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A VERSION OF WEIERSTRASS APPROXIMATIONS THEOREMS' PROOF

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In this paper we present a new proof of the Weierstrass approximation theorem, based on two important results of functional analysis namely, Hahn-Banach and Krein-Milman theorems.

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1. Introduction

If we denote by $C([0,1])$ the space of real continuous functions on $[0,1]$, endowed with the topology of uniform convergence, then it is known that any linear continuous functional on $C([0,1])$ is represented as a Riemann –Stieltjes integral (Theorem 2.1) Theorem 2.2 of this paper states a similar result for the modulus of a such functional. There are numerous proofs of Weierstrass' approximation theorem. In the excelent survey [3], Allan Pinkus classifies these proofs in three groups. The first group contains those proofs based on singular integrals. The original proof of Weierstrass is part of this group. The second group contains the proofs based on the uniform approximation of some particular functions. From this group we recall Kuhn's proof, which is considered by A. Pinkus to be the simplest and the most elegant proof. The third group contains all proofs which do not belong to the first two groups. Such is, for example, Bernstein's proof. The proof presented by us in this paper is part of the third group and is based on the well-known functional analysis theorems: Hahn-Banach and Krein - Milman.
For more details about the approximation function theory, one can consult [1].

2. The Dual Of The Space $C([0,1])$

Let $C([0,1])$ be the Banach space of all real continuous functions defined on $[0,1]$, endowed with the topology of uniform convergence given by the norm:

$$\|f\| = \sup\{|f(x)|; x \in [0,1]\}. \tag{1}$$

We recall that a function $\varphi : [0,1] \to \mathbb{R}$ is said to be *of bounded variation* if there exists $M > 0$ such that:

$$V_\Delta = \sum_{i=0}^{n-1} |\varphi(t_{i+1}) - \varphi(t_i)| < M,$$

for any partition $\Delta : 0 = t_0 < t_1 < \ldots < t_{i-1} < t_i < \ldots < t_n = 1$ of the interval $[0,1]$.
If $\varphi : [0,1] \to \mathbb{R}$ is of bounded variation, then we will denote its total variation by:

$$V_0^1(\varphi) = \sup_\Delta\{V_\Delta(\varphi)\}.$$

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Any monotonically increasing (decreasing) function $\varphi : [0,1] \to \mathbb{R}$ is of bounded variation and we have:

$$V_0^1(\varphi) = \varphi(1) - \varphi(0)(\text{ respectively, } V_0^1(\varphi) = \varphi(0) - \varphi(1)).$$

Theorem 2.1. *For any linear continuous functional $\mu : C([0,1]) \to \mathbb{R}$, there is a function of bounded variation $\varphi_\mu : [0,1] \to \mathbb{R}$ such that:*

$$\mu(f) = \int_0^1 f(t) d\varphi_\mu(t),$$

and

$$\|\mu\| = \sup \|\{|\mu(f)|; \|f\| \leq 1, f \in C([0,1])\} = V_0^1 \varphi_\mu. \tag{2}$$

Moreover, if μ is positive, i.e.: $\mu(f) \geq 0, \forall f \geq 0$, then φ_μ is increasing and:

$$\|\mu = \mu(1) = \varphi_\mu(1) - \varphi_\mu(0).$$

Proof. For the proof see, for example [2], pp. 114-115. □

The linear and continuous functionals on $C([0,1])$ are also called *Radon measure.* We will denote by $M([0,1])$ the set of all those Radon measures.Therefore, $M([0,1])$ is the dual of the space $C([0,1])$, i.e.:

$$M([0,1]) = (C([0,1]))^*.$$

The space $M([0,1])$ is a Banach lattice with respect to the norm given by (2) and the order relation:

$$\mu \leq \nu \Longleftrightarrow \mu(f) \leq \nu(f), \forall f \in C^+([0,1]).$$

The modulus of the Radon measure μ is denoted by $|\mu|$ and it is the smallest element $\mu \in M^+([0,1])$ with the property:

$$|\mu(f)| \leq \nu(|f|), \forall f \in C([0,1]).$$

In fact we have:

$$|\mu|(f) = \sup\{\mu(g); g \in C([0,1]), |g| \leq f\}, \forall f \in C^+([0,1])$$

and

$$\|\mu\| = \||\mu|\| = |\mu|(1) = V_0^1 \varphi_\mu.$$

The following result may be known and therefore we present it without proof.

Theorem 2.2. *If we denote by $\nu_\mu(t) = V_0^1 \varphi_\mu, \forall t \in [0,1]$, then ν_μ is an increasing function on the interval $[0,1]$ and we have:*

$$|\mu|(f) = \int_0^1 f(x) d\nu_\mu(t), \forall f \in C([0,1]).$$

Further, we denote by $\mathcal{P}([0,1])$ the vector subspace of all real polynomials defined on the interval $[0,1]$ and by A^0 the polar of the subset $A \subset C([0,1])$ with respect to the duality $(C([0,1]), M([0,1]))$ i.e.:

$$A^0 = \{\mu \in M([0,1]); \mu(f) \leq 1, \forall f \in A\}.$$

If $E \subset C([0,1])$ is a vector subspace, then:

$$E^0 = \{\mu \in M([0,1]); \mu(f) = 0, \forall f \in E\}.$$

For any $\mu \in C([0,1])$ and any $g \in C([0,1])$ we will denote by $g\mu$ the following Radon measure on $C([0,1])$:

$$g\mu(f) = \mu(g \cdot f) = \int_0^1 g(x) \cdot f(x) d\varphi_\mu(x), \forall f \in C([0,1]).$$

3. The Weierstrass Approximation Theorems

Theorem 3.1. *(Weierstrass) The algebra of all real polynomials defined on the interval* $[0,1]$ *is dense in the space of real continuous functions defined on* $[0,1]$ *i.e.:*

$$\overline{\mathcal{P}([0,1])} = C([0,1]).$$

Proof. The inclusion $\overline{\mathcal{P}([0,1])} \subset C([0,1])$ is obvious.
If we assume that there exists $f \in C([0,1]) \setminus \overline{\mathcal{P}([0,1])}$, then from Hahn-Banach Theorem, it follows that there is a Radon measure $\nu \in M([0,1])$ such that:

$$\nu(f) \neq 0 \text{ and } \nu(p) = 0, \forall p \in \mathcal{P} = \mathcal{P}([0,1]).$$

On the other hand, from Krein-Milman Theorem, we deduce that there exists an extremal element $\mu \in \mathrm{Ext}\,(\mathcal{P}^\circ \cap [-1,1]^\circ)$ such that $\mu(f) \neq 0$. Let $\varphi_\mu : [0,1] \to \mathbb{R}$ be the function of bounded variation with the property:

$$\mu(f) = \int_0^1 f(t) d\varphi_\mu(t), \forall f \in C([0,1]).$$

Since $0, \mu \in [-1,1]^\circ$ and μ is an extremal element, it results that $\|\mu\| = 1$.
Now we will prove that any polynomial $p \in \mathcal{P}$ is constant on the support of the measure μ. Indeed, first let's note that we can assume that $0 \geq p \geq 1$, because we can replace the function p with the function $p' = \frac{p + \|p\|}{2 \cdot \|p\|}$ which has this property and it is obvious that p is constant if and only if p' is constant. We consider now the measures:

$$\mu_1 = \frac{(1+p)\mu}{\|(1+p)\mu\|} \text{ and } \mu_2 = \frac{(1-p)\mu}{\|(1-p)\mu\|}.$$

It's clear that $\mu_1, \mu_2 \in (\mathcal{P}^\circ \cap [-1,1]^\circ)$. On the other hand, we have:

$$\frac{1}{2} \cdot \|(1+p)\mu\| + \frac{1}{2}\|(1-p)\mu\| = \frac{1}{2}\mu(1+p) + \frac{1}{2}\mu(1-p) = \frac{1}{2} \cdot 2 \cdot \mu(1) = 1$$

and further:

$$\frac{1}{2}\|(1+p)\mu\| \cdot \mu_1 + \frac{1}{2}\|(1-p)\mu\| \cdot \mu_2 = \mu.$$

Taking into account that $\mu \in \mathrm{Ext}\,(\mathcal{P}^\circ \cap [-1,1]^\circ)$, it follows that $\mu_1 = \mu_2 = \mu$.
Therefore we have:

$$\mu = \frac{(1+p)\mu}{\|(1+p)\mu\|} = \frac{\mu + p\mu}{(1+p)|\mu|} = \frac{\mu + p\mu}{(1+p)|\mu|} = \frac{\mu + p\mu}{1 + |\mu|(p)},$$

whence it results that:

$$p\mu = |\mu|(p)\mu, \text{ which involves } p|\mu| = |\mu|(p)|\mu|$$

and further that:

$$[p - |\mu|(p)]|\mu| = 0,$$

so:

$$|\mu|[(p - |\mu|(p)) \cdot f] = 0, \text{ for any } f \in C([0,1]).$$

In particular, for $f = p - |\mu|(p)$ we get:

$$|\mu|[(p - |\mu|(p))^2] = 0,$$

whence we deduce that:

$$p = |\mu|(p) = \text{constant, on } S_\mu - \text{ the support of } \mu.$$

Since the set of polynomials $\mathcal{P} = \mathcal{P}([0,1])$ separates the points of the interval $[0,1]$, it follows that the support of μ is a singleton i.e. $S_\mu = \{x_0\}$.Therefore, $\mu = a \cdot \varepsilon_{x_0}$, where:

$$\varepsilon_{x_0}(f) = f(x_0), \forall f \in C([0,1]).$$

Further we have:

$$\mu(1) = a \cdot \varepsilon_{x_0}(1) = a \cdot 1 = a, \text{ so } \mu = \mu(1) \cdot \varepsilon_{x_0}.$$

Since $1 \in \mathcal{P}$, it results that $\mu(1) = 0$, hence $\mu = 0$. We thus reached a contradiction, because $\mu(f) \neq 0$ and the theorem is proved. □

Remark 3.1. *The conclusion of theorem 3.1 remains valid for real valued continuous functions defined on any interval* $[a, b]$.

Indeed, let $f : [a, b] \to \mathbb{R}$ *be a continuous function and let* $h : [0, 1] \to [a, b]$ *be the following homeomorphism:*

$$h(t) = a + (b - a)t, \forall t \in [0, 1].$$

According to Theorem 3.1, for any $\varepsilon > 0$ *there exists an algebraic polynomial* P_ε *such that*

$$|f[h(t)] - P_\varepsilon| < \varepsilon, \forall t \in [0, 1],$$

whence it results

$$|f(x) - P_\varepsilon[h^{-1}(x)]| < \varepsilon, \forall x \in [a, b].$$

Further we denote by $\mathcal{T}([0, 1])$ the set of all trigonometric polynomials defined on $[0, 1]$ namely the set of functions $t : [0, 1] \to \mathbb{R}$ of the following form:

$$t(x) = \sum_{k=0}^{n} (\cos kx + \sin kx), \forall x \in [0, 1], n \in \mathbb{N}.$$

Using some well-known trigonometric formulas, it is immediately shown that the set of trigonometric polynomials $\mathcal{T}([0, 1])$ is an algebra.

Theorem 3.2. *The algebra of all real polynomials defined on the interval* $[0, 1]$ *is dense in the space of real continuous functions defined on* $[0, 1]$*, i.e.*

$$\overline{\mathcal{T}([0, 1])} = C([0, 1]).$$

4. Conclusions

Various proofs of density theorems based on means of functional analysis, for example the Stone-Weierstrass theorem, are known. In this paper a direct proof of Weierstrass approximation theorem based on the well-known functional analysis theorems: Hahn-Banach and Krein –Milman is given.

References

[1] *I. Bucur, G. Păltineanu* , Topics in Uniform Approximation of Continuous Functions, Birkhäuser, 2020.

[2] *R. Cristescu* , Functional Analysis, Didactic and Pedagogic Publishing House, Bucharest,1983.

[3] *A. Pinkus* , The Weierstrass Approximation Theorem, Journal of Approximation Theory, **107**(2005), 1-37.