MIXING AND CHAOS TO MULTIPLE MAPPINGS

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Let (X, d) be a compact metric space and $F = \{f_1, f_2, ..., f_m\}$ be an m-tuple of continuous maps from X to itself. In this paper, we introduce the definitions of transitivity, weakly mixing and mixing of multiple mappings (X, F) from a set-valued perspective, which is the semigroup generated by F based on iterated function system. Firstly, we prove that for multiple mappings, mixing implies distributional chaos in a sequence, Li-Yorke chaos and Kato's chaos. Besides, we demonstrate that F is Kato's chaos if and only if F^k is Kato's chaos for any $k \in \mathbb{N}$. Finally, we construct a symbolic dynamical system to show that distributional chaos may be generated by only two strongly non-wandering points.

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1. Introduction

Chaos is a highly significant research topic in the field of topological dynamical systems. The concept of chaos was first introduced in 1975 by Li and Yorke[1]. Since then, various definitions of chaos have been proposed by researchers in different fields, such as Kato's chaos[2], distributional chaos[3], etc. Therefore, it is an important and significant to understand the relations among the different definitions. Currently, there are numerous research results regarding this field, see, for instance, papers [4, 5, 6] and the references therein.

In 2016, Hou and Wang[7] defined multiple mappings derived from iterated function system (IFS). Iterated function system is a significant branch of fractal theory, reflecting the fundamental facts of the world[8, 9, 10, 11, 12]. Hou and Wang focused primarily on studying the Hausdorff metric entropy of multiple mappings. Then then authors[13] provided a sufficient condition for F to exhibit chaos in the sense of Li-Yorke. They also showed that if F is Hausdorff metric disdributionally chaotic, then there exist at least two strongly nonwandering points of F. In [14], authors proved that Hausdorff metric Li-Yorke chaos and distributional chaos are preserved under topological conjugacy, Hausdorff metric Li-Yorke δ-chaos is equivalent to Hausdorff metric distributional δ-chaos in a sequence and that for Hausdorff metric Li-Yorke chaos, there is non-mutual implication between the multiple mappings $F = \{f_1, f_2, \cdots, f_m\}$ and each element f_i $(i = 1, 2, \cdots, m)$ in F. Recently, Zhao[15] introduced and studied sensitivity, accessibility, and Kato's chaos of multiple mappings. It is worth noting that researchers studying iterated function systems often approach the topic from a group perspective rather than a set-valued perspective. This also establishes a close connection between multiple mappings and set-valued mappings, or we can consider multiple mappings as a special case of set-valued mappings.

The above special set-valued mappings consider the image of one point as a set (a compact set), with the Hausdorff metric of set-valued mappings space, and investigate the

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relationships among the images of points under multiple mappings from a set-valued view. As is known to all, the main idea of the system (X, f) is to point out how the point of X moves. However, when it comes to the numerical simulation, economic variables, target selection, population migration, etc., simply knowing the movement of point of X is not enough. Besides, it is crucial to acknowledge the valuable role of set-valued mappings in addressing complex problems involving uncertainty, ambiguity, or multiple criteria. Set-valued mappings provide versatility and flexibility, making them highly beneficial across various domains. Set-valued mappings also prove their usefulness in decision-making processes that require considering multiple criteria or preferences. In fact, the applications of set-valued mappings are vast and diverse, encompassing numerous fields beyond those mentioned here. Therefore, studying the distance relationship of points iterated under multiple mappings from a set-valued perspective is highly significant. For more recent results about these topics, one is referred to[7, 13, 14, 16] and references therein.

As we all know, it is crucial to study the properties of the chaos to multiple mappings, the relationship among the various chaos to multiple mappings, and the relationship between the chaos to multiple mappings and the chaos in the classic sense. In this paper, we will investigate the relationship between mixing and chaos(distributional chaos in a sequence, Li-Yorke chaos and Kato's chaos) to multiple mappings. For a single continuous self-map, Liao et al. [5] claimed that mixing implies distributively chaotic in a sequence and Li-Yorke chaos, and Wang et al. [6] claimed that weakly mixing implies Kato's chaos. Additionally, we will explore the iteration invariance of Kato's chaos to the multiple mappings. There have been numerous studies on the iteration invariance of chaos in the classic sense(see, for example,[17, 18]). Finally, we will present an example which is Hausdorff metric distributional chaos but only has two strongly non-wandering points. This result is the supplement of the Theorem 1 in [13].

This paper is structured as follows. In Section 2, we we will state some definitions and lemmas about multiple mappings. Section 3 will present the main conclusions. In Section 4, we will give some examples.

2. Preparations and lemmas

Throughout this paper, X is a compact metric space with a metric d, and $\mathbb{N} = \{1, 2, 3, ...\}$. Let $F = \{f_1, f_2, \cdots, f_m\}$ be an m-tuple of continuous self-maps on X. Then $F(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ is a nonempty compact subset of X. Firstly, we will give some definitions about set-valued spaces (see [7]). Denote $\mathcal{K}(X) = \{K \mid K$ is a nonempty compact subset of $X\}$, then F is a map from X to $\mathcal{K}(X)$, and the metric on $\mathcal{K}(X)$ is denoted by d_H, which is called Hausdorff metric and defined by $d_H(A, B) = \max\{dist(A, B), dist(B, A)\}$, where $dist(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$. It is obvious that $(\mathcal{K}(X), d_H)$ is a compact metric space. Given $m \geq 1$, for any $n \in \mathbb{N}$, $F^n : X \to \mathcal{K}(X)$ is defined by $\forall x \in X$, $F^n(x) = \{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x) \mid i_1, i_2, \cdots, i_n = 1, 2, \cdots, m\}$. In fact, for any subset $A \subset X$, one can define $F^n(A) = \bigcup_{a \in A} F^n(a), F^{-n}(A) = \{x \in X \mid F^n(x) \subset A\}$. Specifically, if $A \in \mathcal{K}(X)$, then $F^n(A) \in \mathcal{K}(X)$. It is clear that F naturally induces a map $\widetilde{F}^n : \mathcal{K}(X) \to \mathcal{K}(X)$ denoted by $\widetilde{F}^n(A) = F^n(A), \forall A \in \mathcal{K}(X)$. One can get that F^n and $\widetilde{F}^n$ are both continuous from [7]. Notice that if each element $f_i (i = 1, 2, \cdots, m)$ in $F = \{f_1, f_2, \cdots, f_m\}$ is the same, then the Hausdorff metric dynamical systems of multiple mappings is actually the classical dynamical systems, the former is an extension of the latter. Denote $B(A, \varepsilon) = \{x \in X \mid dist(x, A) < \varepsilon\}$, where $A \subseteq X$.

Next some definitions of Hausdorff metric chaos from a set-valued view were introduced.

Definition 2.1. *The map F is called Hausdorff metric Li-Yorke chaos if there is an uncountable set $S \subset X$ satisfying for any points $x, y \in S$ with $x \neq y$, $\liminf_{n\to\infty} d_H(F^n(x), F^n(y)) = 0$, $\limsup_{n\to\infty} d_H(F^n(x), F^n(y)) > 0$. The map F is called Hausdorff metric Li-Yorke δ-chaos or Hausdorff metric uniformly Li-Yorke chaos if there are $\delta > 0$ and uncountable set $S \subset X$ satisfying for any points $x, y \in S$ with $x \neq y$, $\liminf_{n\to\infty} d_H(F^n(x), F^n(y)) = 0$, $\limsup_{n\to\infty} d_H(F^n(x), F^n(y)) > \delta$.*

Definition 2.2. *For any pair $(x, y) \in X \times X$ and any $n \in \mathbb{N}$, denote distributional function $\phi^n_{xy}(F, \cdot) : R \to [0, 1]$ by*

$$\phi^n_{xy}(F, t) = \frac{1}{n}\sharp\{0 \le i \le n-1 \mid d_H(F^i(x), F^i(y)) < t\},$$

where $\sharp A$ denotes the cardinality of the set A. when $t < 0$, $\phi^n_{xy}(F, t) = 0$. Let

$$\phi_{xy}(F, t) = \liminf_{n\to\infty} \phi^n_{xy}(F, t),\ \ \phi^*_{xy}(F, t) = \limsup_{n\to\infty} \phi^n_{xy}(F, t).$$

Then $\phi_{xy}(F, t)$ and $\phi^*_{xy}(F, t)$ respectively denote lower and upper distributional function. The multiple mappings F is said to be Hausdorff metric distributional chaos if there is an uncountable set $S \subset X$ satisfying for any $x, y \in S$ with $x \neq y$, $\phi^*_{xy}(F, t) = 1$ for all $t > 0$ and $\phi_{xy}(F, \varepsilon) = 0$ for some $\varepsilon > 0$.

Definition 2.3. *Suppose that $\{p_k\}_{k=0}^{\infty}$ is an increasing sequence of positive integers. For any pair $(x, y) \in X \times X$ and any $n \in \mathbb{N}$, put distributional function $\phi^n_{xy}(F, \cdot, p_k) : R \to [0, 1]$ by*

$$\phi^n_{xy}(F, t, p_k) = \frac{1}{n}\sharp\{0 \le i \le n-1 \mid d_H(F^{p_i}(x), F^{p_i}(y)) < t\},$$

when $t < 0$, $\phi^n_{xy}(F, t, p_k) = 0$. Let

$$\phi_{xy}(F, t, p_i) = \liminf_{n\to\infty} \phi^n_{xy}(F, t, p_i),\ \ \phi^*_{xy}(F, t, p_i) = \limsup_{n\to\infty} \phi^n_{xy}(F, t, p_i).$$

Then $\phi_{xy}(F, t, p_i)$ and $\phi^*_{xy}(F, t, p_i)$ are respectively called lower distributional function and upper distributional function. The multiple mappings F is said to be Hausdorff metric distributional chaos in a sequence if there is an uncountable set $S \subset X$ satisfying for any distinct points $x, y \in S$,$\phi^*_{xy}(F, t, p_i) = 1$ for all $t > 0$ and $\phi_{xy}(F, \varepsilon, p_i) = 0$ for some $\varepsilon > 0$. The dynamical system (X, F) is said to be Hausdorff metric distributional δ-chaos in a sequence or Hausdorff metric uniformly distributively chaotic in a sequence, if there exist $\delta > 0$ and uncountable set $S \subset X$ such that for any $x, y \in S$ with $x \neq y$, satisfying $\phi^*_{xy}(F, t, p_i) = 1$ for all $t > 0$ and $\phi_{xy}(F, \delta, p_i) = 0$ for the δ above.

Remark 2.1. *It is obvious that (X, F) is Hausdorff metric distributional δ-chaos in a sequence (Hausdorff metric δ-Li-Yorke chaos, respectively) implies that (X, F) is Hausdorff metric distributional chaos in a sequence (Hausdorff metric Li-Yorke chaos, respectively) by definition.*

Definition 2.4. [15] *The multiple mappings F is said to be (Hausdorff metric) sensitive, if there is $\delta > 0$ such that for any nonempty open set $U \subset X$, there exist $x, y \in U$ and $n \in \mathbb{N}$ such that $d_H(F^n(x), F^n(x)) > \delta$. The multiple mappings F is said to be (Hausdorff metric) accessible, if for any $\varepsilon > 0$ and any nonempty open set $U, V \subset X$, there exist $x \in U, y \in V$ and $n \in \mathbb{N}$ such that $d_H(F^n(x), F^n(x)) < \varepsilon$. The multiple mappings F is said to be (Hausdorff metric) Kato's chaotic, if it is (Hausdorff metric) sensitive and accessible.*

Definition 2.5. *$x \in X$ is said to be a strongly nonwandering point of F if for any open neighborhood V of x, there is $y \in X$ satisfying $\limsup_{n\to\infty} \frac{1}{n}\sharp\{i \mid F^i(y) \cap V \neq \emptyset, 0 \le i \le n-1\} > 0$.*

Transitivity and mixing are important research topics of dynamical systems, next we will define them from a set-valued perspective in the following. Put $R(F^n) = \{F^n(x) \mid x \in X\}$ for all $n \in \mathbb{N}$, that is the range of F^N, which is a subset of $\mathcal{K}(X)$. However, since the multiple mappings is a special set-valued mappings from X to $\mathcal{K}(X)$, there is no natural way to extend the notion of transitivity and mixing. Therefore, in this paper, when discussing the transitivity, weak mixing and mixing, we consider a special multiple mappings, which satisfying $R(F^{N+1}) = R(F^N)$ for some $N \in \mathbb{N}$ and $X = F(X)$. Please remember that term 'map' will always refer to the above special multiple mappings in the rest of the paper without special illustration. For the convenience of the following, we denote the range of F^N above by $R_a(F)$ and the extension of $V \subseteq X$ to $\mathcal{K}(X)$ by $e(V) = \{K \in \mathcal{K}(X) \mid K \subseteq V\}$. It can be shown that if V is an nonempty open set of X, then $e(V)$ is an nonempty open set of $\mathcal{K}(X)$ from [7].

Definition 2.6. *A map F is (topological) transitive if for any nonempty open sets U and V of X with $V \cap R_a(F) \neq \emptyset$, there exists a $n \in \mathbb{N}$ such that $F^n(U) \cap e(V) \neq \emptyset$, where $F^n(U) \cap e(V) \neq \emptyset$ means that $\exists x \in U$ such that $F^n(x) \in e(V)$. A map F is weakly mixing if $F \times F : X \times X \to \mathcal{K}(X) \times \mathcal{K}(X)$ is transitive, that is for any nonempty open sets U_1, U_2 and V_1, V_2 of X with $V_1 \cap R_a(F) \neq \emptyset$ and $V_2 \cap R_a(F) \neq \emptyset$, there exists a $n \in \mathbb{N}$ such that $F^n(U_i) \cap e(V_i) \neq \emptyset, i = 1, 2$. A map F is mixing if for any nonempty open sets U and V of X with $V \cap R_a(F) \neq \emptyset$, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $F^n(U) \cap e(V) \neq \emptyset$.*

Remark 2.2. *By definitions, it is easy to see that mixing implies weakly mixing, weakly mixing imply transitive, and F is (topological) mixing implies F_m is transitive for any $m \in \mathbb{N}$, where F_m is the Cartesian product of m times F.*

For the purposes of the following, we recall some definitions of one-sided symbolic dynamical systems.

Definition 2.7. *The one-sided sequence space $\Sigma_2 := \{\alpha = a_0a_1... \mid a_i = 1 or 2, i \geq 0\}$ is a metric space with the distance $\rho(\alpha, \beta) = \Sigma_{i=0}^{\infty} d(a_i, b_i)/2^i$, where $\alpha = a_0a_1, ..., \beta = b_0b_1, ... \in \Sigma_2, d(a_i, b_i) = 1$ if $a_i \neq b_i$ and $d(a_i, b_i) = 0$ if $a_i = b_i$ for $i \geq 0$. The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is $\sigma(a_0a_1...) = (a_1a_2...)$. It is well know that σ is a continuous map and (Σ_2, σ) is a compact symbolic dynamical system. For $\alpha = (a_0a_1...) \in \Sigma_2$, let $\alpha[i, j] := a_ia_{i+1}...a_j$ be the finite sequence from the $(i + 1)$-th symbol to $(j + 1)$-th symbol of α.*

Lemma 2.1. *The following are equivalent.*

(i) F is transitive.

(ii) For any nonempty open sets U and V of X with $V \cap R_a(F) \neq \emptyset$, there exists a $n \in \mathbb{N}$ such that $F^{-n}(e(V)) \cap U \neq \emptyset$.

(iii) For any nonempty open sets U and V of X with $V \cap R_a(F) \neq \emptyset$ and any $k > 0$, there exists a positive integer $n > k$ such that $F^n(U) \cap e(V) \neq \emptyset$.

Proof. $(iii) \Rightarrow (i)$ is obvious.

$(i) \Rightarrow (ii)$. Since F is transitive, there exist $x_0 \in U$ and $n \in \mathbb{N}$, such that $F^n(x_0) \in e(V)$. Put $A = F^n(x_0)$. Then we have $A \in e(V)$ such that $x_0 \in F^{-n}(e(V))$. So $F^{-n}(e(V)) \cap U \neq \emptyset$.

$(ii) \Rightarrow (iii)$. For any nonempty open sets U and V of X with $V \cap R_a(F) \neq \emptyset$, we have that there exists a $n_0 \in \mathbb{N}$ such that $F^{-n_0}(e(V)) \cap U \neq \emptyset$ by (ii). Since F^n is continuous for any n, $F^{-n_0}(e(V))$ is nonempty open in X, put $V_0 = F^{-n_0}(e(V))$, by the property of $R_a(F)$, we have $V_0 \cap R_a(F) \neq \emptyset$. Again by using (ii) repeatedly, we know that there exists a $n_1 \in \mathbb{N}$ such that $F^{-n_1}(e(V_0)) \cap U \neq \emptyset$, then we easily see that $F^{n_1+n_0}(U) \cap e(V) \neq \emptyset$. Using the same way k times, we have know that there exist positive integer $n_2, n_3, ..., n_k$ such that $F^{n_0+n_1+...+n_k}(U) \cap e(V) \neq \emptyset$. Let $n = n_0 + n_1 + ... + n_k$, so we easily have that $n > k$ and $F^n(U) \cap e(V) \neq \emptyset$. This completes the proof. □

Lemma 2.2. *([19],lemma 2.2) There is an uncountable subset E in Σ_2 such that for any different points $s = s_0s_1..., t = t_0t_1... \in E$, we have $s_n = t_n$ for infinitely many n and $s_m \neq t_m$ for infinitely many m, where Σ_2 denotes the symbolic dynamical system.*

Lemma 2.3. *Supposed $A_1, A_2 \in R_a(F)$ with $A_1 \neq A_2$ and $\{p_k\}_{k=1}^{\infty}$ is a sequence of positive integers. If $\forall C = C_1C_2...$, where $C_k = \overline{B(A_1, \frac{1}{k})}$ or $\overline{B(A_2, \frac{1}{k})}$, there exists $x_c \in X$, such that $\forall k \geq 1$, we have $F^{p_k}(x_c) \in e(C_k)$, then F is uniformly distributively chaotic in a sequence.*

Proof. Take the set E as in lemma 2.2. Then by the assumptions, for any $s = s_0s_1... \in E$, there exists $x_s \in X$, such that for each $k \geq 1$, $n! < k \leq (n+1)!$, we have

$$F^{p_k}(x_s) \in \begin{cases} e(\overline{B(A_1, \frac{1}{k})}), & s_n = 0, \\ e(\overline{B(A_2, \frac{1}{k})}), & s_n = 1. \end{cases}$$

Put $D = \{x_s \mid s \in E\}$. It is easy to see that if $s \neq t$, then $x(s) \neq x(t)$. Since E is uncountable set, D is uncountable set. Let $x(s), y(t) \in D$ and $x(s) \neq y(t)$, where $s = s_0s_1..., t = t_0t_1... \in E$. By Lemma 2.2, there exist sequences of positive integers $n_i \to \infty$ and $m_i \to \infty$ satisfying $s_{n_i} = t_{n_i}, s_{m_i} \neq t_{m_i}$ for infinitely many i. Next we will just prove that $x(s), y(t)$ are uniformly distributively chaotic in a sequence. The whole proof is divided into two steps.

Step 1. For any $\delta > 0$, we take i large enough such that $\frac{1}{n_i} < \frac{\delta}{2}$, and by the property of $F^{p_k}(x)$, for $n_i! < k \leq (n_i+1)!$, we have $d_H(F^{p_k}(x), F^{p_k}(y)) < \delta$. Further, we have

$$\begin{aligned} &\frac{1}{n_i+1}\sharp\{1 \leq k \leq (n_i+1)! \mid d_H(F^{p_k}(x), F^{p_k}(y)) < \delta\} \\ \geq &\frac{(n_i+1)! - n_i!}{(n_i+1)!} = 1 - \frac{1}{n_i+1} \to 1 (i \to \infty). \end{aligned}$$

Therefore we have $\phi^*_{xy}(F, \delta, p_i) = 1$.

Step 2. Put $\varepsilon = \frac{d_H(A_1,A_2)}{2}$. Take i large enough such that $\frac{1}{m_i} < \frac{d_H(A_1,A_2)}{4}$, so $d_H(F^{p_k}(x), F^{p_k}(y)) > \varepsilon$ for $m_i! < k \leq (m_i+1)!$. Thus

$$\begin{aligned} &\frac{1}{m_i+1}\sharp\{1 \leq k \leq (m_i+1)! \mid d_H(F^{p_k}(x), F^{p_k}(y)) < \varepsilon\} \\ \leq &\frac{m_i!}{(m_i+1)!} = \frac{1}{m_i+1} \to 0 (i \to \infty). \end{aligned}$$

Hence $\phi_{xy}(F, t, p_i) = 0$. The entire proof is complete.

□

3. Main results

Theorem 3.1. *If F is mixing, then it must be uniformly distributively chaotic in a sequence.*

Proof. Let $A'_1, A'_2 \in R_a(F)$ with $A'_1 \neq A'_2$ and $U_0 \subset X$ be any nonempty open set such that $\overline{U_0}$ is compact set. Noting that F is mixing, one has that there exists $p_1 > 0$, such that

$$F^{p_1}(U_0) \cap e(\overline{B(A'_1, \frac{1}{k})}) \neq \emptyset \text{ and } F^{p_1}(U_0) \cap e(\overline{B(A'_2, \frac{1}{k})}) \neq \emptyset.$$

So one can find two points x'_1, x'_2 satisfying $F^{p_1}(x'_1) \in e(\overline{B(A'_1, \frac{1}{k})})$,$F^{p_1}(x'_2) \in e(\overline{B(A'_2, \frac{1}{k})})$. Next we will prove the conclusion by induction. Assuming that there exist positive integers $p_1 < p_2 < ... < p_k$ such that for any finite sequence $A_1A_2...A_k$, where $A_i \in \{B(A'_1, \frac{1}{i}), B(A'_2, \frac{1}{i})\}$, there exists $x_k \in U_0$ satisfying $F^{p_i}(x_k) \in e(A_i)$ for $i = 1, 2, ..., k$. We denote the set of all such points by S_k. For any $x \in S_k$, since F^k is continuous for any $k \in \mathbb{N}$, there exists an open nonempty neighborhood $W_x \subset U_0$ such that $F^{p_i}(W_x) \subset e(A_i)$

$(i = 1, 2, ..., k)$. By the Lemma 2.1 and the assumptions that F is mixing, F_m is transitive for any $m \in \mathbb{N}$, then there exists a positive integer p_{k+1} with $p_{k+1} > p_k$, such that

$$F^{p_{k+1}}(W_x) \cap e(\overline{B(A'_1, \frac{1}{k+1})}) \neq \emptyset \text{ and } F^{p_{k+1}}(W_x) \cap e(\overline{B(A'_2, \frac{1}{k+1})}) \neq \emptyset$$

for any $x \in S_k$. So there is a $x_{k+1} \in U_0$ satisfying $F^{p_i}(x_{k+1}) \in e(A_i)$ for $i = 1, 2, ..., k, k+1$. By induction assumption, there is a sequence of positive integers $p_k \to \infty$ such that for each finite sequence $A_1 A_2 ... A_k$, there exists $x_k \in U_0$ satisfying $F^{p_i}(x) \in e(A_i)$ for $i = 1, 2, ..., k$. Let $C = C_1 C_2 ...$ be an infinite sequence, where $C_k = \{e(\overline{B(A_1, \frac{1}{k})}), e(\overline{B(A_2, \frac{1}{k})})\}$. For any k, there exists point $x_k \in \overline{U_0}$ such that $F^{p_i}(x_k) \in e(C_i)$ for $i = 1, 2, ..., k$. Noting that $\overline{U_0}$ is compact, one gets that the sequence $\{x_i\}_{i=1}^{\infty}$ converges to a point x_c in $\overline{U_0}$. It is easily verified that for any $k \in \mathbb{N}$, we have $F^{p_i}(x_c) \in C_k$. Therefore, by Lemma 2.3, F is uniformly distributively chaotic in a sequence.

□

Notice that uniformly distributively chaotic in a sequence implies distributively chaotic in a sequence and Li-Yorke chaos by definitions, so we have the following result by the above theorem.

Corollary 3.1. *If F is mixing, then it must be distributively chaotic in a sequence and Li-Yorke chaos.*

Theorem 3.2. *Mixing implies Kato's chaos.*

Proof. (i) For any $\varepsilon > 0$ and nonempty open sets U, V, W of X with $diam(W) < \varepsilon/2$. Since F is mixing, there exists $n \geq 1$ such that $F^n(U) \cap W \neq \emptyset$ and $F^n(V) \cap W \neq \emptyset$. That is $\exists x \in U, y \in V$ such that $F^n(x) \in e(W)$ and $F^n(y) \in e(W)$. Therefore $d_H(F^n(x), F^n(y)) \leq diam(W) < \varepsilon$.

(ii) Take $x_1, x_2 \in X$ with $x_1 \neq x_2$, Note $r = d(x_1, x_2)$ and $\delta = r/2$. For any nonempty open set U of X. Since F is mixing, there exists $n \geq 1$ such that $F^n(U) \cap e(B(x_1, r/4)) \neq \emptyset$ and $F^n(V) \cap e(B(x_1, r/4)) \neq \emptyset$. That is $\exists y_1 \in U$ such that $F^n(y_1) \in e(B(x_1, r/4))$ and $\exists y_2 \in U$ such that $F^n(y_1) \in e(B(x_2, r/4))$. So we have $d_H(F^n(y_1), x_1) \leq r/4$ and $d_H(F^n(y_2), x_2) \leq r/4$. Therefore, $d_H(F^n(y_1), F^n(y_2)) > r/2 = \delta$. □

Theorem 3.3. *F is Kato's chaos if and only if F^k is Kato's chaos ($k \geq 2$).*

Proof. Sufficiency is obvious. We will prove the necessity in two steps.

Step 1. Since F is Kato's chaos, there exists a $\delta > 0$ such that for any nonempty open set U of X, there exist $x, y \in U, n \geq 1$, such that

$$d_H(F^n(x), F^n(y)) > \delta. \tag{1}$$

Since F is continuous and X is compact metric space, F^j is uniformly continuous ($\forall j = 1, 2, ..., k$). Therefore, for given δ above, there exists $\delta_1 > 0$, such that when $d(u, v) < \delta_1$, we have

$$d_H(F^j(u), F^j(v)) < \delta (j = 1, 2, ..., k). \tag{2}$$

Firstly, we will claim that there exists $n > k$. If not, then take a nonempty open set $U_1 \subseteq U$ with $diam(U_1) < \delta_1$, on the one hand, by (2) we have that $\forall x, y \in U_1, n \leq k$, $d(F^n(x), F^n(x)) < \delta$, on the other hand, there exist $x, y \in U, n \geq 1$, with $d_H(F^n(x), F^n(x)) > \delta$. This is a contradiction. So there exist $x, y \in U, n \geq k$, such that $d_H(F^n(x), F^n(x)) > \delta$. Now, take j satisfying $n = kq + j$, where q, j are positive integer and $1 \leq j \leq k$. Secondly, we will claim that there exists $n_1 \geq 1$, such that $d_H(F^{kn_1}(x), F^{kn_1}(y)) > \delta_1/2$. If not, then for any $n_1 \geq 1$, we have $d_H(f^{kn_1}(x), f^{kn_1}(y)) \leq \delta_1/2$, by (2) we have that $d_H(F^{kn_1+j}(x), F^{kn_1+j}(y)) \leq \delta$, therefore $d(F^n(x), F^n(y)) \leq \delta$, which is a contradiction to (1).

Step 2. For any $\varepsilon > 0$ and nonempty open sets U, V of X. Since $F^{j'}$ is uniformly continuous ($\forall j' = 1, 2, ..., k$) we have that there exists $\delta_2 > 0$, such that when $d(u, v) < \delta_2$, we have

$$d_H(F^{j'}(u), F^{j'}(v)) < \varepsilon(\forall j' = 1, 2, ..., k). \tag{3}$$

Since F is Kato's chaos, for δ_2 above, there exist $x \in U, y \in V, n' \geq 1$ such that $d_H(F^{n'}(x), F^{n'}(y)) < \delta_2$. Take j' satisfying $n' + j' = kq'$, where q', j' are positive integer and $1 \leq j' \leq k$. By (3), $d_H(F^{n'+j'}(x), F^{n'+j'}(y)) < \varepsilon$, that is $d_H(F^{kq'}(x), F^{kq'}(y)) < \varepsilon$. Therefore, for any $\varepsilon > 0$ and nonempty open sets U, V of X, there exist $x \in U, y \in V, q' \geq 1$ such that $d_H(F^{kq'}(x), F^{kq'}(y)) < \varepsilon$.

□

4. **Examples**

Firstly, we provide some examples to show that the definition of mixing (weakly mixing and transitive) of multiple mappings is well.

Example 4.1. *Supposed f_1, f_2 are continuous from $[0, 1]$ to itself. Let $F = \{f_1, f_2\}$, where $f_1(x) \equiv 0, \forall x \in [0, 1]$ and*

$$f_2 = \begin{cases} 2x, & x \in [0, 1/2], \\ 2 - 2x, & x \in (1/2, 1]. \end{cases}$$

Then $R_a(F) = \{\{0, f_2^n(x)\} \mid n \geq 1, x \in [0, 1]\}$. Let U and V be any nonempty open sets of X with $V \cap R_a(F) \neq \emptyset$, As is well-know, f_2 is the tent map, further f_2 is mixing, So it is easily verified that F is mixing.

Remark 4.1. *The Example above can be generalized. That is if $F = \{f_1, f_2\}$, where $f_1(x) \equiv a(a \in X)$, and f_2 is mixing (transitive,respectively) map in the classical sense, then F is mixing (transitive,respectively).*

Example 4.2. *Supposed f_1, f_2 are continuous from $[0, 1]$ to itself. Let $F = \{f_1, f_2\}$, where*

$$f_1(x) = \begin{cases} 2x, & x \in [0, 1/2] \\ 1, & x \in (1/2, 1] \end{cases} \qquad f_2(x) = \begin{cases} 1, & x \in [0, 1/2] \\ 2 - 2x, & x \in (1/2, 1]. \end{cases}$$

And let f be a tent map, that is $f(x) = 2x$ if $x \in [0, 1/2]$ and $f(x) = 2 - 2x$ if $x \in (1/2, 1]$. Then $R_a(F) = \{\{0, 1, f^n(x)\} \mid n > 1, x \in [0, 1]\}$. Let U and V be any nonempty open sets of X with $V \cap R_a(F) \neq \emptyset$, As is well-know, f is the tent map, further f is mixing, So it is easily verified that F is mixing.

Example 4.3. *Supposed f_1, f_2 are continuous from $[0, 1]$ to itself. Let $F = \{f_1, f_2\}$, where*

$$f_1(x) = \begin{cases} 2x, & x \in [0, 1/2] \\ 2 - 2x, & x \in (1/2, 1] \end{cases} \qquad f_2(x) = \begin{cases} 1 - 2x, & x \in [0, 1/2] \\ 2x - 1, & x \in (1/2, 1]. \end{cases}$$

Notice that for any $x \in [0, 1]$ and any $n > 0$, $F^n(x) = \{f_1^n(x), f_2^n(x)\}$ and $f_1^n(x) + f_2^n(x) = 1$. Then $R_a(F) = \{\{f_1^n(x), f_2^n(x)\} \mid n \geq 1, x \in [0, 1]\} = \{\{f_1^n(x), 1 - f_1^n(x)\} \mid n \geq 1, x \in [0, 1]\}$. Let U and V be any nonempty open sets of X with $V \cap R_a(F) \neq \emptyset$, As is well-know, f_1 is the tent map, further f_1 is mixing, So it is easily verified that F is mixing.

Remark 4.2. *The multiple mappings F of Example 1 or Example 3 has been shown to be Hausdorff metric Kato's chaos by definition*[13]*, we can also show that it is Hausdorff metric Kato's chaos by Theorem 3.2. The multiple mappings F of Example 2 has been shown to be Hausdorff metric Li-Yorke chaos by definition*[14]*, we can also show that it is Hausdorff metric Li-Yorke chaos by Corollary 3.1.*

Next we will give an example, which is Hausdorff metric distribution chaos but has only two strongly non-wandering points. We have the conclusion that if F is Hausdorff metric disdributionally chaotic, then there exists at least two strongly nonwandering points of F from [13]. This example indicates that Hausdorff metric distributional chaos may be generated by only two strongly non-wondering points. Firstly, we construct a sequence symbol $\{A_n\}_{n=1}^{\infty}$. Let $A_1 = 10111$. For $n \geq 1$, define $A_{n+1} = A_n O_n A_n I_n A_n$ inductively, where the O_n and I_n have the same length as A_n, and O_n consists only of the symbol 0's while I_n consists only of the symbol 1's. Denote by $|B|$ the length of finite symbol sequence B. Obviously, $|A_n| = 5^n, \forall n \geq 1$. As $n \to \infty$, then A_n enlarge to a one-side infinitely sequence, denote by u. Let X be the ω-limit set of u with the shift map σ, that is $X = \omega(u, \sigma)$, which is a subspace of Σ_2. Let a be the infinitely sequence consists only of the symbol 0's and b be the infinitely sequence consists only of the symbol 1's.

Example 4.4. *The dynamical system $(X, F = \{\sigma, f_0\})$ has only two strongly nonwandering points a and b, but is Hausdorff metric distributional chaos, where σ is a shift map and $f_0 \equiv a$.*

Proof. Firstly, we prove that (X, F) has only two strongly nonwandering points. Obviously, $a, b \in X$ and a, b are strongly nonwandering points. For any infinitely sequence $x \notin \{a, b\}$. We just claim that the x is not strongly nonwandering point of X. Take n large enough such that the finite symbol sequence $x[0, 5^n - 1] \notin \{O_n, I_n\}$. Denote $B_n = x[0, 5^n - 1]$. Take $m = 10n$. Note that for the symbol $A_{m+1} = A_m O_m A_m I_m A_m$, B_n cannot be in O_m and I_m, so we have $\frac{1}{5^{m+1}} \sharp\{0 \leq i \leq 5^{m+1} - 1 \mid A_{m+1}[i, i + 5^n - 1] = B_n\} \leq \frac{4}{5}$, that is the frequency of B_n in A_{m+1} is no more than $\frac{4}{5}$, the frequency denotes by v. Similarly, for the symbol $A_{m+2} = A_{m+1} O_{m+1} A_{m+1} I_{m+1} A_{m+1}$, B_n cannot be in O_{m+1} and I_{m+1}, so $\frac{1}{5^{m+2}} \sharp\{0 \leq i \leq 5^{m+2} - 1 \mid A_{m+1}[i, i + 5^n - 1] = B_n\} \leq \frac{4}{5} v \leq (\frac{4}{5})^2$. By induction, for any $k \geq m + 1$, we have $\frac{1}{5^k} \sharp\{0 \leq i \leq 5^k - 1 \mid A_k[i, i + 5^n - 1] = B_n\} \leq (\frac{4}{5})^{k-m}$. Therefore, $\lim_{k\to\infty} \frac{1}{5^k} \sharp\{0 \leq i \leq 5^k - 1 \mid A_k[i, i + 5^n - 1] = B_n\} = 0$.
By the construction of u, we know that for any $k \geq 1$, u is composed of A_k, O_k and I_k, but B_n can not be in I_k and O_k. So, for the infinite sequence $\{D_r\}_{r=1}^{\infty}$, if each D_r is a finite symbol sequence with length r and is in infinite sequence u, then we have

$$\lim_{r\to\infty} \frac{1}{r} \sharp\{0 \leq i \leq r - 1 \mid D_r[i, i + 5^n - 1] = B_n\} = 0. \tag{4}$$

For any $y \in X$. $D_r = y[0, r - 1]$ denotes finite symbol sequence of y with the length r. Since $y \in \omega(u, \sigma)$, each D_r must be in u. So (4) works. Take the open set $[B_n] = \{z \in X \mid z[0, 5^n - 1] = B_n\}$ of x, then by (4) we have

$$\lim_{r\to\infty} \frac{1}{r} \sharp\{0 \leq i \leq r - 1 \mid \sigma^i(y) \in [B_n]\} = 0. \tag{5}$$

On the other hand, since $f_0 \equiv a \notin [B_n]$, we have

$$\lim_{r\to\infty} \frac{1}{r} \sharp\{0 \leq i \leq r - 1 \mid f_0^i(y) \in [B_n]\} = 0. \tag{6}$$

(5) and (6) imply that $\lim_{r\to\infty} \frac{1}{r} \sharp\{0 \leq i \leq r - 1 \mid F^i(y) \in [B_n]\} = 0$. By the arbitrariness of y, x is not strongly nonwandering point of (X, F).

Secondly, we prove that (X, F) is Hausdorff metric disdributionally chaotic. Let H be the set $\{x = E_1 E_2 ... E_k ...\}$, where $E_k \in \{I_{2^k} A_{2^k}, O_{2^k} A_{2^k} I_{2^k} A_{2^k}\}$, $\forall k \geq 1$. Obviously, H is uncountable set. Denote $s_k(x) = E_1 E_2 ... E_k$. By induction hypotheses, one can easily see that for any $k \geq 1$, the finite symbol sequence $s_k(x)$ happens to be the tail of A_{2^k+1}. So we have $x \in X$, further $H \subset X$. It is obvious that the set H has a similar structure to Σ_2, by Lemma 2.2, there exists an uncountable set $S \subset H$ such that for any different points $x = E_1E_2...E_k...$, $y = F_1F_2...F_k...$ in S, $E_n = F_n$ for infinitely many n and $E_m \neq F_m$ for infinitely many m. Next we will claim that $\{x, y\} \subset S$ must be disdributionally chaotic, hence, (X, F) Hausdorff metric is disdributionally chaotic. For any $k \geq 2$, put $p_k = max\{|s_{k-1}(x)|, |s_{k-1}(y)|\}$, $q_k = 5^{2^k}$. Since both $s_{k-1}(x)$ and $s_{k-1}(y)$ are the tail of $A_{2^{k-1}+1}$, $p_k \leq 5^{2^{k-1}+1}$, further, we have

$$\lim_{k\to\infty} \frac{p_k}{q_k} = 0. \tag{7}$$

Case 1. If $k \geq 2$ satisfying $E_k = F_k$. Without loss of generality assume that $E_k = F_k = I_{2^k}A_{2^k}$. By the construction of x, y, we have that both $x[p_k, q_k] = y[p_k, q_k]$ are the part of A_{2^k}. Take infinitely sequence $\{k_m\}_{m=1}^{\infty}$ with $E_{k_m} = F_{k_m}, \forall m \geq 1$. Notice that (7), we have

$$\lim_{m\to\infty} \frac{1}{q_{k_m}} \sharp\{0 \leq i \leq q_{k_m} - 1 \mid x[i, i+s] = y[i, i+s]\} = 1, \forall s \geq 0.$$

Therefore we have

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \leq i \leq n-1 \mid d(\sigma^i(x), \sigma^i(y)) < t\} = 1, \forall t > 0. \tag{8}$$

Note that $f_0^i(x) = f_0^i(y) = a, \forall i > 0$ and (8), then we have

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \leq i \leq n-1 \mid d_H(F^i(x), F^i(y)) < t\} = 1, \forall t > 0.$$

Case 2. If $k \geq 2$ satisfying $E_k \neq F_k$. Then for $x[p_k, q_k]$ and $y[p_k, q_k]$, one is part of O_{2^k} and the other is part of I_{2^k}. Take infinitely sequence $\{k_m\}_{m=1}^{\infty}$ with $E_{k_m} \neq F_{k_m}, \forall m \geq 1$. By (7), we have $\lim_{m\to\infty} \frac{1}{q_{k_m}} \sharp\{0 \leq i \leq q_{k_m} - 1 \mid d(\sigma^i(x), \sigma^i(y)) = 1\} = 1$. Therefore

$$\liminf_{n\to\infty} \frac{1}{n} \sharp\{0 \leq i \leq n-1 \mid d(\sigma^i(x), \sigma^i(y)) < 1\} = 0. \tag{9}$$

Notice that $f_0^i(x) = f_0^i(y) = a, \forall i > 0$ and (9), we have

$$\liminf_{n\to\infty} \frac{1}{n} \sharp\{0 \leq i \leq n-1 \mid d_H(F^i(x), F^i(y)) < 1\} = 0.$$

That is $\phi_{xy}(F, 1) = 0$. The entire proof is complete.

□

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