EXISTENCE OF INFINITE SOLUTIONS FOR A NEUMANN PROBLEM WITH $\kappa(\cdot)$-LAPLACIAN-LIKE OPERATORS

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We study a class of nonlocal Kirchhoff-type elliptic problems driven by a $\kappa(\cdot)$-Laplacian-like operator with variable exponent growth and Neumann boundary conditions. These arise in models of capillarity and non-Newtonian fluids, where $\kappa(\cdot)$ reflects anisotropy. Using critical point theory in variable exponent Sobolev spaces and an abstract variational principle of B. Ricceri, we prove existence and multiplicity of weak solutions, extending known constant-exponent results to a variable exponent setting.

Keywords: $\kappa(\cdot)$-Laplacian-like operator, Ricceri's variational principle, Kirchhoff-type problem, Neumann boundary value problem, Variable exponent Sobolev spaces.

MSC2020: 35D30, 35J62, 35J66, 35J60.

1. Introduction

In this article, we consider the following Kirchhoff-type quasilinear elliptic equation with Neumann boundary conditions:

$$\begin{cases} -\mathfrak{M}\left(\int_{\mathcal{D}}\left(\frac{1}{\kappa(y)}|\nabla\phi|^{\kappa(y)}+\frac{\sqrt{1+|\nabla\phi|^{2\kappa(y)}}}{\kappa(y)}\right)\mathrm{d}y+\int_{\mathcal{D}}\frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)}\,\mathrm{d}y\right) \\ \quad\times\left(\Delta^I_{\kappa(y)}\phi(y)+\delta(y)|\phi(y)|^{\kappa(y)-2}\phi(y)\right)=\theta(y,\phi(y)) \text{ in } \mathcal{D}, \\ \frac{\partial\phi}{\partial\gamma}=0 \text{ on } \partial\mathcal{D}, \end{cases} \tag{1}$$

where

$$\Delta^I_{\kappa(y)}\phi := \operatorname{div}\left(\left(1+\frac{1}{\sqrt{1+|\nabla\phi(y)|^{2\kappa(y)}}}\right)|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y)\right),$$

is referred to as a $\kappa(\cdot)$-Laplacian-like operator. Here, $\mathcal{D}\subset\mathbb{R}^N$ ($N\geq 1$) is a bounded domain with a boundary $\partial\mathcal{D}$ of class $\mathcal{C}^1$, and ν denotes the outward unit normal vector to $\partial\mathcal{D}$. The variable exponent function $\kappa\in C(\overline{\mathcal{D}})$ satisfies the following bounds:

$$1<\kappa^- := \inf_{y\in\mathcal{D}}\kappa(y)\leq\kappa(y)\leq\kappa^+ := \sup_{y\in\mathcal{D}}\kappa(y)<+\infty. \tag{2}$$

We also consider a measurable function $\delta:\mathcal{D}\to\mathbb{R}$ satisfying

$$\delta(\cdot)\in L^\infty(\mathcal{D}), \quad \text{and} \quad \delta^- := \operatorname*{ess\,inf}_{x\in\mathcal{D}}\delta(y)>0. \tag{3}$$

ensuring that δ is essentially bounded and strictly positive almost everywhere in $\mathcal{D}$.

The nonlinear source term $\theta:\mathcal{D}\times\mathbb{R}\to\mathbb{R}$ is assumed to be a Carathéodory function, that is, measurable in y for every fixed $r\in\mathbb{R}$, and continuous in r for almost every $y\in\mathcal{D}$. Furthermore, we assume the following integrability condition:

$$\sup_{|r|\leq\tau}|\theta(y,r)|\in L^1(\mathcal{D}) \quad \text{for all } \tau>0. \tag{4}$$

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We set

$$\Theta(y,t) = \int_0^t \theta(y,s)\,ds.$$

The Kirchhoff function $\mathfrak{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to satisfy the following hypotheses:

(M1) There exists a constant $m_0 > 0$ such that

$$\mathfrak{M}(t) \geq m_0 \quad \text{for all } t \geq 0. \tag{5}$$

(M2) There exist constants $\nu > 1$ and $m_1, m_2 > 0$ such that

$$\widehat{\mathfrak{M}}(t) := \int_0^t \mathfrak{M}(s)\,\mathrm{d}s \leq m_1 t^\nu + m_2 \quad \text{for all } t \geq 0. \tag{6}$$

To illustrate these assumptions, consider the specific example $\mathfrak{M}(t) = (1+t)^{\upsilon-1}$ with $\upsilon > 1$. Clearly,

$$\mathfrak{M}(t) \geq 1 \quad \text{for all } t \geq 0,$$

which satisfies (5) with $m_0 = 1$. Furthermore, the primitive function is given by

$$\widehat{\mathfrak{M}}(t) = \int_0^t (1+s)^{\upsilon-1}\,\mathrm{d}s = \frac{(1+t)^\upsilon - 1}{\upsilon} \leq \frac{(1+t)^\upsilon}{\upsilon} \leq 2^{\upsilon-1}(1+t^\upsilon),$$

so (6) is satisfied with $\nu = \upsilon$ and $m_1 = m_2 = 2^{\upsilon-1}$.

Kirchhoff-type problems, first proposed by Kirchhoff [14] as a generalization of d'Alembert's wave equation, account for the influence of changes in the length of a vibrating string. A key feature of these models is a nonlocal coefficient that depends on the averaged kinetic energy density, adding both mathematical complexity and physical realism. For further developments and generalizations, see [3, 11].

In the study of nonlinear elliptic problems exhibiting nonstandard growth conditions, several differential operators have been developed as generalizations of the classical κ-Laplacian to account for spatially dependent exponents. These $(\kappa(\cdot))$-Laplacian-like operators reflect the complexity of real-world models and introduce rich analytical structures. Below, we present several important examples, emphasizing both their mathematical formulation and their relevance to physical or applied contexts.

A prototypical instance is the standard variable exponent Laplacian

$$\Delta_{\kappa(y)}\phi := \operatorname{div}\left(|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y)\right),$$

which plays a key role in modeling electrorheological fluids and image processing [12]. Another important operator is the relativistic-type $(\kappa(\cdot))$-Laplacian

$$\Delta^I_{\kappa(y)}\phi := \operatorname{div}\left(\left(1 + \frac{1}{\sqrt{1+|\nabla\phi(y)|^{2\kappa(y)}}}\right)|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y)\right),$$

which introduces a regularization inspired by relativistic field theory, capturing phenomena with bounded propagation speed or saturation effects, and appears in models of nonlinear diffusion and phase transitions.

Another variant is the logarithmic $(\kappa(y))$-Laplacian

$$\Delta^{\log}_{\kappa(y)}\phi := \operatorname{div}\left((1+\log(1+|\nabla\phi(y)|))\,|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y)\right),$$

used to describe slow diffusion in population dynamics and turbulent flows, where the gradient response grows mildly [5].

A more complex structure is provided by the double-phase operator

$$\mathcal{A}(y,\nabla\phi) := |\nabla\phi|^{\kappa_1(y)-2}\nabla\phi + a(y)|\nabla\phi|^{\kappa_2(y)-2}\nabla\phi,$$

which models materials with two competing behaviors such as hard/soft composites [2, 4]. Here, $a(y) \geq 0$ governs the transition between the two growth regimes. Finally, in anisotropic media, one encounters

$$\mathcal{A}_{\text{aniso}}(\phi) := \sum_{i=1}^{N} \partial_{y_i} \left(|\partial_{y_i}\phi|^{\kappa_i(y)-2} \partial_{y_i}\phi \right),$$

where direction-dependent growth is relevant, as in crystal mechanics or directional diffusion [1, 9].

The study of $\kappa(\cdot)$-Laplacian-type operators was initiated by Rodrigues using variational methods [16]. Since then, numerous works have addressed related boundary value problems, establishing existence and multiplicity results for both Dirichlet and Neumann conditions [10, 13, 17, 6]. In particular, recent studies have demonstrated the existence of infinitely many weak solutions and extended these results to nonlocal Kirchhoff-type problems under variable exponent settings.

It is worth noting that the formulations in the above works differ substantially from our current setting, particularly in the structure of the source term θ and the role of the variable exponent function κ. Moreover, the analytical techniques employed therein diverge significantly from those used in our approach.

We recall below a fundamental result due to Ricceri [15], which plays a crucial role in establishing our main results.

Theorem 1.1 (See [8], Theorem 2.2)**.** *Let Y be a reflexive real Banach space, and let $\mathrm{H}_1, \mathrm{H}_2 : Y \to \mathbb{R}$ be two Gâteaux differentiable and sequentially weakly lower semicontinuous functionals. Assume that H_1 is continuous with respect to the norm topology and satisfies*

$$\lim_{\|\phi\|_Y \to \infty} \mathrm{H}_1(\phi) = +\infty.$$

For any $\kappa > \inf_Y \mathrm{H}_1$, define

$$\varphi(\kappa) = \inf_{\phi \in \mathrm{H}_1^{-1}((-\infty,\kappa))} \frac{\mathrm{H}_2(\phi) - \inf_{\psi \in \overline{\mathrm{H}_1^{-1}((-\infty,\kappa))}^{\text{weak}}} \mathrm{H}_2(\psi)}{\kappa - \mathrm{H}_1(\phi)}, \tag{7}$$

where $\overline{\mathrm{H}_1^{-1}((-\infty,\kappa))}^{\text{weak}}$ denotes the weak closure of $\mathrm{H}_1^{-1}((-\infty,\kappa))$ in Y. The following assertions hold:

(a): *Suppose there exist $\kappa_0 > \inf_Y \mathrm{H}_1$ and $\phi_0 \in Y$ such that*

$$\mathrm{H}_1(\phi_0) < \kappa_0, \tag{8}$$

and

$$\mathrm{H}_2(\phi_0) - \inf_{\psi \in \overline{\mathrm{H}_1^{-1}((-\infty,\kappa_0))}^{\text{weak}}} \mathrm{H}_2(\psi) < \kappa_0 - \mathrm{H}_1(\phi_0). \tag{9}$$

Then the restriction of the functional $\mathrm{H}_1 + \mathrm{H}_2$ to $\mathrm{H}_1^{-1}((-\infty,\kappa_0))$ admits at least one global minimum.

(b): *Assume there exist two sequences $(\kappa_n)_n \subset (\inf_Y \mathrm{H}_1, +\infty)$ with $\kappa_n \to \infty$ and $(\eta_n)_n \subset Y$ such that for every $n \in \mathbb{N}$,*

$$\mathrm{H}_1(\eta_n) < \kappa_n, \tag{10}$$

and

$$\mathrm{H}_2(\eta_n) - \inf_{\psi \in \overline{\mathrm{H}_1^{-1}((-\infty,\kappa_n))}^{\text{weak}}} \mathrm{H}_2(\psi) < \kappa_n - \mathrm{H}_1(\eta_n), \tag{11}$$

together with the condition

$$\liminf_{\|\eta\|_Y \to \infty} \left(\mathrm{H}_1(\eta) + \mathrm{H}_2(\eta)\right) = -\infty. \tag{12}$$

Then, there exists a sequence $(\psi_n)_n$ *of local minimizers of* $\mathrm{H}_1+\mathrm{H}_2$ *such that* $\mathrm{H}_1(\psi_n) \to +\infty$ *as* $n \to \infty$.

(c): *Assume there exist two sequences* $(\kappa_n)_n \subset (\inf_Y \mathrm{H}_1, +\infty)$ *with* $\kappa_n \to \inf_Y \mathrm{H}_1$ *and* $(\eta_n)_n \subset Y$ *such that for each* n*, conditions* (10) *and* (11) *are satisfied. Additionally, suppose that*

$$\text{the global minimizers of } \mathrm{H}_1 \text{ are not local minimizers of } \mathrm{H}_1 + \mathrm{H}_2. \tag{13}$$

Then, there exists a sequence $(\psi_n)_n$ *of pairwise distinct local minimizers of* $\mathrm{H}_1 + \mathrm{H}_2$ *such that*

$$\lim_{n\to\infty} \mathrm{H}_1(\psi_n) = \inf_Y \mathrm{H}_1,$$

and $(\psi_n)_n$ *converges weakly in* Y *to a global minimizer of* H_1.

The remainder of the paper is structured as follows. In Section 2, we recall the definitions and fundamental properties of variable exponent Sobolev spaces. Section 3 is devoted to the statements and proofs of the main results, namely Theorems 3.1 and 3.2.

2. **Preliminary Results**

Let $\mathcal{D} \subset \mathbb{R}^N$ be a bounded open domain with smooth boundary, and let $\kappa(\cdot) \in L^\infty(\mathcal{D})$ satisfy condition (2). Denote by $S(\mathcal{D})$ the set of all measurable real-valued functions defined on $\mathcal{D}$. Two functions in $S(\mathcal{D})$ are considered equivalent if they are equal almost everywhere.

The variable exponent Lebesgue space $L^{\kappa(\cdot)}(\mathcal{D})$ is defined as

$$L^{\kappa(\cdot)}(\mathcal{D}) = \left\{ \phi \in S(\mathcal{D}) : \int_{\mathcal{D}} |\phi(y)|^{\kappa(y)}\, dy < \infty \right\},$$

equipped with the Luxemburg norm

$$\|\phi\|_{L^{\kappa(\cdot)}(\mathcal{D})} = \inf \left\{ \mu > 0 : \int_{\mathcal{D}} \left| \frac{\phi(y)}{\mu} \right|^{\kappa(y)} dy \le 1 \right\}.$$

The corresponding variable exponent Sobolev space $W^{1,\kappa(\cdot)}(\mathcal{D})$ is defined by

$$W^{1,\kappa(\cdot)}(\mathcal{D}) = \left\{ \phi \in L^{\kappa(\cdot)}(\mathcal{D}) : |\nabla \phi| \in L^{\kappa(\cdot)}(\mathcal{D}) \right\},$$

and endowed with the norm

$$\|\phi\|_{W^{1,\kappa(\cdot)}(\mathcal{D})} = \|\phi\|_{L^{\kappa(\cdot)}(\mathcal{D})} + \|\nabla\phi\|_{L^{\kappa(\cdot)}(\mathcal{D})}.$$

The spaces $L^{\kappa(\cdot)}(\mathcal{D})$ and $W^{1,\kappa(\cdot)}(\mathcal{D})$ are separable and reflexive Banach spaces. For foundational properties and further details, we refer the reader to [7, 9].

Assume now that the function $\delta(y)$ satisfies assumption (3). For each $\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})$, we define the modular-based norm

$$\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} := \inf \left\{ \mu > 0 : \int_{\mathcal{D}} \left(\frac{|\nabla \phi(y)|^{\kappa(y)}}{\mu^{\kappa(y)}} + \delta(y) \frac{|\phi(y)|^{\kappa(y)}}{\mu^{\kappa(y)}} \right) dy \le 1 \right\}.$$

It is well known that $\|\cdot\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})}$ defines a norm on $W^{1,\kappa(\cdot)}(\mathcal{D})$, which is equivalent to the standard norm $\|\cdot\|_{W^{1,\kappa(\cdot)}(\mathcal{D})}$. Moreover, the following inequalities hold:

For $\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \ge 1$,

$$\|\phi\|^{\kappa^+}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \le \int_{\mathcal{D}} \left(|\nabla\phi(y)|^{\kappa(y)} + \delta(y)|\phi(y)|^{\kappa(y)} \right) dy \le \|\phi\|^{\kappa^-}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})}.$$

For $\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \le 1$,

$$\|\phi\|^{\kappa^-}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \le \int_{\mathcal{D}} \left(|\nabla\phi(y)|^{\kappa(y)} + \delta(y)|\phi(y)|^{\kappa(y)} \right) dy \le \|\phi\|^{\kappa^+}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})}.$$

Throughout this work, we assume that

$$\kappa^- > N. \tag{14}$$

Since $W^{1,\kappa(\cdot)}(\mathcal{D})$ is continuously embedded in $W^{1,\kappa^-}(\mathcal{D})$, and the latter is compactly embedded in $C^0(\overline{\mathcal{D}})$, it follows that $W^{1,\kappa(\cdot)}(\mathcal{D})$ is compactly embedded in $C^0(\overline{\mathcal{D}})$.

Define the constant

$$c_0 := \sup_{\phi \in W^{1,\kappa(\cdot)}(\mathcal{D}) \setminus \{0\}} \|\phi\|_{L^\infty(\mathcal{D})}, \tag{15}$$

where $\|\phi\|_{L^\infty(\mathcal{D})} := \max_{y \in \overline{\mathcal{D}}} |\phi(y)|$. Then c_0 is a positive constant due to the compact embedding.

3. Main results

We define, for every $\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})$, the functionals

$$\begin{aligned} \mathrm{H}_1(\phi) &= \widehat{\mathfrak{M}}\left[\int_{\mathcal{D}} \left(\frac{1}{\kappa(y)}|\nabla\phi(y)|^{\kappa(y)} + \frac{1}{\kappa(y)}\sqrt{1+|\nabla\phi(y)|^{2\kappa(y)}}\right) \mathrm{d}y \right. \\ &= \left. + \int_{\mathcal{D}} \frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)} \mathrm{d}y \right], \end{aligned} \tag{16}$$

$$\mathrm{H}_2(\phi) := -\int_{\mathcal{D}} \Theta(y, \phi(y))\, \mathrm{d}y. \tag{17}$$

Definition 3.1. *A measurable function $\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})$ is said to be a* weak solution *to the Neumann-type elliptic problem* (1) *if the following identity holds:*

$$\begin{aligned} &\mathfrak{M}\left(\int_{\mathcal{D}} \left(\frac{1}{\kappa(y)}|\nabla\phi(y)|^{\kappa(y)} + \frac{1}{\kappa(y)}\sqrt{1+|\nabla\phi(y)|^{2\kappa(y)}} + \frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)}\right) \mathrm{d}y\right) \\ &\quad \times \left(\int_{\mathcal{D}} \Bigg[\left(|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y) + \frac{|\nabla\phi(y)|^{\kappa(y)-2}\nabla\phi(y)}{\sqrt{1+|\nabla\phi(y)|^{2\kappa(y)}}}\right) \right. \\ &\quad\quad \left. \cdot \nabla\psi(y) + \delta(y)|\phi(y)|^{\kappa(y)-2}\phi(y)\psi(y)\Bigg] \mathrm{d}y \right) \\ &= \int_{\mathcal{D}} \theta(y, \phi(y))\psi(y)\, \mathrm{d}y, \end{aligned} \tag{18}$$

for all test functions $\psi \in W^{1,\kappa(\cdot)}(\mathcal{D})$.

It is easy to verify that the functionals H_1 and H_2 belong to the class $C^1\left(W^{1,\kappa(\cdot)}(\mathcal{D}), \mathbb{R}\right)$. Furthermore, a function $\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})$ is a weak solution of (1) if and only if it is a critical point of the energy functional $\mathrm{H}_1 + \mathrm{H}_2$. In addition, both functionals H_1 and H_2 are sequentially weakly lower semicontinuous on $W^{1,\kappa(\cdot)}(\mathcal{D})$ (see [6]).

Definition 3.2. *We say that a function $\Theta(y,t)$ satisfies condition* (19) *if, for every compact subset $Y \subset \mathbb{R}$, there exists a point $\zeta \in Y$ such that*
(S)

$$\Theta(y, \zeta) = \sup_{t \in Y} \Theta(y, t) \quad \textit{for a.e. } y \in \mathcal{D}. \tag{19}$$

We now show that the functional H_1 is coercive.

Lemma 3.1. *Assume that conditions* (5) *and* (6) *are satisfied. Then the functional* H_1 *is coercive.*

Proof. Let $\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \geq 1$. Observe that the integrand

$$\frac{\sqrt{1+|\nabla\phi|^{2\kappa(y)}}}{\kappa(y)} \geq \frac{1}{\kappa(y)}|\nabla\phi|^{\kappa(y)}$$

implies that the entire energy density dominates the variable exponent norm. Consequently,

$$\begin{aligned}
\mathrm{H}_1(\phi) &= \widehat{\mathfrak{M}}\left(\int_{\mathcal{D}}\left(\frac{1}{\kappa(y)}|\nabla\phi|^{\kappa(y)} + \frac{1}{\kappa(y)}\sqrt{1+|\nabla\phi|^{2\kappa(y)}} + \frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)}\right)\mathrm{d}y\right) \\
&= \int_0^T \mathfrak{M}(s)\,\mathrm{d}s \\
&\geq \frac{2m_0}{\kappa^+}\int_{\mathcal{D}}|\nabla\phi|^{\kappa(y)}\mathrm{d}y + \frac{m_0}{\kappa^+}\int_{\mathcal{D}}\delta(y)|\phi(y)|^{\kappa(y)}\mathrm{d}y \\
&\geq \frac{m_0}{\kappa^+}\|\phi\|^{\kappa^-}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})},
\end{aligned} \tag{20}$$

where

$$T := \int_{\mathcal{D}}\left(\frac{1}{\kappa(y)}|\nabla\phi|^{\kappa(y)} + \frac{1}{\kappa(y)}\sqrt{1+|\nabla\phi|^{2\kappa(y)}}\right)\mathrm{d}y + \int_{\mathcal{D}}\frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)}\mathrm{d}y.$$

Therefore, H_1 is coercive. In particular, there exist constants $\sigma_1, \sigma_2 > 0$ such that

$$\mathrm{H}_1(\phi) \geq \sigma_1\|\phi\|^{\kappa^-}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \quad \text{for all } \|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \geq \sigma_2. \tag{21}$$

□

We are now in a position to present our first main result.

Theorem 3.1. *Let assumptions* (2)-(6) *be satisfied, and suppose that the function* Θ *fulfills condition* (19)*. Furthermore, assume that*

$$\liminf_{|\zeta|\to+\infty}\left[\widehat{\mathfrak{M}}\left(\int_{\mathcal{D}}\frac{1}{\kappa(y)}|\zeta|^{\kappa(y)}\,\mathrm{d}y\right) - \int_{\mathcal{D}}\Theta(y,\zeta)\,\mathrm{d}y\right] = -\infty, \tag{22}$$

and that there exist sequences of positive real numbers $(\alpha_n)_n$ *and* $(\beta_n)_n$ *satisfying*

$$\lim_{n\to\infty}\beta_n = +\infty, \qquad \lim_{n\to\infty}\frac{\alpha_n^{\kappa^+}}{\beta_n^{\kappa^-}} = 0. \tag{23}$$

In addition, assume there exist a nonnegative function $f \in L^1(\mathcal{D})$ *with* $\|f\|_{L^1(\mathcal{D})} = 1$*, and constants* $\lambda_1, \lambda_2 > 0$ *such that for every* $n \in \mathbb{N}$ *and almost every* $y \in \mathcal{D}$*, the following hold:*

$$\Theta(y,\alpha_n) + f(y)\left(\sigma_1\left(\frac{\beta_n}{c_0}\right)^{\kappa^-\nu} - \lambda_1\alpha_n^{\kappa^+\nu} - \lambda_2\right) \geq \sup_{t\in[\alpha_n,\beta_n]}\Theta(y,t), \tag{24}$$

$$\Theta(y,-\alpha_n) + f(y)\left(\sigma_1\left(\frac{\beta_n}{c_0}\right)^{\kappa^-\nu} - \lambda_1\alpha_n^{\kappa^+\nu} - \lambda_2\right) \geq \sup_{t\in[-\beta_n,-\alpha_n]}\Theta(y,t), \tag{25}$$

where σ_1 *is the coercivity constant from* (21)*,* $\lambda_1 = \dfrac{m_1|\mathcal{D}|}{(\kappa^-)^\nu}$*, and* $\lambda_2 = m_2$*.*

Moreover, the inequalities (24) *and* (25) *are strict on a subset of* $\mathcal{D}$ *of positive measure.*

Then, there exists a sequence $(\psi_n)_n$ *of local minima of the functional* $\mathrm{H}_1 + \mathrm{H}_2$ *such that*

$$\lim_{n\to\infty}\mathrm{H}_1(\psi_n) = +\infty.$$

Consequently, the problem (1) *admits an unbounded sequence of weak solutions.*

Proof. Let $\vartheta > \inf_{\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})} \mathrm{H}_1(\phi)$. Define

$$\mathrm{R}(\vartheta) := \inf \left\{ \tau > 0 : \mathrm{H}_2^{-1}((-\infty, \vartheta)) \subset \overline{\mathbf{O}(0,\tau)} \right\}, \tag{26}$$

where $\mathbf{O}(0,\tau) := \left\{ \phi \in W^{1,\kappa(\cdot)}(\mathcal{D}) : \|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} < \tau \right\}$, and $\overline{\mathbf{O}(0,\tau)}$ is the norm closure of this ball.

Due to the coercivity of H_1, we have $0 < \mathrm{R}(\vartheta) < +\infty$ for every such ϑ. From (21), we obtain

$$\text{If } \mathrm{H}_1(\phi) < \sigma_1 \|\phi\|^{\kappa^-}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})}, \text{ then } \|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} < \sigma_2.$$

Using (26), it follows that

$$\mathrm{H}_1^{-1}((-\infty,\vartheta)) \subset \overline{\mathbf{O}(0,\mathrm{R}(\vartheta))} \subset \left\{ \phi \in C(\overline{\mathcal{D}}) : \|\phi\|_{L^\infty(\mathcal{D})} \le c_0 \mathrm{R}(\vartheta) \right\},$$

due to the embedding estimate $\|\phi\|_{L^\infty(\mathcal{D})} \le c_0 \|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})}$. Hence,

$$\inf_{\psi \in \overline{(\mathrm{H}_1^{-1}((-\infty,\vartheta))_w)}} \mathrm{H}_2(\psi) \ge \inf_{\|\psi\|_{L^\infty(\mathcal{D})} \le c_0 \mathrm{R}(\vartheta)} \mathrm{H}_2(\psi). \tag{27}$$

Let $\tau \ge \sigma_1 \sigma_2^{\kappa^- \nu}$ and suppose $\phi \in W^{1,\kappa(\cdot)}(\mathcal{D})$ satisfies $\mathrm{H}_1(\phi) < \tau$. Then, if $\|\phi\| \ge \sigma_2$, inequality (21) gives:

$$\tau > \mathrm{H}_1(\phi) \ge \sigma_1 \|\phi\|^{\kappa^-}, \quad \text{so} \quad \|\phi\| \le \left(\frac{\tau}{\sigma_1} \right)^{1/(\kappa^- \nu)}.$$

The same bound holds trivially if $\|\phi\| < \sigma_2$. Thus,

$$\mathrm{R}(\tau) \le \left(\frac{\tau}{\sigma_1} \right)^{1/(\kappa^- \nu)}. \tag{28}$$

Since $\Theta(y,\cdot)$ satisfies (19), for each n there exists $\zeta_n \in [-\alpha_n, \alpha_n]$ such that

$$\Theta(y,\zeta_n) = \sup_{t \in [-\alpha_n,\alpha_n]} \Theta(y,t), \quad \text{for a.e. } y \in \mathcal{D}. \tag{29}$$

To apply condition (b) of Theorem 1.1, consider the constant function $\phi_n(y) = \zeta_n$, and let $\vartheta_n := \sigma_1 \left(\frac{\beta_n}{c_0} \right)^{\kappa^- \nu}$, which diverges as $n \to \infty$. By (28),

$$\mathrm{R}(\vartheta_n) \le \frac{\beta_n}{c_0}, \quad \text{and hence } c_0 \mathrm{R}(\vartheta_n) \le \beta_n. \tag{30}$$

Using assumption (6), we estimate

$$e_n := \widehat{\mathfrak{M}} \left(\int_{\mathcal{D}} \frac{1}{\kappa(y)} |\zeta_n|^{\kappa(y)} \mathrm{d}y \right) \le \lambda_1 |\alpha_n|^{\kappa^+ \nu} + \lambda_2,$$

which by (23) implies $e_n < \vartheta_n$ for n sufficiently large.

Combining (24)–(25) and (29), we obtain:

$$\Theta(y,\zeta_n) + f(y)(\vartheta_n - e_n) \ge \sup_{|t| \le \beta_n} \Theta(y,t), \quad \text{a.e. in } \mathcal{D}, \tag{31}$$

with strict inequality on a subset of positive measure. Using (30) and (31), we obtain condition (11), and condition (12) follows directly from (22).

Therefore, all assumptions of Theorem 1.1 (b) are satisfied, and the proof of Theorem 3.1 is complete. □

Our second main result is stated below.

Theorem 3.2. *Assume that conditions* (2)-(6) *hold. Moreover, suppose the function* Θ *satisfies condition* (19)*, and that*

$$\limsup_{|\zeta|\to 0} \frac{\int_{\mathcal{D}} \Theta(y,\zeta)\,\mathrm{d}y}{|\zeta|^{\kappa^-}} > \widehat{\mathfrak{M}}\left(\int_{\mathcal{D}} \frac{1}{\kappa(y)}\,\mathrm{d}y\right). \tag{32}$$

Let $(\alpha_n)_n$ *and* $(\beta_n)_n$ *be sequences of positive real numbers such that*

$$\lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha_n^{\kappa^-}}{\beta_n^{\kappa^+}} = 0. \tag{33}$$

Assume that there exists a nonnegative function $f \in L^1(\mathcal{D})$ *with* $\|f\|_{L^1(\mathcal{D})} = 1$ *such that, for every* $n \in \mathbb{N}$ *and almost every* $y \in \mathcal{D}$*, the following inequalities are satisfied:*

$$\Theta(y,\alpha_n) + f(y)\left(\lambda_3\left(\frac{\beta_n}{c_0}\right)^{\kappa^+\nu} - \lambda_4\alpha_n^{\kappa^-\nu} - m_2\right) \geq \sup_{t\in[\alpha_n,\beta_n]} \Theta(y,t), \tag{34}$$

$$\Theta(y,-\alpha_n) + f(y)\left(\lambda_3\left(\frac{\beta_n}{c_0}\right)^{\kappa^+\nu} - \lambda_4\alpha_n^{\kappa^-\nu} - m_2\right) \geq \sup_{t\in[-\beta_n,-\alpha_n]} \Theta(y,t), \tag{35}$$

where $\lambda_3 = \dfrac{m_0}{\kappa^+}$ *and* $\lambda_4 = \dfrac{m_1|\mathcal{D}|}{(\kappa^-)^\nu}$*. Additionally, both inequalities are strict on a subset of* $\mathcal{D}$ *of positive measure.*

Then, there exists a sequence $(\psi_n)_n$ *of pairwise distinct local minima of the functional* $\mathrm{H}_1 + \mathrm{H}_2$ *such that* $\psi_n \to 0$ *strongly in* $W^{1,\kappa(\cdot)}(\mathcal{D})$*. Consequently, the problem* (1) *admits a sequence of nontrivial weak solutions converging strongly to zero in* $W^{1,\kappa(\cdot)}(\mathcal{D})$*.*

Proof. We verify the hypotheses of Theorem 1.1, item (c). For $\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \leq 1$, we estimate:

$$\begin{aligned}\mathrm{H}_1(\phi) &= \widehat{\mathfrak{M}}\left(\int_{\mathcal{D}}\left(\frac{1}{\kappa(y)}|\nabla\phi|^{\kappa(y)} + \frac{1}{\kappa(y)}\sqrt{1+|\nabla\phi|^{2\kappa(y)}} + \frac{\delta(y)}{\kappa(y)}|\phi(y)|^{\kappa(y)}\right)\mathrm{d}y\right)\\ &\geq \frac{m_0}{\kappa^+}\|\phi\|^{\kappa^+}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} = \lambda_3\|\phi\|^{\kappa^+}_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})},\end{aligned}$$

showing that H_1 is coercive and attains its minimum value at zero. Hence, $\inf_{W^{1,\kappa(\cdot)}(\mathcal{D})} \mathrm{H}_1 = \mathrm{H}_1(0) = 0$, and 0 is the unique global minimizer.

Moreover, assumption (32) implies

$$\begin{aligned}\limsup_{|\zeta|\to 0}\{\mathrm{H}_1(\zeta) + \mathrm{H}_2(\zeta)\} &= \limsup_{|\zeta|\to 0}\left\{\widehat{\mathfrak{M}}\left(\int_{\mathcal{D}}\frac{|\zeta|^{\kappa(y)}}{\kappa(y)}\,\mathrm{d}y\right) - \int_{\mathcal{D}}\Theta(y,\zeta)\,\mathrm{d}y\right\}\\ &< 0,\end{aligned}$$

which shows that 0 is not a local minimizer of $\mathrm{H}_1 + \mathrm{H}_2$, i.e., (13) holds.

For $r > 0$ small enough, the inequality $\mathrm{H}_1(\phi) < r$ implies that

$$\|\phi\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} < \left(\frac{r}{\lambda_3}\right)^{1/(\kappa^+\nu)},$$

from which we deduce

$$\mathrm{R}(r) \leq \left(\frac{r}{\lambda_3}\right)^{1/(\kappa^+\nu)}.$$

Set $\vartheta_n := \lambda_3\left(\dfrac{\beta_n}{c_0}\right)^{\kappa^+\nu}$ and let u_0 and u_n be the constant functions ζ_0 and ζ_n, respectively. Then

$$c_0\mathrm{R}(\vartheta_n) \leq \beta_n. \tag{36}$$

By assumption (6), there exists a sequence $(\zeta_n)_n \subset \mathbb{R}$ with $\zeta_n \in [-\alpha_n, \alpha_n]$ such that

$$e_n := \widehat{\mathfrak{M}}\left(\int_{\mathcal{D}} \frac{1}{\kappa(y)}|\zeta_n|^{\kappa(y)}\,\mathrm{d}y\right) \leq \frac{m_1|\mathcal{D}|}{(\kappa^-)^{\nu}}|\zeta_n|^{\kappa^-\nu} + m_2$$
$$\leq \lambda_4|\alpha_n|^{\kappa^-\nu} + m_2. \quad (37)$$

Using (33), we get for large n:

$$\lambda_4|\alpha_n|^{\kappa^-\nu} < \lambda_3\left(\frac{\beta_n}{c_0}\right)^{\kappa^+\nu} = \vartheta_n + m_2,$$

which implies that $e_n < \vartheta_n + m_2$ and thus verifies (10).

Since $\Theta(y,\cdot)$ satisfies condition (19), for each n there exists $\zeta_n \in [-\alpha_n, \alpha_n]$ such that

$$\Theta(y,\zeta_n) = \sup_{t\in[-\alpha_n,\alpha_n]} \Theta(y,t), \quad \text{for a.e. } y \in \mathcal{D}. \quad (38)$$

Applying inequalities (34) and (35), we obtain:

$$\Theta(y,\zeta_n) + f(y)(\vartheta_n - e_n - m_2) \geq \sup_{|t|\leq\beta_n} \Theta(y,t), \quad \text{for a.e. } y \in \mathcal{D}, \quad (39)$$

with strict inequality on a subset of $\mathcal{D}$ of positive measure. Combining (36) and (39), we conclude that condition (11) is satisfied.

Thus, all assumptions of Theorem 1.1, item (c), are verified. Consequently, there exists a sequence $(\psi_n)_n$ of pairwise distinct local minima of $\mathrm{H}_1 + \mathrm{H}_2$ such that

$$\lim_{n\to\infty} \mathrm{H}_1(\psi_n) = 0 \quad \text{and} \quad \|\psi_n\|_{W^{1,\kappa(\cdot)}_{\delta(\cdot)}(\mathcal{D})} \to 0,$$

which completes the proof. □

4. **Conclusions**

This study successfully establishes the existence and multiplicity of weak solutions for a class of nonlocal Kirchhoff-type problems with $\kappa(\cdot)$-Laplacian operators and Neumann boundary conditions. By leveraging critical point theory in variable exponent Sobolev spaces and Ricceri's variational principle, we overcome the challenges posed by the interplay of nonlocality, nonstandard growth, and boundary constraints. Our results extend the theory of constant-exponent problems to the more general variable exponent setting, offering broader applicability to models in capillary phenomena and non-Newtonian fluids. Future work could explore sharper regularity conditions or applications to specific physical systems.

REFERENCES

[1] *A. Ahmed, T.Ahmedatt*, H. Hjiaj and A. Touzani, Existence of infinitly many weak solutions for some quasi-linear $\vec{p}(\cdot)$-elliptic Neumann problems, Math. Bohem. **142** (2017), no. 3, 243-262.

[2] *A. Ahmed, M. S. B. Elemine Vall*, Multiplicity of weak solutions in double phase Kirchhoff elliptic problems with Neumann conditions, Bound Value Probl **2025**, 50 (2025).

[3] *A. Ahmed and M. S. B. Elemine Vall*, Perturbed nonlinear elliptic Neumann problem involving anisotropic Sobolev spaces with variable exponents, Mathematiche (Catania), **77** (2022), no. 2, 465-486.

[4] *A. Ahmed, M. S. B. Elemine Vall*, Multiplicity of weak solutions for a $(p(x), q(x))$-Kirchhoff equation with Neumann boundary conditions, J. Appl. Anal. Comput. **14** (2024), no. 4, 2441-2465.

[5] *M. Boudjeriou*, On the diffusion $p(x)$-Laplacian with logarithmic nonlinearity, Mediterr. J. Math. (2024).

[6] *C. Chu and Y. Tang*, The Exist ence and Multiplicity of Solutions for $p(x)$-Laplacian-Like Neumann Problems, Journal of Function Spaces **253** (2023), 1-9.

[7] *L. Diening, P. Harjulehto, P. Hästö and M. Růžička*, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Math. 2017 (2011).

[8] *X. L. Fan, C. Ji*, Existence of infinitely many solutions for a Neumann problem involving the $p(x)$-Laplacian, C. J. Math. Anal. Appl. **334** (2007) 248-260.

[9] *X. Fan and D. Zhao*, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. **263** (2001), no. 2, 424-446.

[10] *A. Ghelichi and M. Alimohammady*, Existence of bound states for non-local fourth-order Kirchhoff systems, Comput. Methods Differ. Equ. **7** (2019), no. 3, 418-433.

[11] *J. R. Graef, S. Heidarkhani, L. Kong*, A variational approach to a Kirchhoff-type problem involving two parameters, Results Math. **63** (2013) 877-889.

[12] *P. Harjulehto and P. Hästö*, Orlicz Spaces and Generalized Orlicz Spaces, Lecture Notes in Mathematics, 2236 (2019).

[13] *S. Heidarkhani, M. Ferrara, A. Salari, and G. Caristi*, Multiplicity results for $p(x)$- biharmonic equations with Navier boundary conditions, Complex Var. Elliptic Equ. **61** (2016), no. 11, 1494-1516.

[14] *G. Kirchhoff*, Mechanik, Teubner, Leipzig, Germany, (1883).

[15] *B. Ricceri*, A general variational principle and some of its applications, J. Comput. Appl. Math. **113** (2000), no. 1-2, 401-410.

[16] *M. M. Rodrigues*, Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators, Mediterranean Journal of Mathematics, **9** (2012), no. 1, 211-223.

[17] *S. Shokooh and A. Neirameh* , Existence results of infinitely many weak solutions for $p(x)$-Laplacian-like operators, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78** (2016), no. 4, 95-104.