U.P.B. Sci. Bull., Series A, Vol. 88, Iss. 1, 2026 ISSN 1223-7027

ITERATIVE ALGORITHMS FOR GENERAL MIXED BIVARIATIONAL INEQUALITIES

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We introduce and study some new classes of general mixed bivariational inequalities. The equivalence between the general mixed bivariational inequalities and fixed point problems using the resolvent operator technqiue is establised . This equivalent formulation is used to several iterative algorithms. We also use the auxiliary principle technique for solving mixed bivariational inequalities involving nonlinear term. The convergence of these new methods requires either monotonicity or pseudomonotonicity of the operator. Our new methods differ from the existing iterative methods for solving bivariational inequalities and complementarity problems. The new results are versatile and are easy to implement. Our results represent a significant improvement and refinement of the previously known results.

Keywords: Variational inequalities, splitting methods, auxilialry principle, algorithms, convergence criteria

2020 Mathematics Subject Classification: 26D15, 49J40, 90C33.

1. **Introduction**

Variational inequality theory has emerged as an effective and powerful tool for studying a wide class of unrelated problems arising in various branches of social, physical, engineering, pure and applied sciences in a unified and general framework. Variational inequalities can be viewed as a novel generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. Variatitional principles have played a leading role in the developments of computational methods for solving complicated and complex problems Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas, both for their own sake and for their applications. An important and useful generalization is called the mixed variational inequality or the variational inequality of the second kind. For the applications, formulation, generalizations, numerical methods and other aspects of variational inequalities, see [1, 2, 5, 6, 7, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]and the references therein. In recent years, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, linear approximation, auxiliary principle, and descent framework for solving variational inequalities and related optimization problems. It is well known that the projection methods and its variant forms cannot be used to suggest and analyze iterative methods for solving

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mixed variational inequalities due to the presence of the nonlinear term. These facts motivated us to use the technique of resolvent operators. In this technique, the given operator is decomposed into the sum of two (or more) maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to the development of very efficient methods, since one can treat each part of the original operator independently. The operator splitting methods and related techniques have been analyzed and studied by many authors in recent years. In the context of the mixed variational inequalities, Noor [22, 24, 27, 28] has used the resolvent operator technique to suggest some splitting type methods. A useful feature of the forward-backward splitting method for solving the mixed variational inequalities is that the resolvent step involves the subdifferential of the proper, convex and lower semicontinuous part only and the other part facilitates the problem decomposition.
In the operator splitting approach, one has to evaluate the resolvent operator, which is itself a difficult problem. To overcome these drawbacks, the auxiliary principle technique has been developed, the origin of which can be traced back to Lions and Stampacchia [15] and Glowinski et al.[9, 10]. The main idea involving this technique is to first consider an auxiliary thsolutionary problem and then to show that the solution of the auxiliary problem is of the original problem by using the fixed-point approach. Noor [21] has modified the auxiliary principle technique involving an operator and discussed its applications in information theory and machine leare nsed to suggest and analyze various iterative methods for solving various classes of variational inequalities. In recent years, this technique has been ucan be shown that several numerical methods including the projection , extragragient, and Newton, can be obtained as special cases from this technique

Motivated and inspired by the research activities in these active fields, we introduce and analyze some new calsses of general mixed bivariational inequalities involving the nonlinear term. For the proper, lower semi-continuous nonlinear term, we show that the general mixed bivariational inequalites are equivalent to the fixed point problem using the resolvent operator technique. This equivalent formulation is used to suggest some splitting type iterative methods. For the nonlinear continuous term, we apply the auxiliary principle technqique to explore some iterative methods for solving the general mixed bivariational inequalities. We also consider the convergence criteria of these newmethods. The convergence of these methods requires only either the monotonicity or pseudomonotonicity of the operator, which are conditions much weaker than the requirements of other iterative methods. Consequently our results represent an improvement and refinement of previously known results. It is interesting to compare the efficiency and practicality of the proposed methods with the other known methods and this is the subject of future research.

2. **Preliminaries and Basic Results**

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Let $N(.,.) : H \times H \longrightarrow H$ be a bifunction and A, T, g be nonlinear operators.

For a given nonlinear operator $g, T, A : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle N(Tu, Au), g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \forall v \in H. \tag{1}$$

The inequality of type (1) is called the *general mixed bivariational inequality* or the general bivariational inequality of the second kind. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the general mixed bivariational inequalities (1).

Special Cases: We now consider some important special cases of the general mixed bivariational inequalities.

(1) For $N(Tu, Au) = Tu$, where $T : H \to H$ is a single valued operator, problem (1) is equivalent to finding $u \in H$ such that

$$\langle Tu, g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \quad \text{for all } v \in H, \tag{2}$$

which is called the general mixed variational inequality. For the applications and numerical methods, see [22, 24, 28] and the references therein.

(2) If $g = I$, the identity operator, then the problem (2) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in H, \tag{3}$$

which are called the mixed variational inequalities. For the applications, numerical methods and formulations, see [1, 9, 14, 15, 18, 22, 24, 27, 28, 29, 36, 37, 39] and the references therein.

(3) φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the general mixed bivariational inequality (1) is equivalent to finding $u \in K$ such that

$$\langle N(Tu, Au), g(v) - g(u)\rangle \geq 0, \qquad \forall v \in K. \tag{4}$$

The inequality of the type (4) is known as the *generalized bivariational inequality.*

(4) If $N(Tu, Au) = Tu + Au$, then problem (4) reduces to finding $u \in K$ such that

$$\langle Tu + Au, g(v) - g(u)\rangle \geq 0, \qquad \forall v \in K. \tag{5}$$

The inequality of the type (5) is known as the *general mildly nonlinear variational inequality* involving the sum(difference) of two operators. For different and suitable choice of the oprerators, general mildly nonlinear variational inequalities include Aabsolute value, hemivariational inequalities and representing theorems such as Lax-Milgram lemma and Reisz-Frechet theorem as special cases. It has been shown that the minimumof the difference of two convex functions can be studied in the unified frame work of mildly variational inequalities. Also, see [11, 12, 18] for the applications of the DC-problems and optimization problems. It turned out that a class of unrelated odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied by the general variational inequality (5), see [18, 19, 22, 23, 26, 31, 32, 34] and the references therein.

(5) If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \forall v \in K\}$ is a polar (dual) cone of the convex cone K in H, then problem (4) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad N(Tu, Au) \in K^*, \quad \langle g(u), N(Tu, Au)\rangle = 0, \tag{6}$$

which is known as the general bicomplementarity problem. Note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem (6) is known as the quasi(implicit) bicomplementarity problem. If $N(Tu, Au) = Tu, g = I$, the identity operator, then problem (6) is the generalized complementarity problem [19, 25], which has been studied extensively.

(6) For $N(Tu, Au) = Tu$ and $g = I$, the identity operator, the general variational inequality (4) becomes : find $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \text{forall } v \in K, \tag{7}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [38] in 1964. For the recent state-of-the-art, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

We now recall the following well known concepts and results.

Definition 2.1. [5] *If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by*

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|.$$

Remark 2.1. *It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator. We denote by*

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \text{for all } u \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H.

Lemma 2.1. *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \text{for all } v \in H, \tag{8}$$

if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

This property of the resolvent operator J_φ plays an important role in obtaining our results.

Definition 2.2. *For all $u, v \in H$, the operator $N(.,.) : H \times H \longrightarrow H$ is said to be:*

(i) *g-monotone, if*

$$\langle N(Tu, Au) - N(Tv, Av), g(u) - g(v)\rangle \geq 0$$

(ii)*g-pseudomonotone, if*

$$\langle N(Tu, Au), g(v) - g(u)\rangle \geq 0 \quad \text{implies} \quad \langle N(Tv, Av), g(v) - g(u)\rangle \geq 0.$$

Note that for $g \equiv I$, the identity operator and $N(Tu, Au) = Tu$, where $T; H \longrightarrow H$ is a single -valued operator, Definition 2.2 reduces to the standard definition of monotonicity and pseudomonotonicity of the operator T. It is well known [4] that monotonicity implies pseudomonotonicity, but not conversely.

3. Main Results

In this section, we suggest and analyze some new iterative methods for solving general mixed bivariational inequality (1). One can prove that the general mixed bivariational inequality (1) is equivalent to a fixed-point problem by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in H$ is a solution of the general mixed variational inequality (2.1), if and only if , $u \in H$ satisfies the relation*

$$g(u) = J_{\varphi}[g(u) - \rho N(Tu, Au)], \tag{9}$$

where $J_{\varphi} = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the general mixed bivariational inequality (1) is equivalent to the fixed-point problem. This alternate equivalent formulation is very useful from the numerical point of view.
From now onward, we denote

$$w = J_{\varphi}[g(u) - \rho N(Tu, Au)], \tag{10}$$

unless otherwise specified.

The fixed-point formulation (9) enables us to suggest and analyze the following iterative method.

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_{\varphi}[g(u_n) - \rho N(Tu_n, Au_n)].$$

It is known that the convergence analysis of Algorithm 3.1 requires that both the operators g and $N(.,.)$ must be strongly monotone and Lipschitz continuous. These strict conditions rule out many applications of Algorithm 3.1 These facts motivated us to modify these resolvent type iterative methods.

We define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - J_{\varphi}[g(u) - \rho N(Tu, Au)] \equiv g(u) - w. \tag{11}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of the general mixed variational inequality (1), if and only if, it is a zero of the equation

$$R(u) = 0. \tag{12}$$

For a constant $\gamma \in (0, 2)$, equation (12) can be written as

$$g(u) + \rho N(Tu, Au) = g(u) + \rho N(Tu, Au) - \gamma R(u).$$

This formulation is used to suggest a new implicit method for solving the general mixed bivariational inequality (1).

Algorithm 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) + \rho N(Tu_n, Au_n) - \rho N(Tu_{n+1}, Au_{n+1}) - \gamma R(u_n), \quad n = 0, 1, 2, \ldots. \tag{13}$$

We remark that if φ is the indicator function of a closed convex set K in H, then the resolvent operator $J_{\varphi} \equiv P_K$, the projection of H onto K. Consequently, the relation (13) becomes

$$R_K(u) = g(u) - P_K[g(u) - \rho N(Tu, Au)], \tag{14}$$

and Algorithm 3.2 becomes:

Algorithm 3.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) + \rho N(Tu_n, Au_n) - \rho N(Tu_{n+1}, Au_{n+1})) - \gamma R_K(u_n),$$

If $g \equiv I$, the identity operator, then Algorithm 3.2 reduces to:

Algorithm 3.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n + \rho N(Tu_n, Au_n) - \rho N(Tu_{n+1}, Au_{n+1}) - \gamma R(u_n).$$

where

$$R(u_n) = u_n - J_\varphi[u_n - \rho N(Tu_n, Au_n)].$$

If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently Algorithm 3.2 reduces to:

Algorithm 3.5. *For a given $u_0 \in K$,compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n + \rho N(Tu_n, Au_n) - \rho N(Tu_{n+1}, Au_{n+1}) - \gamma\{u_n - P_K[u_n - \rho N(Tu_n, Au_n)]\}.$$

For $N(u, u) = Tu, T : H \longrightarrow H$, Algorithm 3.1-Algorithm3.5 have been suggested and analyzed for general mixed variational inequalities of type (2). In brief, for suitable and appropriate choice of the operators $T, A, g, N(.,.)$ and the space H, one can obtain a large number of new and known algorithms for solving various classes of variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 3.2.

Lemma 3.2. *Let $\bar{u} \in H$ be a solution of (1). If $N(.,.) : H \times H \to H$ is a g-monotone operator, then*

$$\langle g(u) - g(\bar{u}) + \rho(N(Tu, Au) - N(T\bar{u}), A\bar{u}), R(u)\rangle \geq \|R(u)\|^2, \quad \textit{for all } u \in H. \tag{15}$$

Proof. Let $\bar{u} \in H$ be a solution of (1), then

$$\langle N(T\bar{u}, A\bar{u}), g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geq 0, \quad \forall v \in H. \tag{16}$$

Taking $g(v) = J_\varphi[g(u) - \rho N(Tu, Au)] = w$ in (16), we have

$$\rho\langle N(T\bar{u}, A\bar{u}), w - g(\bar{u})\rangle + \rho\varphi(w) - \rho\varphi(g(\bar{u})) \geq 0. \tag{17}$$

Setting $z = g(u) - \rho N(Tu, Au), u = J_\varphi[g(u) - \rho N(Tu, Au)] = w, v = g(\bar{u})$ in (8), we obtain

$$\langle g(u) - \rho N(Tu, Au) - w, w - g(\bar{u})\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(w) \geq 0. \tag{18}$$

Adding (17), (18) and using (11), we have

$$\langle R(u) - \rho(N(Tu, Au) - N(T\bar{u}, A\bar{u})), g(u) - g(\bar{u}) - R(u)\rangle \geq 0. \tag{19}$$

From (19), since $N(.,.)$ is g-monotone, it follows that

$$\begin{aligned}
&\langle g(u) - g(\bar{u}) + \rho(N(Tu, Au) - N(T\bar{u}, A\bar{u}), R(u)\rangle \\
&\quad \geq \lambda R(u), R(u)\rangle + \rho\langle N(Tu, Au) - N(T\bar{u}, A\bar{u}), g(u) - g(\bar{u})\rangle \\
&\quad \geq \langle R(u), R(u)\rangle,
\end{aligned}$$

which implies that

$$\langle g(u) - g(\bar{u}) + \rho(N(Tu, Au) - N(T\bar{u}, A\bar{u})), R(u)\rangle \geq \|R(u)\|^2,$$

the required result. □

Lemma 3.3. *Let $\bar{u} \in H$ be the solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 3.2, then*

$$\|g(u_{n+1}) - g(\bar{u}) + \rho(N(Tu_{n+1}, Au_{n+1}) - N(T\bar{u}, A\bar{u}))\|^2 \leq \|g(u_n) - g(\bar{u}) \\ +\rho(N(Tu_n, Au_n) - N(T\bar{u}, A\bar{u}))\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \quad (20)$$

Proof. Since $\bar{u}$ is a solution of (1) and u_{n+1} satisfies the relation (13), so

$$\begin{aligned} &\|g(u_{n+1}) - g(\bar{u}) + \rho(N(Tu_{n+1}, Au_{n+1}) - N(T\bar{u}, A\bar{u}))\|^2 = \|g(u_n) - g(\bar{u}) \\ &\quad +\rho(N(Tu_n, Au_n) - N(T\bar{u}, A\bar{u})) - \gamma R(u_n)\|^2 \\ &\quad \leq \|g(u_n) - g(\bar{u}) + \rho(N(Tu_n, Au_n) - N(T\bar{u}, A\bar{u}))\|^2 \\ &\quad -2\gamma\|R(u_n)\|^2 + \gamma^2\|R(u_n)\|^2, \text{ by using (15).} \\ &\quad = \|g(u_n) - g(\bar{u}) + \rho(N(Tu_n, Au_n) - N(T\bar{u}, A\bar{u}))\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \end{aligned}$$

□

Theorem 3.1. *Let $g : H \to H$ be invertible and H be a finite dimensional space, then the approximate solution u_{n+1} obtained from Algorithm 3.2 converges to a solution $\bar{u}$ of the general mixed bivariational inequality (1).*

Proof. Let $\bar{u} \in H$ be a solution of (1). From (3.12), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|g(u_0) - g(\bar{u}) + \rho(N(Tu_0, Au_0) - N(T\bar{u}, A\bar{u}))\|^2,$$

and consequently

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and suppose that the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u}$. Since $R(u)$ is continuous, it follows that

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

and $\bar{u}$ is the solution of the general mixed bivariational inequality (1) by invoking Lemma 3.1 and

$$\begin{aligned} &\|g(u_{n+1}) - g(\bar{u}) + \rho(N(Tu_{n+1}, Au_{n+1}) - N(T\bar{u}, A\bar{u}))\|^2 \\ &\quad \leq \|g(u_n) - g(\bar{u}) + \rho(N(Tu_n, Au_n) - N(T\bar{u}, A\bar{u}))\|^2. \end{aligned}$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}).$$

Since g is invertible, so

$$\lim_{n\to\infty} (u_n) = \bar{u},$$

which is the solution of the general mixed bivariational inequality (1). □

To implement Algorithm 3.2, one has to find the solution implicitly, which may create some problems. To overcome this difficulty, we suggest another iterative method, the convergence of which requires only the pseudomonotonicity of the operator, which is a weaker condition than monotonicity.

For a stepsize $\gamma \in (0, 2)$, equation (12) can be written as

$$g(u) = g(u) - \gamma R(u).$$

This fixed-point formulation is used to suggest the following iterative method.

Algorithm 3.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \gamma R(u_n) = g(u_n) - \gamma\{g(u_n) - w_n\}.$$

Note that for $\gamma = 1$, Algorithm 3.6 becomes Algorithm 3.1.
We now consider the convergence criteria of Algorithm 3.6.

Lemma 3.4. *Let $\bar{u} \in H$ be a solution of (1) and $T : H \to H$ be a g-pseudomonotone operator. Then*

$$\langle g(u) - g(\bar{u}), R(u)\rangle \geq \|R(u)\|^2, \quad \forall u \in H. \tag{21}$$

Proof. Since T is g-pseudomonotone, from (16), we have

$$\langle N(Tv, Av), g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geq 0. \tag{22}$$

Taking $g(v) = w$ in (22), we have

$$\langle N(T(g^{-1}w), A(g^{-1}w)), w - g(\bar{u})\rangle + \varphi(w) - \varphi(g(\bar{u})) \geq 0. \tag{23}$$

Setting $z = u - \rho N(T(g^{-1}w), A(g^{-1}w))$,$u = w$, and $v = g(\bar{u})$ in (8), we obtain

$$\langle w - g(u) + \rho N(T(g^{-1}w), A(g^{-1}w)), g(\bar{u}) - w\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(w) \geq 0. \tag{24}$$

Adding (23), (24) and using (11), we have

$$\langle g(u) - g(\bar{u}), R(u)\rangle \geq \|R(u)\|^2,$$

the required result. □

Lemma 3.5. *The sequence $\{u_n\}$ generated by Algorithm 3.6 for general mixed bivariational inequalities (1) satisfies the inequality*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2, \quad \forall \bar{u} \in H.$$

Proof. From (21), and Algorithm 3.6, we have

$$\begin{aligned}\|g(u_{n+1}) - g(\bar{u})\|^2 &= \|g(u_n) - g(\bar{u}) - \gamma R(u_n)\|^2 \\ &\leq ||g(u_n) - g(\bar{u})||^2 - 2\gamma\langle g(u_n) - g(\bar{u}), R(u_n)\rangle + \gamma^2||R(u_n)||^2 \\ &\leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2.\end{aligned}$$

□

Theorem 3.2. *Let $\{u_n\}$ be the approximate solution obtained from Algorithm 3.12 and $\bar{u} \in H$ be a solution of (2.1). Then $\lim_{n\to\infty}(u_n) = \bar{u}$.*

Proof. Its proof follows from Theorem 3.1. □

4. **Auxiliary principle technique**

In this section, we consider the mixed bivariational inequalities (1) involving the nonlinear function φ, which is not a proper, lower semicontinous function. In these cases, the technique discussed in the previous section cannot be used to propose and suggest the projection, resolvent, descent methods for solving the general mixed bivariational inequalities and their variant forms. To overcome these drawbacks, we use the auxiliary principle technique, which is mainly due to Lions et al. [15] and Glowinski et al [9] as developed in [18, 21, 25, 28, 31, 32, 33, 36] . Noor [21] has modified the auxilialry priciple technique involving an arbitrary function. To be more precise, for an arbitrary strongly monontone operator M, we define the distance function as

$$\begin{aligned} \mathtt{M}(v,u) &= M(v)-M(u), v-u\rangle, \forall u,v\in K. \\ &\geq \zeta\|v-u\|^2, \quad u,v\in K, \end{aligned} \tag{25}$$

where ζ is the strongly monononicity constant. It is important to emphasize that various types of function $\mathtt{M}$ gives different modified distance function.
We apply the auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the problem (1).
For a given $u\in H$ satisfying (1), find $w\in H$ such that

$$\begin{aligned} &\langle \rho N(T(w+\eta(u-w)), A(w+\eta(u-w))+g(w)-g(u)), g(v)-g(w)\rangle \\ &\quad +\rho(\varphi(g(v))-\varphi(g(w)))+\langle M(w)-M(u), v-w\rangle \geq 0, \quad \forall v\in H, \end{aligned} \tag{26}$$

where $\rho>0, \eta\in[0,1]$ are constants and M is an arbitrary operator. The inequality (26) is called the auxiliary bivariational inequality.
If $w=u$, then w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving (1).

Algorithm 4.1. *For a given $u_0\in H$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho N(T(u_{n+1}+\eta(u_n-u_{n+1})), Au_{n+1}+\eta(u_n-u_{n+1}))+g(u_{n+1})-g(u), g(v)-g(u_{n+1})\rangle \\ &\quad +\rho(\varphi(g(v))-\varphi(g(u_{n+1})))+\langle M(\mu_{n+1})-M(\mu_n), \nu-\mu_{n+1}\rangle \geq 0, \quad \forall v\in H. \end{aligned} \tag{27}$$

Algorithm 4.1 is called the hybrid proximal point algorithm for solving the bivariational inequalities (1).
Special Cases: We now discuss some special cases.
(I). For $\eta=0$, Algorithm 4.1 reduces to

Algorithm 4.2. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho N(T(\mu_{n+1}), A(\mu_{n+1}))+g(\mu_{n+1})-g(\mu_n), g(v)-g(\mu_{n+1})\rangle \\ &\quad +\rho(\varphi(g(v))-\varphi(g(u_{n+1})))+\langle M(u_{n+1})-M(u), v-u_{n+1}\rangle \geq 0, \quad \forall v\in H, \end{aligned} \tag{28}$$

is called the implicit iterative methods for solving the problem (1).
(II). For $\eta=\frac{1}{2}$, Algorithm 4.1 becomes:

Algorithm 4.3. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho N(T(\frac{u_{n+1}+u_n}{2}), A(\frac{u_{n+1}+u_n}{2}))+g(u_{n+1})-g(u), g(v)-g(u_{n+1})\rangle \\ &\quad +\rho(\varphi(g(v))-\varphi(g(u_{n+1})))+\langle M(u_{n+1})-M(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall v\in H, \end{aligned}$$

is known as the mid-point proximal method for solving the problem (1).

(III). For $\eta=1$, Algorithm 4.1 reduces to

Algorithm 4.4. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho N(T(u_n), A(u_n))+g(u_{n+1})-g(u_n), g(v)-g(u_{n+1})\rangle \\ &\quad +\rho(\varphi(g(v))-\varphi(g(u_{n+1})))+\langle M(u_{n+1})-M(u), v-u_{n+1}\rangle \geq 0, \quad \forall v\in H, \end{aligned}$$

which is an explicit method for solving the general mixed bivariational inequality (1).

For the convergence analysis of Algorithm 4.2, we need the following concepts.

Definition 4.1. *An operator g is said to be pseudomontone with respect to the function φ, if*

$$\langle \rho N(Tu, Au) + g(u) - g(v), g(v) - g(u)\rangle + \rho(\varphi(g(v)) - \varphi(g(u))) \geq 0, \quad \forall v \in H,$$

implies that

$$-\langle \rho N(Tv, Av) + g(v) - g(u), g(u) - g(v)\rangle + \rho(\varphi(g(u)) - \varphi(g(v))) \geq 0, \quad \forall v \in H.$$

Theorem 4.1. *Let the operator g be a pseudo-monotone. Let the approximate solution u_{n+1} obtained in Algorithm 4.2 converges to the exact solution $u \in H$ of the problem (1). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|u_{n+1} - u_n\| \leq \zeta\|u - u_n\|. \tag{29}$$

Proof. Let $u \in H$ be a solution of the problem (1). Then

$$-\langle \rho N(Tv, Av) + g(v) - g(u), g(u) - g(v)\rangle + \rho(\varphi(g(u)) - \varphi(g(v))) \geq 0, \quad \forall v \in H. \tag{30}$$

since the operator g is a pseudo-monotone with respect to the function φ.
Takin $v = u_{n+1}$ in (30), we obtain

$$\begin{aligned} &-\langle \rho N(T(u_{n+1}), A(u_{n+1})) + g(u_{n+1}) - g(u), g(u) - g(\mu_{n+1}) \\ &\quad +\rho(\varphi(g(u)) - \varphi(g(u_{n+1})))\rangle \geq 0. \end{aligned} \tag{31}$$

Setting $v = u$ in (28), we have

$$\begin{aligned} &\langle \rho N(T(u_{n+1}), A(u_{n+1})) + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\quad +\rho(\varphi(g(u)) - \varphi(g(u_{n+1}))) + \langle M(u_{n+1}) - M(u), u - u_{n+1}\rangle \geq 0, \quad \forall v \in H, \end{aligned} \tag{32}$$

Combining (32), (31) and (30), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), u - u_{n+1}\rangle \geq -\langle \rho N(Tu_{n+1}, Au_{n+1}), u - u_{n+1}\rangle \geq 0. \tag{33}$$

From the equation (33), we have

$$\begin{aligned} 0 &\leq \langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle = \langle M(u_{n+1}) - M(u_n), u - u_n + u_n - u_{n+1}\rangle \\ &= \langle M(u_{n+1}) - M(u_n), u - u_n\rangle + \langle M(u_{n+1} - M(u_n), u_n - u_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(u_{n+1} - M(u_n), u_{n+1} - u_n\rangle \leq \langle M(u_{n+1}) - M(u_n), u - u_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|u_{n+1} - u_n\|^2 \leq \zeta\|u_{n+1} - u_n\|\|u_n - u\|,$$

which implies that

$$\xi\|u_n - u_{n+1}\| \leq \zeta\|u_n - u\|,$$

the required result (29). □

Theorem 4.2. *Let H be a finite dimensional space and all the assumptions of Theorem 4.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 4.2 converges to the exac solution $\mu \in H$ of (1).*

Proof. Let $\mu \in \Omega$ be a solution of (1). From (29), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi \sum_{n=0}^{\infty} \|u_{n+1} - u_n\| \leq \zeta \|u_n - u\|,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{34}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; whose subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (28), taking the limit $n_j \longrightarrow \infty$ and using (34), we have

$$\langle \rho N(T\hat{u}, A\hat{u} + g(\hat{u}) - \hat{u}, g(v) - g(\hat{u})\rangle + \rho(\varphi(v)) - \varphi(g(u))) \geq 0, \qquad \forall v \in H,$$

which implies that $\hat{u}$ solves the problem (1) and

$$\|u_{n+1} - u\| \leq \|u_n - u\|.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result. □

We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the bivariational inequalities (1).
For a given $u \in H$ satisfying (1), find $w \in H$ such that

$$\begin{aligned}&\langle \rho N(T(w + \eta(u - w)), Aw + \eta(u - w)), g(v) - g(w)\rangle + \rho(\varphi(g(v)) - \varphi(g(w)))\\ &\quad + \langle M(w) - M(u) + \alpha(u - u), v - w\rangle \geq 0, \quad \forall v \in H,\end{aligned} \tag{35}$$

where $\rho > 0, \eta, \alpha \in [0, 1]$ are constants and M is a nonlinear operator.
Clearly $w = \mu$, implies that w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving (1).

Algorithm 4.5. *For given u_0, u_1, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(T(u_{n+1} + \eta(u_n - u_{n+1})), A(u_{n+1} + \eta(u_n - u_{n+1})), g(v) - g(u_{n+1})\rangle\\ &\quad + \rho(\varphi(g(v)) - \varphi(g(u_{n+1})))\\ &\qquad + \langle M(u_{n+1}) - M(u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in H.\end{aligned}$$

Algorithm 4.5 is called the hybrid proximal point algorithm for solving the mixed bivariational inequalities (1). For $\alpha = 0$, Algorithm 4.5 is exactly the$Algorithm 4.1$.
$If M = 0$, then Algorithm 4.5 reduces to:

Algorithm 4.6. *For given u_0, u_1, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(T\mu_{n+1} + \eta(\mu_n - u_{n+1})), A(u_{n+1} + \eta(u_n - u_{n+1})), g(v) - g(u_{n+1})\rangle\\ &\quad + \rho(\varphi(g(v)) - \varphi(g(u_{n+1}))) + \alpha\langle (u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in H,\end{aligned}$$

which is called the inertial iterative method for solving the general mixed bivariational inequalities (1).

Remark 4.1. *For different and suitable choice of the parameters ρ, η, α, operators T, A, g, M and convex-valued sets, one can recover new and known iterative methods for solving bivariational inequalities, bicomplementarity problems and related optimization problems. Using the technique and ideas of Theorem 4.1 and Theorem 4.2, one can analyze the convergence of Algorithm 4.5 and its special cases.*

.

Conclusion: We have suggested and analyzed a number of new resolvent iterative methods for solving general mixed bivariational inequalities. Convergence of some of these methods requires the pseudomonotonicity of the operator, which is weaker than the monotonicity. In particular, we have proved the modified projection method and projection method converge for pseudomonotone monotone operators, thereby, improving the earlier results. In this respect, our results represent an improvement and refinement of the previous results. The comparison of these new methods with the other standard techniques for solving the general mixed variational inequalities is an interesting problem for further research.

Authors contributions: All authors contributed equally to the conception, design of the work, analysis, interpretation of data, reviewing it critically and final approval of the version for publication.

Acknowledgments: The authors sincerely thank their respected Professors, teachers, students, colleagues, collaborators, referees, editors and friends, who have contributed, directly or indirectly to this research.

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