone of our study.

The conformable derivative, introduced by Khalil et al. [29], has advanced fractional calculus beyond classical definitions [33, 30] and enabled diverse applications (see eg. [9, 11]). Recent work has extended this framework to complex settings: Liang et al. [32] and Bouaouid et al. [21, 20] studied impulsive differential equations using semigroup theory, while Bouaouid et al. [12, 8] and Atraoui et al. [13] applied fixed-point theorems to prove existence and controllability. Further contributions by Baghli et al. [14] and Agarwal et al. [7] extended this approach to controllability, also, see [1, 16, 17, 18, 19, 4, 5, 6, 26, 31, 34]. Researchers have also utilized measures of noncompactness to address solution existence challenges across various contexts (see eg. [10, 15, 27]).

Firstly, we study in Section 3 the conformable fractional order functional evolution equations with local conditions of the form:

$$D^c[D^c(\psi(s) - \mathcal{Y}(s, \psi_s))] = \mathfrak{P}(\psi(s) - \mathcal{Y}(s, \psi_s)) + \Psi(s, \psi_s), \quad \text{a.e. } s \in I := [0, +\infty); \tag{1}$$

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$$\psi(s) = \eta(s), \quad s \in \mathcal{H} := [-b, 0], \ where\ 0 < b < +\infty, \quad D^c\psi(0) = \vartheta \in \mathcal{D}; \tag{2}$$

Let $\Psi$ and $\mathcal{Y}$: $I \times C([-b, 0], \mathcal{D}) \to \mathcal{D}$ denote given functions, $\eta$: $[-b, 0] \to \mathcal{D}$ represent a continuous function, $\mathfrak{P}$: $D(\mathfrak{P}) \subset \mathcal{D} \to \mathcal{D}$ serves as the infinitesimal generator of a strongly continuous cosine function, represented by a family of bounded linear operators $(\mathfrak{C}(s))_{s\in\mathbb{R}}$. This elegant framework ensures that $\mathfrak{S}(s) = \int_0^s \mathfrak{C}(x)dx$, weaving a seamless integration of the operators over the real line, and $\mathcal{D}$ denote a real separable Banach space equipped with norm $|\cdot|$, while $D^c$ represents a fractional conformable derivative of order $0 < c \leq 1$.

Let $\psi_s$ denote, for all $s \geq 0$, the function in $C([-b, 0], \mathcal{D})$ defined as $\psi_s(\theta) = \psi(s+\theta)$. Here, $\psi_s(\cdot)$ captures the state history from $s - b$ up to the current time $s$. Additionally, we will explore the attractiveness of mild solutions to conformable fractional-order functional evolution equations subject to local conditions $(1)-(2)$. Moreover, we will delineate adequate conditions to guarantee the controllability of mild solutions across the semi-infinite interval $I = [0, +\infty)$ for conformable fractional-order functional evolution equations characterized by the following conditions

$$D^c[D^c(\psi(s) - \mathcal{Y}(s, \psi_s))] = \mathfrak{P}(\psi(s) - \mathcal{Y}(s, \psi_s)) + \Psi(s, \psi_s) + \mathcal{B}\mathcal{U}(s); \tag{3}$$

$$\psi(s) = \eta(s), \quad s \in \mathcal{H}, \quad D^c\psi(0) = \vartheta \in \mathcal{D}; \tag{4}$$

where $\mathfrak{P}$, $\Psi$, $\mathcal{Y}$, and $\eta$ are defined as in problem $(1)-(2)$, the control function $\mathcal{U}(\cdot)$ is provided in $L^2(I, \mathcal{D})$, representing the Banach space of admissible control functions, and $\mathcal{B}$ stands as a bounded linear operator mapping from $\mathcal{D}$ to $\mathcal{D}$.

Ultimately, we furnish an illustrative example demonstrating the abstract theory expounded in the preceding results.

## 2. Introductory concepts

In this part, we present symbols, explanations, and fundamental principles drawn from multivalued analysis.

The notation $BC(I, \mathcal{D})$ represents the Banach space comprising all functions that are both bounded and continuous from $I$ to $\mathcal{D}$, where the norm is defined as: $\|\psi\|_{BC} = \sup\{\ |\psi(s)|\ :\ s \in I\}$.

Consider the space $BC_\infty$ defined as $\{\psi : [-b, +\infty) \to \mathcal{D},\ \psi|_{[0,s]}$ is bounded and continuous for $s > 0\}$, with the norm: $\|\psi\|_{BC_\infty} = \sup\{\ |\psi(s)|\ :\ s \in [0, T]\}$, here $T = \sup\{s > 0 :\ \psi|_{[0,s]}$ is bounded and continuous$\}$.

**Definition 2.1.** *(Khalil et al.* [29]*) The conformable fractional derivative* of order $0 < c \leq 1$ for a function $\psi(\cdot)$ is defined as

$$D^c\psi(s) = \lim_{t\to 0} \frac{\psi(s + ts^{1-c}) - \psi(s)}{t}, \quad s > 0;$$

$$D^c\psi(0) = \lim_{t\to 0} D^c\psi(t),$$

**Definition 2.2.** *(see eg.* [10, 15, 27]*)* Let $\mathcal{F}_{\mathcal{D}}$ the bounded subsets of $\mathcal{D}$ so the map $\mathfrak{A} : \mathcal{F}_{\mathcal{D}} \to [0, +\infty)$ denotes *the Kuratowski measure of noncompactness which is given by*

$$\mathfrak{A}(\mathcal{F}) = \inf\{\alpha > 0 : \mathcal{F} \subseteq \bigcup_{j+1}^{k} \mathcal{F}_j \ and\ diam(\mathcal{F}_j) \leq \alpha\}, \ here\ \mathcal{F} \in \mathcal{F}_{\mathcal{D}}.$$

**Definition 2.3.** Let's say we have a nonempty subset $\mathcal{F}$ within the Banach space $\mathcal{D}$, and consider any arbitrary measure of noncompactness $\mathfrak{A}$ defined on $\mathcal{D}$. We define $\mathfrak{M} : \mathcal{F} \to \mathcal{D}$ as a *Meir-Keeler condensing operator* if it meets the following criteria: $\mathfrak{M}$ is both continuous and bounded, and for any given $\beta > 0$, there exists $\mu > 0$ such that if $\beta < \mathfrak{A}(\mathcal{R}) < \beta + \mu$, then $\mathfrak{A}(\mathfrak{M}(\mathcal{R})) \leq \beta$ holds true for every bounded subset $\mathcal{R}$ of $\mathcal{F}$.

**Lemma 2.1.** *(see [25]) Consider $\mathcal{D}$ as a Banach space, and let $\mathcal{F}$ be a subset of $C(I,\mathcal{D})$ that is both bounded and equicontinuous. Then, the function $\mathfrak{A}(\mathcal{F}(s))$ remains continuous over the interval $I$, and $\mathfrak{A}_I(\mathcal{F})$ equals the maximum value of $\mathfrak{A}(\mathcal{F}(s))$ for $s$ in $I$.*

**Theorem 2.1.** *(Meir-Keeler's Theorem [8]) Let $\mathcal{F}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{D}$. If $\mathfrak{M} : \mathcal{F} \to \mathcal{F}$ is a continuous Meir-Keeler condensing operator, then $\mathfrak{M}$ guarantees at least one fixed point, and the collection of all such fixed points within $\mathcal{F}$ forms a compact set.*

**Definition 2.4.** *(see [23]) We characterize solutions of Equations (1) − (2) as locally attractive if there exists a closed ball $\overline{B}(\psi^*,\sigma)$ within the space $BC$, where $\psi^* \in BC$, such that for any solutions $\psi$ and $\widetilde{\psi}$ of Equations (1) − (2) within $\overline{B}(\psi^*,\sigma)$, the following convergence occurs:* $\lim_{s\to+\infty}(\psi(s)-\widetilde{\psi}(s)) = 0$.

## 3. Existence results

In this section, we reveal our main findings regarding the existence of solutions to problems (1) − (2). Before presenting and verifying this result, we introduce the notion of its mild solution.

**Definition 3.1.** *We define the mild solution $\psi \in C([-b,+\infty),\mathcal{D})$ of the problem (1) − (2) as follows*

$$\psi(s) = \begin{cases} \eta(s), & \text{if } s \in \mathcal{H}; \\ \mathfrak{C}\left(\frac{s^c}{c}\right)\,[\eta(0) - \mathcal{Y}(0,\eta(0))] + \mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta + \mathcal{Y}(s,\psi_s) & \\ + \int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\,\Psi(t,\psi_t)\,dt, & \text{if } s \in I; \end{cases}$$

We must introduce the following hypotheses, which will be utilized subsequently:

$(i)$ The function $\Psi : I \times C(\mathcal{H},\mathcal{D}) \to \mathcal{D}$ is carathéodory function and there exist a continuous function $\mathcal{O} : I \to I$ such that

$$|\Psi(s,u)| \leq \mathcal{O}(s)\|u\|,$$

$$\mathfrak{A}(\Psi(s,\mathcal{F})) \leq \mathcal{O}(s)\mathfrak{A}(\mathcal{F}),$$

and $\mathcal{O}^* := \sup_{s\in I}\int_0^s t^{c-1}\mathcal{O}(t)dt < \infty$, for all $s \in I,\ u \in C(\mathcal{H},\mathcal{D})$, *bounded set* $\mathcal{F} \subset C([-b,+\infty),\mathcal{D})$ *and* $0 < c \leq 1$ ;

$(ii)$ The cosine operator $\mathfrak{C}(s)_{s\in\mathbb{R}}$ is uniformly continuous and there exist constants $\mathcal{M}_c^{\mathfrak{C}}$, $\mathcal{M}_c^{\mathfrak{S}}$ both greater than zero, such that

$$\sup_{s\in I}\|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\| \leq \mathcal{M}_c^{\mathfrak{C}} \ \textit{and} \ \sup_{s\in I}\|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\| \leq \mathcal{M}_c^{\mathfrak{S}}.$$

$(iii)$ The function $\mathcal{Y} : I \times C([-b,0],\mathcal{D}) \to \mathcal{D}$ is carathéodory function and there exist $\mathcal{Y}^* > 0$ such that

$$|\mathcal{Y}(s,u)| \leq \mathcal{Y}^*\|u\|,$$

$$\mathfrak{A}(\mathcal{Y}(s,\mathcal{F})) \leq \mathcal{Y}^*\ \mathfrak{A}(\mathcal{F}),$$

$$\{s \mapsto \mathcal{Y}(s,u),\ u \in \mathcal{F}\} \text{ is equicontinuous on each compact interval of } I,$$

for all $s \in I,\ u \in C(\mathcal{H},\mathcal{D})$, *bounded set* $\mathcal{F} \subset C([-b,+\infty),\mathcal{D})$.

**Theorem 3.1.** *Given assumptions $(i)-(iii)$, if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^* < 1$, then problem (1) − (2) admits at least one mild solution over $BC$.*

**Proof.** We initiate the transformation of problem (1) − (2) into a fixed-point problem. Let's examine the operator $\mathfrak{M} : BC([-b,+\infty),\mathcal{D}) \to BC([-b,+\infty),\mathcal{D})$, which is delineated as follows:

$$\mathfrak{M}(\psi)(s) = \begin{cases} \eta(s), & \text{if } s \in [-b,0]; \\ \mathfrak{C}\Big(\frac{s^c}{c}\Big)\ [\eta(0) - \mathcal{Y}(0,\eta(0))] + \mathfrak{S}\Big(\frac{s^c}{c}\Big)\vartheta + \mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt, & \text{if } s \in I; \end{cases}$$

The operator $\mathfrak{M}$ maps $BC$ into $BC$. Specifically, for $\psi \in BC$ and for any $s \in I$ we have:

$$\begin{aligned} |\mathfrak{M}(\psi)(s)| &\le \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\||\eta(0) + \mathcal{Y}(0,\eta(0))| + \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\|\|\vartheta\| + |\mathcal{Y}(s,\psi_s)| \\ &+ \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\|\ |\Psi(t,\psi_t)|dt \\ &\le \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)\|\psi\|_{BC}. \end{aligned}$$

which imply $\mathfrak{M} \in BC$.

Furthermore, suppose $l \ge \frac{\mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*)+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$, and let $B_l$ denote the closed ball in $BC$ centered at the origin with radius $l$. consider $\psi \in B_l$ and $s \in I$, we get

$$|\mathfrak{M}(\psi)(s)| \le \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)l$$

Thus, $\|\mathfrak{M}(\psi)\|_{BC} \le l$.

Now we prove that $\mathfrak{M} : B_l \to B_l$ satisfies the assumptions of Meir-Keeler's fixed point Theorem.

Firstly, we establish that $\mathfrak{M}$ exhibits continuity within $B_l$. Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$ in $B_l$. We have

$$|\mathfrak{M}(\psi_n)(s) - \mathfrak{M}(\psi)(s)| \le |\mathcal{Y}(s,(\psi_s)_n) - \mathcal{Y}(s,\psi_s)| + \mathcal{M}_c\int_0^s t^{c-1}|\Psi(t,(\psi_t)_n) - \Psi(t,\psi_t)|dt$$

and by $(i)$ and $(iii)$ we get $\Psi(t,(\psi_t)_n) \to \Psi(t,\psi_t)$ and $\mathcal{Y}(t,(\psi_t)_n) \to \mathcal{Y}(t,\psi_t)$ as $n \to +\infty$ for ae. $t \in I$ and by the Lebesgue dominated convergence Theorem we conclude that

$$\|\mathfrak{M}(\psi_n) - \mathfrak{M}(\psi)\|_{BC} \to 0,\ as\ n \to \infty.$$

Thus, $\mathfrak{M}$ is continuous.

Secondly, we observe that $\mathfrak{M}(B_l) \subset B_l$, which is evident.

Moving on, we note that $\mathfrak{M}(B_l)$ demonstrates equicontinuity on every compact interval $X'$ of $I$, let $x_1,\ x_2 \in X'$ with $x_2 > x_1$ we have

$$\begin{aligned} |\mathfrak{M}(\psi)(x_1) - \mathfrak{M}(\psi)(x_2)| &\le \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ (1+\mathcal{Y}^*)\|\eta\| \\ &+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\vartheta\| + |\mathcal{Y}(x_1,\psi_{x_1}) - \mathcal{Y}(x_2,\psi_{x_2})| \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c-t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c-t^c}{c}\Big)\|_{B(\mathcal{D})}\ |\Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\int_{x_1}^{x_2} t^{c-1}\ |\Psi(t,\psi_t)|dt \end{aligned}$$

As $x_1 \to x_2$, the uniformly continuity property of the operators $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous enequality converges to zero. This confirms the equicontinuity of $\mathfrak{M}$.

Additionally, we establish the equiconvergence of $\mathfrak{M}(B_l)$. For $s \in I$ and $\psi \in B_l$, we find

$$|\mathfrak{M}(\psi)(s)| \leq \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}} \int_0^s x^{c-1}\mathcal{O}(x)dx + \mathcal{Y}^*)l$$

Consequently, $|\mathfrak{M}(\psi)(s)| \to l',\ as\ s \to +\infty$. Where $l' \leq \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*)+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)l$. Here $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{M}(\psi)(s) - \mathfrak{M}(\psi)(+\infty)| \to 0,\ \ s \to +\infty.$$

Finally, we confirm that the Meir-Keeler type condition is satisfied.

For any given $\beta > 0$, there exists $\mu > 0$ such that if $\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu$, then $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta$ for any $\mathcal{R} \subset B_l$ where $\mathfrak{A}_I(\mathcal{R}) = \max_{x\in I}\mathfrak{A}(\mathcal{R}(x))$.
We have

$$\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)) \leq (\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}).$$

Since $\mathfrak{M}(\mathcal{R})$ is bounded and equicontinuous of all $\mathcal{R} \subset B_l$. Then

$$\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) = \max_{s\in I}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)).$$

Therefore $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq (\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}) \leq \beta \Rightarrow \mathfrak{A}_I(\mathcal{R}) \leq \frac{\beta}{\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}$.
Then for any given $\beta > 0$ and taking $\mu = \left(\frac{1-\mathcal{Y}^*-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}{\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}\right)\beta - \epsilon$ such that $\epsilon > 0$, we obtain

$$\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu \Rightarrow \mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta,\ \textit{for any}\ \mathcal{R} \subset B_l$$

Hence $\mathfrak{M}$ is a Meir-Keeler condensing operator.
Through these steps, we ensure that the conditions required for Meir-Keeler' s fixed-point Theorem [8] are satisfied by $\mathfrak{M} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{M}$ has a fixed point $\psi$ that provides a mild solution to the problem $(1) - (2)$.

**3**.1. **Uniqueness results.** Subsequently, we present our main finding concerning the existence and uniqueness of solutions to problem $(1) - (2)$. Before proceeding with the demonstration of this outcome, we establish the following conditions.

$(i)'$ The function $\Psi : I \times C(\mathcal{H}, \mathcal{D}) \to \mathcal{D}$ is carathéodory function and there exist a continuous function $\mathcal{O} : I \to I$ such that

$$|\Psi(s,u) - \Psi(s,v)| \leq \mathcal{O}(s)\|u-v\|,$$

$$\Psi^* = \sup_{s\in I}\int_0^s t^{c-1}\Psi(t,0)dt < \infty$$

$$\mathfrak{A}(\Psi(s,\mathcal{F})) \leq \mathcal{O}(s)\mathfrak{A}(\mathcal{F}),$$

and $\mathcal{O}^* := \sup_{s\in I}\int_0^s t^{c-1}\mathcal{O}(t)dt < \infty$, for all $s \in I,\ u, v \in C(\mathcal{H}, \mathcal{D})$, *bounded set* $\mathcal{F} \subset C([-b, +\infty), \mathcal{D})$ *and* $0 < c \leq 1$ ;

$(iii)'$ The function $\mathcal{Y} : I \times C([-b, 0], \mathcal{D}) \to \mathcal{D}$ is carathéodory function, continuous according to its first variable and there exist $\mathcal{Y}^* > 0$ such that

$$|\mathcal{Y}(s,u) - \mathcal{Y}(s,v)| \leq \mathcal{Y}^*\|u-v\|,$$

$$\mathfrak{A}(\mathcal{Y}(s,\mathcal{F})) \leq \mathcal{Y}^*\mathfrak{A}(\mathcal{F}),$$

$$\{s \mapsto \mathcal{Y}(s,u),\ u \in \mathcal{F}\} \text{ is equicontinuous on each compact interval of } I,$$

$$\mathcal{Y}' = \sup_{s\in I}|\mathcal{Y}(s,0)| < +\infty,$$

for all $s \in I,\ u \in C(\mathcal{H}, \mathcal{D})$, *bounded set* $\mathcal{F} \subset C([-b, +\infty), \mathcal{D})$.

**Theorem 3.2.** *Given assumptions* $(i)' - (ii)$ *and* $(iii)'$, *if* $\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* < 1$, *then problem* $(1) - (2)$ *possesses a unique mild solution over BC.*

**Proof.** By following analogous procedures as those in the proof of Theorem 3.2, we confirm the presence of a sole mild solution. Particularly noteworthy is the adjustment of the radius estimation to $l \geq \frac{\mathcal{M}_c^{\mathfrak{C}}[\|\eta\|(1+\mathcal{Y}^*)+\mathcal{Y}']+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*+\mathcal{Y}'}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$.

Now, we proceed to demonstrate uniqueness. Suppose $\psi$ and $\psi^*$ are both mild solutions of the problem $(1)-(2)$, then,

$$\begin{aligned}|\psi(s)-\psi^*(s)| &= |\mathfrak{M}\psi(s)-\mathfrak{M}\psi^*(s)| \\ &\leq |\mathcal{Y}(s,\psi_s)-\mathcal{Y}(s,\psi_s^*)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,\psi_t)-\Psi(t,\psi_t^*)|dt \\ &\leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\|\psi-\psi^*\|_{BC}\end{aligned}$$

then $(1-(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*))\|\psi-\psi^*\|_{BC} \leq 0$ therefore $\psi=\psi^*$. Hence the uniqueness of the mild solution.

**3.2. Attractivity of Mild Solutions.** In this section, we explore the local attractiveness of solutions to problem $(1)-(2)$.

**Theorem 3.3.** *Given assumptions $(i)'-(ii)$ and $(iii)'$, if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*<1$, and let $\psi^*$ be a solution of $(1)-(2)$, and $\overline{B}(\psi^*,\sigma)$ represent the closed ball in BC such that: $\sigma \geq \frac{\mathcal{M}_c^{\mathfrak{C}}[\|\eta\|(1+\mathcal{Y}^*)+\mathcal{Y}']+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*+\mathcal{Y}'}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$ then the problem $(1)-(2)$ exhibits attractivity.*

**Proof.** For $\psi \in \overline{B}(\psi^*,\sigma)$, by $(i)'-(ii)$ *and* $(iii)'$, we get

$$\begin{aligned}|\mathfrak{M}(\psi)(s)-\psi^*(s)| &= |\mathfrak{M}(\psi)(s)-\mathfrak{M}(\psi^*)(s)| \\ &\leq |\mathcal{Y}(s,\psi_s^*)-\mathcal{Y}(s,\psi_s)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,\psi_t^*)-\Psi(t,\psi_t)|dt \\ &\leq \mathcal{Y}^*\|\psi_s^*-\psi_s\| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}\mathcal{O}(t)\|\psi_t^*-\psi_t\|dt \\ &\leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\sigma \leq \sigma\end{aligned}$$

consequently, $\mathfrak{M}(\overline{B}(\psi^*,\sigma)) \subset \overline{B}(\psi^*,\sigma)$ then for each solutions $\psi,\ \widetilde{\psi} \in \overline{B}(\psi^*,\sigma)$ of $(1)-(2)$ and $s \in I$, we have

$$|\psi(s)-\widetilde{\psi}(s)| \leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\ \|\widetilde{\psi}-\psi\|_{BC}$$

hence

$$\|\widetilde{\psi}-\psi\|_{BC}=0$$

As a result, the problem solutions $(1)-(2)$ are locally attractive.

**3.3. Controllability results.** This section delineates the controllability outcomes for the system $(3)-(4)$. Before delving into this, we introduce a specific type of solutions for problem $(3)-(4)$.

**Definition 3.2.** *We define the mild solution $\psi \in C([-b,+\infty),\mathcal{D})$ of the problem $(3)-(4)$ as follows*

$$\psi(s)=\begin{cases}\eta(s), & \text{if } s\in\mathcal{H};\\ \mathfrak{C}\left(\frac{s^c}{c}\right)\ [\eta(0)-\mathcal{Y}(0,\eta(0))]+\mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta+\mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\ \Psi(t,\psi_t)\ dt,+\int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\ \mathcal{B}\mathcal{U}(t)dt, & \text{if } s\in I;\end{cases}$$

**Definition 3.3.** *The system $(3)-(4)$ is considered controllable if, for every initial function $\eta \in C([-b,0],\mathcal{D})$ and $\widehat{\psi} \in \mathcal{D}$, there exists some natural number $n \in \mathbb{N}$ and a control function $\mathcal{U} \in L^2([0,n],\mathcal{D})$ such that the resulting mild solution $\psi(\cdot)$ satisfies the terminal condition $\psi(n)=\widehat{\psi}$.*

We will adopt the assumptions $(i)-(iii)$ from Section 3, along with the introduction of the following additional assumption, which will be consistently assumed hereafter:

$(iv)$ For all $n$ integer, the linear operator $\mathfrak{V} : L^2([0,n],\mathcal{D}) \to \mathcal{D}$ defined by

$$\mathfrak{V}\mathcal{U} = \int_0^n x^{c-1}\mathfrak{S}\Big(\frac{n^c - x^c}{c}\Big)\ \mathcal{B}\mathcal{U}(x)dx,$$

possesses a pseudo-invertible operator $\widetilde{\mathfrak{V}}^{-1}$, which maps functions from $L^2([0,n],\mathcal{D})$ to the space $L^2([0,n],\mathcal{D})$ excluding the kernel of $\mathfrak{V}$, and is bounded. Additionally, $\mathcal{B}$ is bounded, satisfying:

$$\|\mathcal{B}\| \le \widetilde{\mathcal{N}}\ and\ \|\widetilde{\mathfrak{V}}^{-1}\| \le \widetilde{\mathcal{N}}_1.$$

$(v)$ There exists a continuous function function $\mathcal{K}_{\mathfrak{V}} : [0,n] \to \mathbb{R}_+$ such that: for any bounded subset $\mathcal{F} \subset \mathcal{D}$, we have : $\mathfrak{A}(\widetilde{\mathfrak{V}}^{-1}(\mathcal{F})(s)) \le \mathcal{K}_{\mathfrak{V}}(s)\mathfrak{A}(\mathcal{F})$, $s \in I$ and $\mathcal{K}' := \sup_{s\in I}\int_0^s t^{c-1}\mathcal{K}_{\mathfrak{V}}(t)dt < \infty$ for all $0 < c \le 1$.

**Theorem 3.4.** *Assume that* $(i)-(v)$ *hold. If* $\max\{\mathcal{M}_c\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}+(\mathcal{Y}^*+\mathcal{M}_c\mathcal{O}^*)[1+\mathcal{M}_c\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}],\ (\mathcal{Y}^*+\mathcal{M}_c\mathcal{O}^*)(1+\mathcal{M}_c\widetilde{\mathcal{N}}\ \mathcal{K}')\} < 1$, *then the problem* $(3)-(4)$ *is controllable on* $[-b,+\infty)$.

**Proof.** Convert problem $(3)-(4)$ into a fixed-point problem. We examine the operator $\mathfrak{M} : BC_\infty([-b,+\infty),\mathcal{D}) \to BC_\infty([-b,+\infty),\mathcal{D})$, defined as:

$$\mathfrak{M}(\psi)(s) = \begin{cases} \eta(s), & \text{if } s\in[-b,0];\\ \mathfrak{C}\Big(\frac{s^c}{c}\Big)\ [\eta(0)-\mathcal{Y}(0,\eta(0))] + \mathfrak{S}\Big(\frac{s^c}{c}\Big)\vartheta + \mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt, + \int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \mathcal{B}\mathcal{U}(t)dt, & \text{if } s\in I; \end{cases}$$

Using assumption $(iv)$, for arbitrary function $\psi(\cdot)$, we define the control

$$\mathcal{U}_\psi(s) = \widetilde{\mathfrak{V}}^{-1}\Big[\widehat{\psi} - \mathfrak{C}\Big(\frac{n^c}{c}\Big)\ [\eta(0)-\mathcal{Y}(0,\eta(0))] - \mathfrak{S}\Big(\frac{n^c}{c}\Big)\vartheta - \mathcal{Y}(n,\psi_n)$$
$$- \int_0^n t^{c-1}\mathfrak{S}\Big(\frac{n^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt\Big](s)$$

Noting that, we have

$$|\mathcal{U}_\psi(s)| \le \widetilde{\mathcal{N}}_1\Big[|\widehat{\psi}| + \mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)\ \|\psi\|_{BC_\infty}\Big]$$

The operator $\mathfrak{M}$ maps $BC_\infty$ into $BC_\infty$. Specifically, the mapping $\mathfrak{N}(\psi)$ is continuous on $[-b,n]$ for any $\psi \in BC_\infty$ we have:

$$\begin{aligned}|\mathfrak{M}(\psi)(s)| &\le \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\|[|\eta(0)| + |\mathcal{Y}(0,\eta(0))|] + \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\|\|\vartheta\| + |\mathcal{Y}(s,\psi_s)|\\ &+ \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\||\Psi(t,\psi_t)|dt + \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\|\|\mathcal{B}\|\ |\mathcal{U}_\psi(t)|dt\\ &\le (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + \|\psi\|_{BC_\infty}\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}\\ &+ (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big].\end{aligned}$$

Which imply $\mathfrak{M} \in BC_\infty$.

Furthermore, suppose $l \ge \frac{(\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})}{1-\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}+(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big]}$, and let $B_l$ denote the closed ball in $BC_\infty$ centered at the origin with radius $l$. Let $\psi \in B_l$ and $s \in I$, we get

$$|\mathfrak{M}(\psi)(s)| \leq l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c} + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big] + (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}).$$

Thus, $\|\mathfrak{M}(\psi)\|_{BC_\infty} \leq l$.

We now aim to demonstrate that $\mathfrak{M} : B_l \to B_l$ fulfills the prerequisites of Meir-Keeler's fixed-point Theorem.

Firstly, we establish that $\mathfrak{M}$ exhibits continuity within $B_l$. Let $\{\psi_k\}$ be a sequence such that $\psi_k \to \psi$ in $B_l$. We have

$$\begin{aligned}|\mathfrak{M}(\psi_k)(s) - \mathfrak{M}(\psi)(s)| &\leq |\mathcal{Y}(s,(\psi_s)_k) - \mathcal{Y}(s,\psi_s)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,(\psi_t)_k) - \Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s t^{c-1}|\mathcal{U}_{\psi_k}(t) - \mathcal{U}_\psi(t)|dt \\ &\leq |\mathcal{Y}(s,(\psi_s)_k) - \mathcal{Y}(s,\psi_s)| \\ &+ \mathcal{M}_c^{\mathfrak{S}}\Big(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}\Big)\int_0^n t^{c-1}|\Psi(t,(\psi_t)_k) - \Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}\Big[|\widehat{\psi}_k - \widehat{\psi}| + |\mathcal{Y}(n,(\psi_n)_k) - \mathcal{Y}(n,\psi_n)|\Big]\end{aligned}$$

Using $(i)$ and $(iii)$, we have $\Psi(s,(\psi_s)_k) \to \Psi(s,\psi_s)$ and $\mathcal{Y}(s,(\psi_s)_k) \to \mathcal{Y}(s,\psi_s)$ as $k \to +\infty$ for almost every $s \in [0,n]$. Then, by the Lebesgue dominated convergence Theorem: $\|\mathfrak{M}(\psi_n) - \mathfrak{M}(\psi)\|_{BC_\infty} \to 0,\ as\ n \to \infty$. Thus, $\mathfrak{M}$ is continuous.

Secondly, we observe that $\mathfrak{M}(B_l) \subset B_l$, which is evident.

Moving on, we note that $\mathfrak{M}(B_l)$ demonstrates equicontinuity on every compact interval $X' = [0,n]$ , let $x_1,\ x_2 \in X'$ with $x_2 > x_1$ we have

$$\begin{aligned}|\mathfrak{M}(\psi)(x_1) - \mathfrak{M}(\psi)(x_2)| &\leq \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ (1+\mathcal{Y}^*)\|\eta\| \\ &+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\vartheta\| + |\mathcal{Y}(x_1,\psi_{x_1}) - \mathcal{Y}(x_2,\psi_{x_2})| \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c - t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - t^c}{c}\Big)\|_{B(\mathcal{D})}\ |\Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\int_{x_1}^{x_2} t^{c-1}\ |\Psi(t,\psi_t)|dt \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c - t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - t^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\mathcal{B}\|\|\mathcal{U}_\psi(t)\|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_{x_1}^{x_2} t^{c-1}\ \|\mathcal{U}_\psi(t)\|dt\end{aligned}$$

As $x_1 \to x_2$, the uniformly continuity property of $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous inequality converges to zero. This confirms the equicontinuity of $\mathfrak{M}$.

Additionally, we establish the equiconvergence of $\mathfrak{M}(B_l)$. For $s \in X'$ and $\psi \in B_l$, we find

$$|\mathfrak{M}(\psi)(s)| \leq (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c} + (\mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)dx + \mathcal{Y}^*)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big].$$

Consequently, $|\mathfrak{M}(\psi)(s)| \to l', \ as \ s \to +\infty$. Where $l' \le (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c} + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*dx + \mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big]$. Here $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{M}(\psi)(s) - \mathfrak{M}(\psi)(+\infty)| \to 0, \ s \to +\infty.$$

Finally, we confirm that the Meir-Keeler type condition is satisfied.
For any given $\beta > 0$, there exists $\mu > 0$ such that if $\beta < \mathfrak{A}_I(\mathcal{R}) < \beta+\mu$, then $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \le \beta$ for any $\mathcal{R} \subset B_l$ where $\mathfrak{A}_I(\mathcal{R}) = \max_{s\in I}\mathfrak{A}(\mathcal{R}(s))$. We have

$$\mathfrak{A}(\mathcal{U}_{\mathcal{R}}(s)) \le \mathcal{K}_{\mathfrak{V}}(s)(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}),$$

which imply

$$\begin{aligned}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)) &\le \mathcal{Y}^*\mathfrak{A}_I(\mathcal{R}) + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}\mathcal{O}(t)\mathfrak{A}_I(\mathcal{R})dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s t^{c-1}\mathcal{K}_{\mathfrak{V}}(t)(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R})dt \\ &\le (1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R})\end{aligned}$$

Since $\mathfrak{M}(\mathcal{R})$ is bounded and equicontinuous of all $\mathcal{R} \subset B_l$. Then

$$\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) = \max_{s\in I}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)).$$

Therefore $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \le (1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}) \le \beta \Rightarrow \mathfrak{A}_I(\mathcal{R}) \le \frac{\beta}{(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}$.
Then for any given $\beta > 0$ and taking $\mu = \left(\frac{1-(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}{(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\,\mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}\right)\beta - \epsilon$ such that $\epsilon > 0$, we obtain

$$\beta < \mathfrak{A}_I(\mathcal{R}) < \beta+\mu \Rightarrow \mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \le \beta, \ for \ any \ \mathcal{R} \subset B_l$$

Hence $\mathfrak{M}$ is a Meir-Keeler condensing operator.
Through these steps, we ensure that the conditions required for Meir-Keeler' s fixed-point Theorem [8] are satisfied by $\mathfrak{M} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{M}$ has a fixed point $\psi$ that provides the controllability of the problem $(3)-(4)$.

## 4. Examples

**Example 4.1.** *To showcase the practical application of our results, let $\mathcal{E}$ denote a nonempty bounded open set in $\mathbb{R}^2$. We explore the following conformable fractional differential equation:*

$$D_s^{\frac{2}{3}}[D_s^{\frac{2}{3}}\psi(s,x) - \mathcal{Y}(s,\psi(s-b,x))] = D_x^2[\psi(s,x) - \mathcal{Y}(s,\psi(s-b,x))] + \Psi(s,\psi(s-b,x)),$$

$$x \in \mathcal{E}, \ s \in [0,+\infty); \tag{5}$$

$$\psi(s,x) = 0, \ s \in [0,+\infty), \ x \in \partial\mathcal{E}; \tag{6}$$

$$\psi(s,x) = \eta(s,x); \quad D_s^{\frac{2}{3}}[\psi(0,x)] = \vartheta, \ s \in [-b,0], \ x \in \mathcal{E}. \tag{7}$$

*Here, $b > 0$ and we have*

$$\Psi(s,\psi(s-b,x)) = \frac{\exp -s}{7}\sin\psi(s-b,x),$$

$$\mathcal{Y}(s,\psi(s-b,x)) = \frac{\exp -s}{2}\tanh\psi(s-b,x),$$

*taking $\mathcal{D} = L^2(\mathcal{E})$ and defining $\mathfrak{P}$ as follows: $\mathfrak{P}\psi = D_x^2\psi, \ \psi \in D(\mathfrak{P})$ and*

$$D(\mathfrak{P}) = \{\psi \in \mathcal{H}(\mathcal{D}), \ \psi(x)|_{x\in\partial\mathcal{E}} = 0\}$$

*It is well know the operator $\mathfrak{P}$ generates a cosine family $((\mathfrak{C}(s))_{s\in\mathbb{R}}, (\mathfrak{S}(s))_{s\in\mathbb{R}})$. Additionally, it follows that*

$$\|\mathfrak{C}(s)\| \le 1 \ and \ \|\mathfrak{S}(s)\| \le 1, \ for \ all \ s \in [0,+\infty).$$

*Thus, to apply our Theorems on existence and attractivity, we require* $\mathcal{Y}^* + \mathcal{O}^* < 1$.
*The function* $\Psi(s, \psi(s-b,x)) = \frac{\exp -s}{7} \sin\psi(s-b,x)$ *is carathéodory and*

$$|\Psi(s,\psi_1(s-b,x)) - \Psi(s,\psi_2(s-b,x))| \leq \frac{\exp -s}{7}|\psi_1(s-b,x) - \psi_2(s-b,x)|$$

*thus* $\mathcal{O}(s) = \frac{\exp -s}{7}$. *Moreover, we have*

$$\mathcal{O}^* = \sup\{\int_0^s x^{-\frac{1}{3}} \frac{\exp -x}{7} dx,\ s \in [0,+\infty)\} = \frac{\Gamma(\frac{2}{3})}{7} \backsimeq 0.19302,\ \Psi_0 = 0.$$

*Also,* $\mathcal{Y}(s,\psi(s-b,x)) = \frac{\exp -s}{2} \tanh\psi(s-b,x)$ *is carathéodory and*

$$|\mathcal{Y}(s,\psi_1(s-b,x)) - \mathcal{Y}(s,\psi_2(s-b,x))| \leq \frac{1}{2}|\psi_1(s-b,x) - \psi_2(s-b,x)|$$

*thus* $\mathcal{Y}^* = \frac{1}{2}$. *Moreover, we have*

$$\mathcal{Y}(s,0) = \frac{\exp -s}{2}\tanh(0) = 0 = \mathcal{Y}'.$$

*Thus* $\mathcal{Y}^* + \mathcal{O}^*\mathcal{M}_c^{\mathfrak{S}} \leq \frac{1}{2} + \frac{\Gamma(\frac{2}{3})}{7} \backsimeq 0.693 < 1$.
*Then, by* [15, 24]*, the problem* (1)-(3) *is an abstract formulation of the problem* (5)-(7)*, and conditions* $(i)-(iii)$ *are satisfied. Theorem 3.3 implies that the problem* (5)-(7) *has a unique mild solution on* $BC$*, which is attractive by Theorem 3.4.*

## REFERENCES


[1] *S. Abbas, B. Ahmad, M. Benchohra and A. Salim*, Fractional Difference, Differential Equations, and Inclusions: Analysis and Stability, Elsevier.

[2] *T. Abdeljawad*, On conformable fractional calculus, Comput. Appl. Math. **279**(2015), 57-66.

[3] *T. Abdeljawad, M. Al Horani and R. Khalil*, Conformable fractional semigroups of operators, J. Semigr. Theor. Appl. **7**(2015), 1-9.

[4] *R. S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan*, On the solution of a boundary value problem associated with a fractional differential equation, Math. Methods Appl. Sci.

[5] *R. S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan*, On The Solutions Of Fractional Differential Equations Via Geraghty Type Hybrid Contractions, Appl. Comput. Math. **20**(2021), 313-333.

[6] *R. S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan*, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, RACSAM **115**(2021), 155.

[7] *R. P. Agarwal, S. Baghli and M. Benchohra*, Controllability of Mild Solutions on Semiinfinite Interval for Classes of Semilinear Functional and Neutral Functional Evolution Equations with Infinite Delay, Appl Math Optim **60**(2009), 253-274.

[8] *A. Aghajani, M. Mursaleen, A. Shole Haghighi*, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, Acta Math. Sci. Ser. B Engl. Ed. **35**(2015), 552-566.

[9] *S. A. Ahmed, R. Saadeh, A. Qazza and T. M. Elzaki*, Modified conformable double Laplace-Sumudu approach with applications, Heliyon **9**(2023), e15891.

[10] *R.R. Akhmerov, M.I. Kamenskii, A.S. Patapov, A.E. Rodkina, B.N. Sadovskii*, Measures of Noncompactness an Condensing Operators, Birkhauser Verlag, Basel, 1992.

[11] *A.A. Al-Shawba, F.A. Abdullah, A. Azmi et al*, Compatible extension of the (G'/G)-expansion approach for equations with conformable derivative, Heliyon **9**(2023), e15717.

[12] *M. Atraoui and M. Bouaouid*, On the existence of mild solutions for nonlocal differential equations of the second order with conformable fractional derivative, Adv. Differ. Equ. **447**(2021).

[13] *M. Atraoui, M. Bouaouid and Mustapha Johri*, Controllability of mild solutions for a nonlocal fractional conformable cauchy problem of differential equations of the second order, Advanced Mathematical Models and Applications **8**(2023).

[14] *S. Baghli, M. Benchohra and Kh. Ezzinbi*, Controllability Results for Semilinear Functional and Neutral Functional Evolution Equations with Infinite Delay, Surveys in Mathematics and its Applications, **4**(2009), 15-39.

[15] *J. Banaś, K. Goebel*, Measures of Noncompactness in Banach Spaces, Marcel Dekker, New York, 1980.

[16] *M. Benchohra, S. Bouriah, A. Salim, and Y. Zhou*, Fractional differential equations: a coincidence degree approach, Walter de Gruyter GmbH and Co KG, 2023.

[17] *M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim*, Advanced Topics in Fractional Differential Equations.

[18] *M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim*, Fractional Differential Equations, 2023.

[19] *M. Benchohra, A. Salim and Y. Zhou*, Integro-Differential Equations: Analysis, Stability and Controllability, Walter de Gruyter GmbH and Co KG.

[20] *M. Bouaouid, Kh. Hilal and M. Hannabou*, Integral solutions of nondense impulsive conformable-fractional differential equations with nonlocal condition, Journal of Applied Analysis, **27**(2021), No. 2, 187-197.

[21] *M. Bouaouid, A. Kajouni, K. Hilal and S. Melliani*, A class of nonlocal impulsive differential equations with conformable fractional derivative, CUBO, **24**(2022), No. 3, 439-455.

[22] *A. Boukenkoul and M. Ziane*, Conformable functional evolution equations with nonlocal conditions in banach spaces, Surveys in Mathematics and its Applications, **18**(2023), 83-95.

[23] *B. C. Dhage, V. Lakshmikantham*, On global existence and attractivity results for nonlinear functional integral equations, Nonlinear Anal. **72**(2010), 2219-2227.

[24] *K. Ezzinbi, M. Ziat*, Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces, Differ Equ Dyn Syst **30**(2022), 315-333.

[25] *H. P. Heinz*, On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Analysis. Theory, Method and Applications. **7**(1983), 1351-1371.

[26] *E. Karapinar, R. S. Adigüzel, Ü. Aksoy, İ. M. Erhan*, A new approach to the existence and uniqueness of solutions for a class of nonlinear q-fractional boundary value problems, Appl. Comput. Math. **24**(2025), 235-249.

[27] *M. Kamenskii, V. Obukhovskii and P. Zecca*, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, De Gruyter, Berlin, 2001.

[28] *R. Khalil, M. Al Horani and M. A. Hamma*, Geometric meaning of conformable derivative via fractional cords, J. Math. Comput. Sci. **19**(2019), 241-245.

[29] *R. Khalil, M. Al Horani, A. Yousef and M. Sababheh*, A new definition of fractional derivatives, J. Comput. Appl. Math. **264**(2014), 65-70.

[30] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*, Theory and applications of fractional differential equations, Elsevier, 2006.

[31] *J. E. Lazreg, S. Abbas, M. Benchohra, and E. Karapinar*, Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces, Open Math. **19**(2021), 363-372.

[32] *J. Liang, Y. Mu, and T.J. Xiao*, Impulsive differential equations involving general conformable fractional derivative in Banach spaces, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **116**(2022), 114.

[33] *I. Podlubny*, Fractional differential equations, 1999.

[34] *A. Salim, B. Benchohra, E. Karapinar, J. E. Lazreg*, Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations, Adv. Differ. Equ. **2020**(2020), 601.

[35] *I.I. Vrabie*, $C_0$-semigroups and applications, North-Holland Mathematics Studies 191, North-Holland Publishing Co., Amsterdam, 2003.