

ON COBALANCING ELLIPTIC QUATERNIONS

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In this paper, we introduce and study cobalancing and Lucas-cobalancing elliptic quaternions. Using properties of cobalancing and Lucas-cobalancing numbers, we obtain similar properties of defined quaternions, among others, Binet type formulas and generating functions. Moreover, we prove formulas which generalize many identities.

Keywords: quaternions, elliptic quaternions, cobalancing numbers, Diophantine equation

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1. Introduction

A positive integer n is a cobalancing number if it is the solution of the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$$

for some positive integer r . Here, r is called the cobalancer corresponding to the cobalancing number n .

The sequence $\{b_n\}$ of cobalancing numbers was introduced in [9]. The sequence $\{c_n\}$ of Lucas-cobalancing numbers was investigated in [6, 7]. The n th cobalancing number b_n and the n th Lucas-cobalancing number c_n are in the relation $(c_n)^2 = 8b_n^2 + 8b_n + 1$.

The cobalancing and Lucas-cobalancing numbers are among many numbers defined in connection with Diophantine equations. The balancing numbers, the solution of the Diophantine equation $1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$, were introduced in [1]. The almost balancing numbers, introduced in [8], are solutions of the equation $|\{(n + 1) + (n + 2) + \cdots + (n + r)\} - \{1 + 2 + \cdots + (n - 1)\}| = 1$ for some natural number r . In [3], as a solution of $1 + 2 + 3 + \cdots + (n - 1) = (n - 1) + (n + 0) + (n + 1) + (n + 2) + \cdots + (n + r)$, neo balancing numbers were defined.

The cobalancing and Lucas-cobalancing numbers fulfill the following recurrence relations

$$b_n = 6b_{n-1} - b_{n-2} + 2 \text{ for } n \geq 2, \text{ with } b_0 = 0, b_1 = 0, \quad (1)$$

$$c_n = 6c_{n-1} - c_{n-2} \text{ for } n \geq 2, \text{ with } c_0 = -1, c_1 = 1.$$

Note that the cobalancing and Lucas-cobalancing numbers were originally defined for $n \geq 1$. Defining $b_0 = 0$ and $c_0 = -1$ we obtain the same correctly defined sequences.

The Binet type formulas for these sequences have the following forms

$$b_n = \frac{\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2}, \quad (2)$$

$$c_n = \frac{\alpha^{n-\frac{1}{2}} + \beta^{n-\frac{1}{2}}}{2}, \quad (3)$$

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for $n \geq 0$, where

$$\alpha = 3 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2}, \alpha^{\frac{1}{2}} = 1 + \sqrt{2}, \beta^{\frac{1}{2}} = 1 - \sqrt{2}. \quad (4)$$

Note that additionally defined b_0 and c_0 satisfy (2)-(3).

The Table 1 includes the initial terms of the cobalancing numbers b_n and Lucas-cobalancing numbers c_n , for $n = 0, 1, \dots, 7$.

TABLE 1. The cobalancing and Lucas-cobalancing numbers.

n	0	1	2	3	4	5	6	7
b_n	0	0	2	14	84	492	2870	16730
c_n	-1	1	7	41	239	1393	8119	47321

In this paper, we use the cobalancing and Lucas-cobalancing numbers in the theory of elliptic quaternions.

In [5], Özdemir defined elliptic quaternions as follows. The set of elliptic quaternions, denoted by $\mathbb{H}_{a_1, a_2, a_3}$, is defined by

$$\mathbb{H}_{a_1, a_2, a_3} = \{\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

where $a_1, a_2, a_3 \in \mathbb{R}^+$ and basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfy the equalities

$$\mathbf{i}^2 = -a_1, \mathbf{j}^2 = -a_2, \mathbf{k}^2 = -a_3, \quad (5)$$

$$\mathbf{ij} = -\mathbf{ji} = \frac{\Delta}{a_3}\mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \frac{\Delta}{a_1}\mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \frac{\Delta}{a_2}\mathbf{j}, \quad (6)$$

and $\Delta = \sqrt{a_1 a_2 a_3}$.

Two elliptic quaternions $\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $\mathbf{y} = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ are equal if and only if $x_0 = y_0$, $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$. The sum and subtraction of two elliptic quaternions are defined by $\mathbf{x} \pm \mathbf{y} = (x_0 \pm y_0) + (x_1 \pm y_1)\mathbf{i} + (x_2 \pm y_2)\mathbf{j} + (x_3 \pm y_3)\mathbf{k}$. Using (5) and (6), we get the multiplication Table 2. Using rules given in the Table 2, the multiplication of elliptic quaternions can be made analogously as multiplication of algebraic expressions.

TABLE 2. The elliptic quaternions multiplication.

\cdot	1	\mathbf{i}	\mathbf{j}	\mathbf{k}
1	1	\mathbf{i}	\mathbf{j}	\mathbf{k}
\mathbf{i}	\mathbf{i}	$-a_1$	$\Delta\mathbf{k}/a_3$	$-\Delta\mathbf{j}/a_2$
\mathbf{j}	\mathbf{j}	$-\Delta\mathbf{k}/a_3$	$-a_2$	$\Delta\mathbf{i}/a_1$
\mathbf{k}	\mathbf{k}	$\Delta\mathbf{j}/a_2$	$-\Delta\mathbf{i}/a_1$	$-a_3$

The elliptic quaternions generalize the classical quaternions, introduced by Hamilton in [4]. Putting $a_1 = a_2 = a_3 = 1$, we obtain the set \mathbb{H} of quaternions with the basic elements $\{1, i, j, k\}$ which satisfy $i^2 = j^2 = k^2 = ijk = -1$. Quaternions and elliptic quaternions are defined with coefficients of the real numbers and basic elements, and they belong to the class of hypercomplex numbers. In the literature, we can find many papers in which some hypercomplex numbers are associated with selected sequences. The balancing quaternions and Lucas-balancing quaternions were introduced independently in [2] and [11]. Dual bicomplex balancing and Lucas-balancing numbers were investigated in [13]. The bicomplex balancing quaternions and bicomplex Lucas-balancing quaternions are defined in [12].

2. Main results

Let $n \geq 0$ be an integer. The n th cobalancing elliptic quaternion $\mathbb{E}b_n$ and the n th Lucas-cobalancing elliptic quaternion $\mathbb{E}c_n$ are defined as

$$\begin{aligned}\mathbb{E}b_n &= b_n + b_{n+1}\mathbf{i} + b_{n+2}\mathbf{j} + b_{n+3}\mathbf{k}, \\ \mathbb{E}c_n &= c_n + c_{n+1}\mathbf{i} + c_{n+2}\mathbf{j} + c_{n+3}\mathbf{k},\end{aligned}\tag{7}$$

respectively, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are units that satisfy (5) and (6).

Using initial terms of cobalancing and Lucas-cobalancing sequences, we have

$$\begin{aligned}\mathbb{E}b_0 &= 2\mathbf{j} + 14\mathbf{k}, \\ \mathbb{E}b_1 &= 2\mathbf{i} + 14\mathbf{j} + 84\mathbf{k},\end{aligned}\tag{8}$$

$$\begin{aligned}\mathbb{E}c_0 &= -1 + \mathbf{i} + 7\mathbf{j} + 41\mathbf{k}, \\ \mathbb{E}c_1 &= 1 + 7\mathbf{i} + 41\mathbf{j} + 239\mathbf{k}.\end{aligned}\tag{9}$$

The cobalancing and Lucas-cobalancing numbers can also be written in recursive form.

Theorem 2.1. *Let $n \geq 2$ be an integer. Then*

$$\mathbb{E}b_n = 6\mathbb{E}b_{n-1} - \mathbb{E}b_{n-2} + 2(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where $\mathbb{E}b_0, \mathbb{E}b_1$ are given by (8).

Proof. For $n = 2$ we have

$$\begin{aligned}\mathbb{E}b_2 &= 6\mathbb{E}b_1 - \mathbb{E}b_0 + 2(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= 6(2\mathbf{i} + 14\mathbf{j} + 84\mathbf{k}) - (2\mathbf{j} + 14\mathbf{k}) + 2 + 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ &= 2 + 14\mathbf{i} + 84\mathbf{j} + 492\mathbf{k}.\end{aligned}$$

Let $n \geq 3$. By (7) and (1), we have

$$\begin{aligned}6\mathbb{E}b_{n-1} - \mathbb{E}b_{n-2} + 2(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= 6(b_{n-1} + b_n\mathbf{i} + b_{n+1}\mathbf{j} + b_{n+2}\mathbf{k}) - (b_{n-2} + b_{n-1}\mathbf{i} + b_n\mathbf{j} + b_{n+1}\mathbf{k}) \\ &\quad + 2 + 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ &= b_n + b_{n+1}\mathbf{i} + b_{n+2}\mathbf{j} + b_{n+3}\mathbf{k} = \mathbb{E}b_n,\end{aligned}$$

which ends the proof. \square

In the same way, one can prove the following theorem.

Theorem 2.2. *Let $n \geq 2$ be an integer. Then*

$$\mathbb{E}c_n = 6\mathbb{E}c_{n-1} - \mathbb{E}c_{n-2},$$

where $\mathbb{E}c_0, \mathbb{E}c_1$ are given by (9).

The next theorems give the Binet type formulas for the cobalancing and Lucas-cobalancing elliptic quaternions.

Theorem 2.3. *Let $n \geq 0$ be an integer. Then*

$$\mathbb{E}b_n = \frac{\alpha^{n-\frac{1}{2}}}{\alpha - \beta} \hat{\alpha} - \frac{\beta^{n-\frac{1}{2}}}{\alpha - \beta} \hat{\beta} - \frac{1}{2} \hat{\mathbf{i}},\tag{10}$$

where α, β are given by (4) and

$$\hat{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}, \quad \hat{\beta} = 1 + \beta\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}, \quad \hat{\mathbf{i}} = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}.\tag{11}$$

Proof. Using (7) and (2), we have

$$\begin{aligned}
\mathbb{E}b_n &= b_n + b_{n+1}\mathbf{i} + b_{n+2}\mathbf{j} + b_{n+3}\mathbf{k} \\
&= \left(\frac{\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) + \left(\frac{\alpha^{n+1-\frac{1}{2}} - \beta^{n+1-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{i} \\
&\quad + \left(\frac{\alpha^{n+2-\frac{1}{2}} - \beta^{n+2-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{j} + \left(\frac{\alpha^{n+3-\frac{1}{2}} - \beta^{n+3-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{k} \\
&= \left(\frac{\alpha^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{\beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) + \left(\frac{\alpha \cdot \alpha^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{\beta \cdot \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{i} \\
&\quad + \left(\frac{\alpha^2 \cdot \alpha^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{\beta^2 \cdot \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{j} + \left(\frac{\alpha^3 \cdot \alpha^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{\beta^3 \cdot \beta^{n-\frac{1}{2}}}{\alpha - \beta} - \frac{1}{2} \right) \mathbf{k} \\
&= \frac{\alpha^{n-\frac{1}{2}}}{\alpha - \beta} (1 + \alpha\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}) - \frac{\beta^{n-\frac{1}{2}}}{\alpha - \beta} (1 + \beta\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}) \\
&\quad - \frac{1}{2} (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}),
\end{aligned}$$

which ends the proof. \square

Using (7) and (3), one can easily prove the following theorem.

Theorem 2.4. *Let $n \geq 0$ be an integer. Then*

$$\mathbb{E}c_n = \frac{1}{2}\alpha^{n-\frac{1}{2}}\hat{\alpha} + \frac{1}{2}\beta^{n-\frac{1}{2}}\hat{\beta}, \quad (12)$$

where α, β and $\hat{\alpha}, \hat{\beta}$ are given by (4) and (11), respectively.

Using (10) and (12), we can prove the following theorems – general bilinear index-reduction formulas for the cobalancing and Lucas-cobalancing elliptic quaternions.

Theorem 2.5. *Let $a \geq 0, b \geq 0, c \geq 0, d \geq 0$ be integers such that $a + b = c + d$. Then*

$$\begin{aligned}
\mathbb{E}b_a \cdot \mathbb{E}b_b - \mathbb{E}b_c \cdot \mathbb{E}b_d &= \\
&= \frac{1}{32} \left[(\alpha^c \beta^d - \alpha^a \beta^b) \hat{\alpha} \hat{\beta} + (\beta^c \alpha^d - \beta^a \alpha^b) \hat{\beta} \hat{\alpha} \right] \\
&\quad + \frac{2 - \sqrt{2}}{16} \left[(\alpha^c - \alpha^a) \hat{\alpha} \hat{\mathbf{1}} + (\alpha^d - \alpha^b) \hat{\mathbf{1}} \hat{\alpha} \right] \\
&\quad - \frac{2 + \sqrt{2}}{16} \left[(\beta^a - \beta^c) \hat{\beta} \hat{\mathbf{1}} + (\beta^b - \beta^d) \hat{\mathbf{1}} \hat{\beta} \right],
\end{aligned}$$

where α, β and $\hat{\alpha}, \hat{\beta}, \hat{\mathbf{1}}$ are given by (4) and (11), respectively.

Proof. By (10), we have

$$\begin{aligned}
& \mathbb{E}b_a \cdot \mathbb{E}b_b - \mathbb{E}b_c \cdot \mathbb{E}b_d = \\
& = \left(\frac{\alpha^{a-\frac{1}{2}}}{\alpha-\beta} \hat{\alpha} - \frac{\beta^{a-\frac{1}{2}}}{\alpha-\beta} \hat{\beta} - \frac{1}{2} \hat{\mathbf{1}} \right) \cdot \left(\frac{\alpha^{b-\frac{1}{2}}}{\alpha-\beta} \hat{\alpha} - \frac{\beta^{b-\frac{1}{2}}}{\alpha-\beta} \hat{\beta} - \frac{1}{2} \hat{\mathbf{1}} \right) \\
& - \left(\frac{\alpha^{c-\frac{1}{2}}}{\alpha-\beta} \hat{\alpha} - \frac{\beta^{c-\frac{1}{2}}}{\alpha-\beta} \hat{\beta} - \frac{1}{2} \hat{\mathbf{1}} \right) \cdot \left(\frac{\alpha^{d-\frac{1}{2}}}{\alpha-\beta} \hat{\alpha} - \frac{\beta^{d-\frac{1}{2}}}{\alpha-\beta} \hat{\beta} - \frac{1}{2} \hat{\mathbf{1}} \right) \\
& = -\frac{\alpha^{a-\frac{1}{2}} \beta^{b-\frac{1}{2}}}{(\alpha-\beta)^2} \hat{\alpha} \hat{\beta} - \frac{\alpha^{a-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\alpha} \hat{\mathbf{1}} - \frac{\beta^{a-\frac{1}{2}} \alpha^{b-\frac{1}{2}}}{(\alpha-\beta)^2} \hat{\beta} \hat{\alpha} + \frac{\beta^{a-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\beta} \hat{\mathbf{1}} \\
& - \frac{\alpha^{b-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\mathbf{1}} \hat{\alpha} + \frac{\beta^{b-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\mathbf{1}} \hat{\beta} + \frac{\alpha^{c-\frac{1}{2}} \beta^{d-\frac{1}{2}}}{(\alpha-\beta)^2} \hat{\alpha} \hat{\beta} + \frac{\alpha^{c-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\alpha} \hat{\mathbf{1}} \\
& + \frac{\beta^{c-\frac{1}{2}} \alpha^{d-\frac{1}{2}}}{(\alpha-\beta)^2} \hat{\beta} \hat{\alpha} - \frac{\beta^{c-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\beta} \hat{\mathbf{1}} + \frac{\alpha^{d-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\mathbf{1}} \hat{\alpha} - \frac{\beta^{d-\frac{1}{2}}}{2(\alpha-\beta)} \hat{\mathbf{1}} \hat{\beta}
\end{aligned}$$

and after calculations we get the result. \square

Theorem 2.6. (General bilinear index-reduction formula for Lucas-cobalancing hybrid numbers) Let $a \geq 0, b \geq 0, c \geq 0, d \geq 0$ be integers such that $a + b = c + d$. Then

$$\mathbb{E}c_a \cdot \mathbb{E}c_b - \mathbb{E}c_c \cdot \mathbb{E}c_d = \frac{1}{4} \left[(\alpha^a \beta^b - \alpha^c \beta^d) \hat{\alpha} \hat{\beta} + (\beta^a \alpha^b - \beta^c \alpha^d) \hat{\beta} \hat{\alpha} \right],$$

where α, β and $\hat{\alpha}, \hat{\beta}$ are given by (4) and (11), respectively.

For special values of a, b, c, d , by Theorems 2.5-2.6, we can obtain some identities for the cobalancing and Lucas-cobalancing elliptic quaternions:

- d'Ocagne type identity – for $a = n, b = m + 1, c = n + 1, d = m$,
- Vajda type identity – for $a = m + r, b = n - r, c = m, d = n$,
- first Halton type identity – for $a = m + r, b = n, c = r, d = m + n$,
- second Halton type identity – for $a = n + k, b = n - k, c = n + s, d = n - s$,
- Catalan type identity – for $a = n + r, b = n - r, c = d = n$,
- Cassini type identity – for $a = n + 1, b = n - 1, c = d = n$.

Moreover, by simple calculations we obtain

$$\begin{aligned}
\hat{\alpha} \hat{\beta} &= 1 - a_1 - a_2 - a_3 + \left(6 - \frac{4\sqrt{2}\Delta}{a_1} \right) \mathbf{i} \\
&+ \left(34 + \frac{24\sqrt{2}\Delta}{a_2} \right) \mathbf{j} + \left(198 - \frac{4\sqrt{2}\Delta}{a_3} \right) \mathbf{k},
\end{aligned}$$

$$\begin{aligned}
\hat{\beta} \hat{\alpha} &= 1 - a_1 - a_2 - a_3 + \left(6 + \frac{4\sqrt{2}\Delta}{a_1} \right) \mathbf{i} \\
&+ \left(34 - \frac{24\sqrt{2}\Delta}{a_2} \right) \mathbf{j} + \left(198 + \frac{4\sqrt{2}\Delta}{a_3} \right) \mathbf{k},
\end{aligned}$$

$$\begin{aligned}
\hat{\alpha}\hat{\mathbf{i}} &= 1 - (3 + 2\sqrt{2})a_1 - (17 + 12\sqrt{2})a_2 - (99 + 70\sqrt{2})a_3 \\
&\quad + \left(4 + 2\sqrt{2} - \frac{(82 + 58\sqrt{2})\Delta}{a_1} \right) \mathbf{i} \\
&\quad + \left(18 + 12\sqrt{2} + \frac{(92 + 68\sqrt{2})\Delta}{a_2} \right) \mathbf{j} \\
&\quad + \left(100 + 70\sqrt{2} - \frac{(14 + 10\sqrt{2})\Delta}{a_3} \right) \mathbf{k}, \\
\hat{\mathbf{i}}\hat{\alpha} &= 1 - (3 + 2\sqrt{2})a_1 - (17 + 12\sqrt{2})a_2 - (99 + 70\sqrt{2})a_3 \\
&\quad + \left(4 + 2\sqrt{2} + \frac{(82 + 58\sqrt{2})\Delta}{a_1} \right) \mathbf{i} \\
&\quad + \left(18 + 12\sqrt{2} - \frac{(92 + 68\sqrt{2})\Delta}{a_2} \right) \mathbf{j} \\
&\quad + \left(100 + 70\sqrt{2} + \frac{(14 + 10\sqrt{2})\Delta}{a_3} \right) \mathbf{k}, \\
\hat{\beta}\hat{\mathbf{i}} &= 1 - (3 - 2\sqrt{2})a_1 - (17 - 12\sqrt{2})a_2 - (99 - 70\sqrt{2})a_3 \\
&\quad + \left(4 - 2\sqrt{2} + \frac{(-82 + 58\sqrt{2})\Delta}{a_1} \right) \mathbf{i} \\
&\quad + \left(18 - 12\sqrt{2} - \frac{(-92 + 68\sqrt{2})\Delta}{a_2} \right) \mathbf{j} \\
&\quad + \left(100 - 70\sqrt{2} + \frac{(-14 + 10\sqrt{2})\Delta}{a_3} \right) \mathbf{k}, \\
\hat{\mathbf{i}}\hat{\beta} &= 1 - (3 - 2\sqrt{2})a_1 - (17 - 12\sqrt{2})a_2 - (99 - 70\sqrt{2})a_3 \\
&\quad + \left(4 - 2\sqrt{2} - \frac{(-82 + 58\sqrt{2})\Delta}{a_1} \right) \mathbf{i} \\
&\quad + \left(18 - 12\sqrt{2} + \frac{(-92 + 68\sqrt{2})\Delta}{a_2} \right) \mathbf{j} \\
&\quad + \left(100 - 70\sqrt{2} - \frac{(-14 + 10\sqrt{2})\Delta}{a_3} \right) \mathbf{k}.
\end{aligned}$$

At the end, we give the generating functions for the cobalancing and Lucas-cobalancing elliptic quaternions.

Theorem 2.7. *The generating function for the cobalancing elliptic quaternion sequence $\{\mathbb{E}b_n\}$ is*

$$g(t) = \frac{2\mathbf{j} + 14\mathbf{k} + (2\mathbf{i} - 14\mathbf{k})t + (2 + 2\mathbf{k})t^2}{(1 - 6t + t^2)(1 - t)}.$$

Proof. Assume that the generating function of the cobalancing elliptic quaternion sequence $\{\mathbb{E}b_n\}$ has the form $g(t) = \sum_{n=0}^{\infty} \mathbb{E}b_n t^n$. Then

$$g(t) = \mathbb{E}b_0 + \mathbb{E}b_1 t + \mathbb{E}b_2 t^2 + \dots$$

Hence, we get

$$\begin{aligned} -6t \cdot g(t) &= -6\mathbb{E}b_0t - 6\mathbb{E}b_1t^2 - 6\mathbb{E}b_2t^3 - \dots \\ t^2 \cdot g(t) &= \mathbb{E}b_0t^2 + \mathbb{E}b_1t^3 + \mathbb{E}b_2t^4 + \dots \end{aligned}$$

By adding these three equalities above, we get

$$\begin{aligned} g(t)(1 - 6t + t^2) &= \mathbb{E}b_0 + (\mathbb{E}b_1 - 6\mathbb{E}b_0)t + (\mathbb{E}b_2 - 6\mathbb{E}b_1 + \mathbb{E}b_0)t^2 \\ &+ (\mathbb{E}b_3 - 6\mathbb{E}b_2 + \mathbb{E}b_1)t^3 + \dots \end{aligned}$$

As we know, $t^2 + t^3 + \dots = \frac{t^2}{1-t}$, so adding this equality multiplied by $-2 \cdot \hat{\mathbf{i}}$ to the above we obtain

$$\begin{aligned} g(t)(1 - 6t + t^2) - \frac{2t^2\hat{\mathbf{i}}}{1-t} &= \mathbb{E}b_0 + (\mathbb{E}b_1 - 6\mathbb{E}b_0)t \\ &+ (\mathbb{E}b_2 - 6\mathbb{E}b_1 + \mathbb{E}b_0 - 2\hat{\mathbf{i}})t^2 + (\mathbb{E}b_3 - 6\mathbb{E}b_2 + \mathbb{E}b_1 - 2\hat{\mathbf{i}})t^3 + \dots \end{aligned}$$

Then we have

$$g(t) = \frac{[\mathbb{E}b_0 + (\mathbb{E}b_1 - 6\mathbb{E}b_0)t](1-t) + 2t^2\hat{\mathbf{i}}}{(1-6t+t^2)(1-t)}$$

since $\mathbb{E}b_n = 6\mathbb{E}b_{n-1} - \mathbb{E}b_{n-2} + 2\hat{\mathbf{i}}$ (see Theorem 2.1) and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, by simple calculations, we have

$$g(t) = \frac{\mathbb{E}b_0 + (\mathbb{E}b_1 - 7\mathbb{E}b_0)t + (2\hat{\mathbf{i}} - \mathbb{E}b_1 + 6\mathbb{E}b_0)t^2}{(1-6t+t^2)(1-t)}$$

and

$$\mathbb{E}b_1 - 7\mathbb{E}b_0 = 2\mathbf{i} - 14\mathbf{k}, \quad 2\hat{\mathbf{i}} - \mathbb{E}b_1 + 6\mathbb{E}b_0 = 2 + 2\mathbf{k}.$$

□

In the same way, we can prove the following theorem.

Theorem 2.8. *The generating function for the Lucas-cobalancing elliptic quaternion sequence $\{\mathbb{E}c_n\}$ is*

$$G(t) = \frac{-1 + \mathbf{i} + 7\mathbf{j} + 41\mathbf{k} + (7 + \mathbf{i} - \mathbf{j} - 7\mathbf{k})t}{1 - 6t + t^2}.$$

3. Concluding Remarks

By modifying the Diophantine equation defining the cobalancing numbers, we can obtain other cobalancing type numbers, for example, co-cobalancing numbers, introduced in [10]. Using these numbers, we can introduce co-cobalancing elliptic quaternions and then study their properties and relations to other elliptic quaternions of balancing type.

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