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APPROXIMATE CONNES AMENABILITY OF ℓ^1-MUNN ALGEBRAS WITH APPLICATION

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In this paper, we study approximate Connes amenability of ℓ^1-Munn algebras. As an application, we give a necessary and sufficient condition for approximate Connes amenability of Brandt semigroup algebras.

Keywords: Approximate Connes-amenability, ℓ^1-Munn algebras, Brandt semigroup, Banach algebras

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1. Introduction and Preliminaries

The concept of approximate Connes amenability for Banach algebras was introduced and investigated by G. H. Esslamzadeh *et al.* in [5]. This notion was defined by inspiration of two important notions, namely Connes amenability and approximate amenability. For more study on these two notions see [11] and [6], respectively. After that, several authors studied and modified this notion by relaxing some of the restrictions on the definition. For instance, approximate Connes amenability of some Banach algebras associated to (discrete) semigroups and groups and also matrix algebras was studied in [10]. Mahmoodi in [9] introduced a stronger notion of bounded approximate Connes amenability and obtained some hereditary properties of that. Johnson pseudo-Connes amenable Banach algebras was introduced in [13] and its relation to approximate Connes-amenability was studied. H. Samea in [14] investigated approximate amenability of ℓ^1-Munn algebras and established that $\mathcal{L}M_I(A)$ is approximately amenable if and only if A is approximately amenable and the index set I is finite.

Inspired by these results, this paper is devoted to investigate approximate Connes amenability of ℓ^1-Munn algebras $\mathcal{L}M_I(A)$ to extend results in [14]. Finally, we give a complete characterization of approximate Connes amenability of semigroup algebras associated to a Brandt semigroup.

We remind some homological and cohomological properties of Banach algebras. Let A be a Banach algebra. We denote $A\otimes_p A$ for the projective tensor product of A with A which is a

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Banach A-bimodule by the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca, \qquad (a, b, c \in A).$$

The map $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$ for each $a, b \in A$.
Recall that a Banach algebra A is called a dual Banach algebra if there exists a closed submodule A_* of A^* such that $A \cong (A_*)^*$. If A is a dual Banach algebra and X is a dual Banach A-bimodule, an element $x \in X$ is normal if the maps from A into X defined by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^*-w^*-continuous. X is called normal, if any element of X is normal. A dual Banach algebra A is called Connes-amenable if every w^*-w^*-continuous derivation from A into any normal, dual Banach A-bimodule X is inner. Here, a derivation is a linear map $D : A \to X$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$ and a derivation D is called inner whenever there exists $x \in X$ such that $D(a) = ad_x(a) = ax - xa$ for all $a \in A$. A natural genralization of connes amenability was provided by Esslamzadeh *et al.* in [5], namely approximate Connes amenability. A dual Banach algebra A is called approximately Connes amenable precisely when for every normal dual Banach A-bimodule X, every w^*-w^*-continuous derivation from A into X is approximately inner. For a Banach A-bimodule X, we write $\sigma wc(X)$ for the set of all elements $x \in X$ such that the maps $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^*-w-continuous. The set $\sigma wc(X)$ is a closed submodule of X. It is shown in [12, Corollary 4.6] that $\pi_A^*(A_*) \subseteq \sigma wc(A \otimes_p A)^*$. Taking adjoints, we can extend π_A to an A-bimodule homomorphism $\pi_A^{\sigma wc}$ from $\sigma wc((A \otimes_p A)^*)^*$ to A.

2. Approximate Connes-amenability of matrix algebras

In this section, we investigate Connes-amenability and approximate Connes-amenability of ℓ^1-Munn algebras.

For a Banach algebra A, we recall that the multiplier norm on A is defined by

$$\|a\|_{mul} := \max\{\|\lambda_a\|, \|\rho_a\|\},$$

where $\lambda_a : A \longrightarrow A$, given by $x \longrightarrow ax$ is the left multiplication by a and $\rho_a : A \longrightarrow A$, given by $x \longrightarrow xa$ is the right multiplication by a. The following definition is adapted from [1, Definition 2.2] to obtain a criterion for non-approximately amenable Banach algebras.

Definition 2.1. *Let A be a Banach algebra. A separated, unbounded, multiplier-bounded configuration (or SUM configuration for short) in A consists of two sequences $(u_n) \subseteq A$, $(p_n) \subseteq A$ which satisfy the following properties:*
(Separated) $u_n p_n = p_n u_n = u_n$ for all n; and $u_j p_k = p_k u_j = 0$, whenever $j \neq k$.
(Unbounded) $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.
(Multiplier-bounded) The sequences $(\|u_n\|_{mul})$ and $(\|p_n\|_{mul})$ are bounded.

Theorem 2.1. *Let A be a dual Banach algebra and let A contains a SUM configuration. Then A is not approximately Connes-amenable.*

Proof. Let A be approximately Connes-amenable. Then $A^\sharp$ is approximately Connes-amenable [5, Proposition 2.3]. We set $u = e \otimes e$. Because of the normality of the dual module $\sigma wc((A^\sharp \otimes_p A^\sharp)^*)^*$, $ad_u : A^\sharp \to A^\sharp \otimes_p A^\sharp$ is a w^*-w^*-continuous derivation with range in $\ker \pi_{A^\sharp}^{\sigma wc}$. Since $A^\sharp$ is approximately Connes-amenable, there is a net $(e_\alpha) \subseteq \ker \pi_{A^\sharp}^{\sigma wc}$ such that $ad_u(a) = \lim_\alpha ad_{e_\alpha}(a)$, for every $a \in A^\sharp$. We set $M_\alpha = u - e_\alpha$. So we have

$$a \cdot M_\alpha - M_\alpha \cdot a = (a \cdot u - u \cdot a) - (a \cdot e_\alpha - e_\alpha \cdot a) \longrightarrow 0,$$

and

$$\pi_A^{\sigma wc}(M_\alpha) = \pi_A^{\sigma wc}(u) = e,$$

for all $a \in A^\sharp$. Since $(M_\alpha) \subseteq (\sigma wc(A^\sharp \otimes_p A^\sharp)^*)^*$, by the similar argument in [8, Theorem 2.7], we can write

$$M_\alpha = M_\alpha^{**} - F_\alpha \otimes e - e \otimes G_\alpha + c_\alpha e \otimes e,$$

where $(M_\alpha^{**}) \subseteq \sigma wc((A \otimes_p A)^*)^*$ and $(F_\alpha), (G_\alpha) \subseteq \sigma wc(A^*)^*$ and $(c_\alpha) \subseteq \mathbb{C}$. Applying $\pi_A^{\sigma wc}$, we have

$$\pi_A^{\sigma wc}(M_\alpha^{**}) - F_\alpha - G_\alpha + c_\alpha e = e.$$

Therefore $c_\alpha = 1$, $\pi_A^{\sigma wc}(M_\alpha^{**}) - F_\alpha - G_\alpha = 0$ and for every $a \in A$, we have

$$a \cdot M_\alpha^{**} - a \cdot F_\alpha \otimes e - a \otimes G_\alpha + a \otimes e - M_\alpha^{**} \cdot a + F_\alpha \otimes a + e \otimes G_\alpha \cdot a - e \otimes a \longrightarrow 0.$$

So

$$a \cdot M_\alpha^{**} - M_\alpha^{**} \cdot a + F_\alpha \otimes a - a \otimes G_\alpha \longrightarrow 0,$$

and

$$a \cdot F_\alpha \longrightarrow a, \quad G_\alpha \cdot a \longrightarrow a.$$

By the Goldstine's Theorem, Since A is w^*-dense in $\sigma wc(A^*)^*$ and $A \otimes_p A$ is w^*-dense in $\sigma wc((A \otimes_p A)^*)^*$, there is a net $(\vartheta_\lambda)_{\lambda \in \Lambda} \subseteq A \otimes_p A$ and there are nets $(H_\gamma)_{\gamma \in \Gamma} \subseteq A$ and $(L_\gamma)_{\gamma \in \Gamma} \subseteq A$ with

$$\lim_\lambda \lim_\gamma (a \cdot \vartheta_\lambda - \vartheta_\lambda \cdot a + H_\gamma \otimes a - a \otimes L_\gamma) = 0, \tag{1}$$

and

$$\lim_\gamma a \cdot H_\gamma = a, \quad \lim_\gamma L_\gamma \cdot a = a, \tag{2}$$

convergence in the weak topology of $A \otimes_p A$ and A, respectively.

Suppose that $Z = \Lambda \times \Gamma^\Lambda$ is directed by the product ordering. Then for any $\theta = (\lambda, (\gamma_{\lambda'})_{\lambda' \in \Lambda}) \in Z$, the equations (1) and (2) imply that

$$\lim_\theta (a \cdot \vartheta_\theta - \vartheta_\theta \cdot a + H_\theta \otimes a - a \otimes L_\theta) = 0,$$

and

$$\lim_\theta a \cdot H_\theta = a, \quad \lim_\theta L_\theta \cdot a = a,$$

in the weak topology, by the iterated limit theorem [7, P. 69].

On the other hand, by Mazur's Theorem we can obtain a net $(\mu_\beta) \subseteq A \otimes_p A$ and $(H'_\beta) \subseteq A$ and $(L'_\beta) \subseteq A$ such that

$$a \cdot \mu_\beta - \mu_\beta \cdot a + H'_\beta \otimes a - a \otimes L'_\beta \longrightarrow 0,$$

and

$$a \cdot H'_\beta \longrightarrow a, \quad L'_\beta \cdot a \longrightarrow a,$$

in the norm topology. Following the similar argument as in [1, Theorem 2.5], we can conclude that A is not approximately Connes-amenable [5, Proposition 2.3]. □

Definition 2.2. *Suppose that A is a Banach algebra and I is an index set. Then we denote by $\mathcal{LM}_I(A)$ the vector space of all $I \times I$-matrices $E = (E_{ij})_{i,j\in I}$ over A such that $\|E\|_1 = \sum_{i,j\in I}\|E_{ij}\| < \infty$. Certainly, $\mathcal{LM}_I(A)$ with the matrix multiplication as its product and the norm $\|\cdot\|_1$ is a Banach algebra, called ℓ^1-Munn algebra over A with identity as sandwich matrix. For more information on these algebras see* [3].

In the sequel, for each $i, j \in I$, the symbol $x\delta_{ij} = (e_{kl})$ describes the element of $\mathcal{LM}_I(A)$ with $e_{kl} = 0$ if $(k,l) \neq (i,j)$ and $e_{ij} = x$.

Proposition 2.1. *Suppose that A is a unital dual Banach algebra. If $\mathcal{LM}_I(A)$ is approximately Connes-amenable then I is finite and A is approximately Connes-amenable.*

Proof. Let I be infinite. Then there exists an infinite countable number of mutually disjoint non-empty subsets I_n of I with $|I_n| \to \infty$. Suppose that p_n is an $I \times I$ matrix with entries given by

$$(p_n)_{ij} = \begin{cases} e_A & \text{if } i = j \in I_n, \\ 0 & \text{otherwise,} \end{cases}$$

where e_A is the identity of A. Then (p_n) is a sequence of mutually orthogonal idempotents with $\|p_n\|_1 = |I_n| \longrightarrow \infty$ and $\|p_n\|_{mul} = 1$, where $\|p_n\|_{mul} = \max(\|\lambda_{p_n}\|, \|\rho_{p_n}\|)$ such that $\lambda_{p_n} : \mathcal{LM}_I(A) \longrightarrow \mathcal{LM}_I(A)$, $x \longrightarrow p_n x$ is the left multiplication by p_n and $\rho_{p_n} : \mathcal{LM}_I(A) \longrightarrow \mathcal{LM}_I(A)$, $x \longrightarrow x p_n$ is the right multiplication by p_n. So, putting $u_n = p_n$ implies $\mathcal{LM}_I(A)$ has a SUM configuration. Hence $\mathcal{LM}_I(A)$ is not approximate Connes-amenable by Theorem 2.1, which is construction. Therefore I is finite. Suppose that E is a normal dual Banach A-bimodule and $D : A \longrightarrow E$ is a weak*-continuous derivation. We define the weak*-continuous derivation $\tilde{D} : \mathcal{LM}_I(A) \longrightarrow \mathcal{LM}_I(E)$ by $\tilde{D}((a_{i,j})) = (D(a_{j,i}))$ [2]. Since $\mathcal{LM}_I(A)$ is approximately Connes-amenable, there is a net $(x_\alpha) = ((x_{i,j})_\alpha) \subseteq \mathcal{LM}_I(E)$ such that $\tilde{D}(a) = \lim a \cdot x_\alpha - x_\alpha \cdot a$ for every $a \in \mathcal{LM}_I(A)$. Then we identify a with the matrix $a\delta_{11}$. So $x_{1,1} \in E$ and

$$D(a) = \tilde{D}(a\delta_{11})\delta_{1,1} = (a\delta_{11} \cdot x - x \cdot a\delta_{11})\delta_{1,1} = a \cdot x_{1,1} - x_{1,1} \cdot a.$$

Hence D is approximately inner and so A is approximately Connes-amenable. □

Applying the above result we give a complete characterization of approximate Connes-amenability of Brandt semigroup algebras. For a non-empty set I and a group G, let $(g)_{jj'}$ represents the $I \times I$-matrix with $g \in G$ is in the (j, j') position and zero in the other positions. We define the set

$$\mathcal{M}^0(G, I) = \{(g)_{jj'} : g \in G, j, j' \in I\} \cup \{0\}.$$

$\mathcal{M}^0(G, I)$ with the product rule defined by

$$(g)_{ii'}(h)_{jj'} = \begin{cases} (gh)_{ij'} & \text{if } j = i' \\ 0 & \text{if } j \neq i' \end{cases} \qquad (g, h \in G,\ i, j, i', j' \in I),$$

forms an inverse semigroup and is known as the Brandt semigroup over G and I. In this case, $\ell^1(S) \cong \mathcal{LM}_I(\ell^1(G)) \bigoplus \mathbb{C}$ and so $\ell^1(S)$ is a dual Banach algebra.

Corollary 2.1. *Let G be a group, I be a non-empty set and let $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup over G with index set I. Then $\ell^1(S)$ is approximately Connes-amenable if and only if I is finite and G is an amenable group.*

Proof. Assume that $\ell^1(S)$ is approximately Connes-amenable. Using the isomorphism $\ell^1(S) \cong \mathcal{LM}_I(\ell^1(G)) \bigoplus \mathbb{C}$ and [5, Proposition 2.3], it implies that $\mathcal{LM}_I(\ell^1(G))$ is approximately Connes-amenable. Now, by Proposition 2.1 $\ell^1(G)$ is approximately Connes-amenable and I is finite. Then G is amenable by [5, Theorem 5.2]. Conversely, since G is amenable then so is $\ell^1(G)$ and therefore $\ell^1(G)$ is Connes-amenable. Since I is finite then by [4, Theorem 3.1] $\mathcal{LM}_I(\ell^1(G))$ is Connes-amenable. Thus applying [5, Proposition 2.3], $\ell^1(S)$ is approximately Connes-amenable. □

Example 2.1. *Let $G = \{e\}$ be the trivial group and $I = \{1, 2, \dots, n\}$ be a finite set. Since every finite group is amenable, so Corollary 2.1 implies that $\ell^1(\mathcal{M}^0(G, I)) \cong M_{n\times n}(\mathbb{C})$, the algebra of $n \times n$ matrices with entries in $\mathbb{C}$, is approximately Connes-amenable.*

Example 2.2. *Let $G = \{e\}$ be the trivial group and $I = \mathbb{N}$ be the set of natural numbers. Since I is infinite, Corollary 2.1 implies that $\ell^1(\mathcal{M}^0(G, I)) \cong M_{\mathbb{N}}(\mathbb{C})$, the infinite dimensional matrix algebra, is not approximately Connes-amenable.*

Example 2.3. *Let $G = \mathbb{F}_2$ be the free group on two generators and $I = \{1, 2\}$. Since $\mathbb{F}_2$ is not amenable, Corollary 2.1 implies that the matrix algebra $M_{2\times 2}(\ell^1(\mathbb{F}_2))$ fails to be approximately Connes-amenable.*

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4. Conclusions

In this paper, we studied approximate Connes amenability of ℓ^1-Munn algebras and as an application we show that for a Brandt semigroup $S = \mathcal{M}^0(G, I)$, the semigroup algebra $\ell^1(S)$ is approximately Connes amenable if and only if G is an amenable group and I is a finite set.

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