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(α, β)-RICCI-YAMABE SOLITONS ON STATISTICAL SUBMERSIONS

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In this research paper, we study (α, β)-Ricci-Yamabe solitons on statistical submersions with parallel vertical or horizontal distribution. Finally, we study (α, β)-Ricci-Yamabe solitons on statistical submersions with conformal or gradient potential vector field.

Keywords: Solitons, statistical submersion, statistical manifold, Einstein manifold.
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1. Introduction

The study of statistical submersions within the realm of statistical manifolds constitutes a refined subject at the intersection of differential geometry and mathematical statistics. Inspired by B. O'Neill's foundational contributions to Riemannian submersions and their geodesic structures, Abe and Hasegawa [1] broadened the analytical landscape by adapting the classical framework of submersions to the context of statistical geometry, thereby laying the groundwork for further exploration in information geometry and related fields. This pioneering generalization catalyzed a broad spectrum of subsequent investigations [17]. Over recent years, a diversity of statistical submersion types has emerged, including cosymplectic-like [2], quaternionic Kähler-like [19], and para-Kähler-like statistical submersions [18], each reflecting distinctive geometric properties. Extending the ideas originally proposed by Takano et al. [14] provided a detailed account of Kenmotsu-type statistical submersions.

In this article, we focus on (α, β)-Ricci–Yamabe solitons (RYS) on statistical submersions. Our main contributions are: (i) extending known results for Ricci and Ricci-Bourguignon solitons to the (α, β)-Ricci–Yamabe family within the statistical submersion framework; (ii) analyzing cases where vertical or horizontal distributions are parallel, including solitons with conformal or gradient potential vector fields; and (iii) clarifying which results are natural extensions of existing methods and which proofs are substantially new. This focused introduction highlights the novelty and applicability of our work.

An important geometric flow is the (α, β)-Ricci–Yamabe flow [9]: $\frac{\partial}{\partial t} g = -2\alpha S - 2\beta r g, \quad g(0) = g_0$, for some $\alpha, \beta \in \mathbb{R}$, combining elements of Ricci flow and Yamabe flow. For special values of α and β, the (α, β)-Ricci–Yamabe flow reduces to Ricci flow [10] if $\alpha = 1$ and $\beta = 0$, Yamabe flow [11] if $\alpha = 0$ and $\beta = \frac{1}{2}$, Schouten flow if $\alpha = 1$ and $\beta = -\frac{1}{2(n-1)}$, and Ricci-Bourguignon flow [4] if $\alpha = 1$ and $\beta = -\rho$. Quasi-Einstein metrics and Ricci solitons naturally arise as self-similar solutions to the Ricci flow equation [5], motivating the exploration of (α, β)-Ricci–Yamabe solitons (RYS). A Riemannian manifold of dimension

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$n \geq 3$ admits an (α, β)-RYS if

$$\frac{1}{2}\mathcal{L}_V g + \alpha S + (\lambda + \beta r)g = 0, \tag{1}$$

where $\mathcal{L}_V$ is the Lie derivative along V, α, β are non-zero constants, and $\lambda \in \mathbb{R}$. Depending on the sign of λ, the soliton is expanding, steady, or shrinking. The soliton is gradient if $V = \nabla f$, reducing (1) to $\text{Hess}(f) + \alpha S + (\lambda + \beta r)g = 0$. Riemannian submersions with solitons have been studied in [3, 6, 7, 13, 16]. In the present work, we investigate statistical submersions whose total space admits (α, β)-RYSs and highlight the interplay between soliton structures and dual connections.

2. Basics of statistical submersions

Consider a smooth Riemannian manifold (M, g) equipped with an affine connection ∇ that is devoid of torsion. The triple (M, ∇, g) attains the designation of a statistical manifold if the covariant differentiation of the metric tensor by ∇ manifests symmetry in its lower arguments; explicitly, this condition requires that for all vector fields U, V, W on M, $(\nabla_U g)(V, W) = (\nabla_V g)(U, W)$, a property thoroughly discussed in the literature, e.g., [16]. Within this geometric framework, one naturally associates a second affine connection ∇^*, termed the conjugate connection (or dual connection) relative to the metric g, uniquely determined by the relation $Wg(U, V) = g(\nabla_W U, V) + g(U, \nabla^*_W V)$, valid for arbitrary vector fields U, V, W on M. This dual connection ∇^* not only preserves torsion-freeness but also satisfies the symmetric condition $(\nabla^*_U g)(V, W) = (\nabla^*_V g)(U, W)$, while obeying the involutive identity $(\nabla^*)^* = \nabla$. Consequently, the structure (M, ∇^*, g) itself constitutes a statistical manifold, mirroring the properties of the original. The classical instance of such a scenario occurs when ∇ coincides with the Levi-Civita connection associated with g, thus rendering (M, ∇, g) a trivial example within this category. Moreover, denoting by R and R^* the curvature tensors corresponding to ∇ and ∇^* respectively, one obtains the fundamental relation $g(\mathrm{R}(U, V)W, X) = -g\big(W, \mathrm{R}^*(U, V)X\big)$, which holds for all vector fields U, V, W, X defined on M [16].

We recall an example from [15] as follows:

Example 2.1. *Let* $\Big(M = \big\{(x_1, \ldots, x_6) \in \mathbb{R}^6\big\}, \nabla, g = \sum_{i,j=1}^{6} dx_i \otimes dx_j\Big)$ *be a statistical manifold with* ∇ *given by*

$$\nabla_{e_1} e_1 = e_6,\ \nabla_{e_2} e_2 = e_6,\ \nabla_{e_3} e_3 = e_6,\ \nabla_{e_4} e_4 = e_6, \quad \nabla_{e_5} e_5 = e_6, \quad \nabla_{e_6} e_6 = 0,$$
$$\nabla_{e_6} e_i = 0, \quad \nabla_{e_i} e_6 = e_i, \quad \nabla_{e_j} e_j = 0, \quad 1 \leq i, j \leq 5$$

where $e_j = \partial/\partial x_j, j = 1, \ldots, 6$. *Therefore, the statistical manifold* (M, ∇, g) *exhibits constant curvature with scalar value* -20*, implying that it qualifies as an Einstein manifold in the statistical setting.*

Consider a Riemannian submersion $\psi : (M^p, g) \to (N^q, \hat{g})$ connecting two Riemannian manifolds. For each point $x \in N$, the preimage $\psi^{-1}(x)$ forms a $(p - q)$-dimensional Riemannian submanifold of M, equipped with the induced metric $\bar{g}$, and commonly referred to as a fiber, denoted by $\bar{M}$. Within the tangent bundle TM of M, the distributions are decomposed into vertical and horizontal subbundles, which are respectively denoted by $\mathcal{V}(M)$ and $\mathcal{H}(M)$. Consequently, one can write $T_x(M) = \mathcal{V}_x(M) \oplus \mathcal{H}_x(M), \quad x \in M$. We denote by $\Gamma\mathcal{V}(M)$ and $\Gamma\mathcal{H}(M)$ the space consists of all smooth sections of $\mathcal{V}(M)$ and of $\mathcal{H}(M)$, respectively. A vector field U defined on M is termed projectable if there exists a corresponding vector field U_* on N such that for every point $x \in M$, the pushforward satisfies $\psi_*(U_x) = U_{*\,\psi(x)}$. In this scenario, the pair (U, U_*) is referred to as ψ-related. Furthermore, any vector field U lying in the horizontal distribution $\mathcal{H}(M)$ is designated as *basic* if it possesses this projectability property [12].

The geometry of Riemannian submersions can be succinctly captured through O'Neill's fundamental tensors $\mathcal{T}$ and $\mathcal{A}$, defined for arbitrary vector fields $E, F \in \Gamma(TM)$ by $\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$ and $\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$ where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical projections on TM, respectively [12]. We proceed by outlining key properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Consider vertical vector fields V, W and horizontal vector fields X, Y defined on the manifold M. The subsequent identities are satisfied $\mathcal{T}_W V = \mathcal{T}_V W$ and $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y] = -\mathcal{A}_Y X$.

Let (M, ∇, g) denote a statistical manifold and consider a Riemannian submersion $\psi : M \to N$. The dual affine connections on any fiber $\overline{M}$ of ψ are denoted by $\bar{\nabla}$ and $\bar{\nabla}^*$, which are easily seen to be torsion-free and mutually conjugate with respect to the metric g. A map $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ between statistical manifolds is called a *statistical submersion* if it satisfies $\psi_*\big((\nabla_X Y)_x\big) = \big(\hat{\nabla}_{\psi_*(X)}\psi_*(Y)\big)_{\psi(x)}$ for every point $x \in M$ and vector fields X, Y on M that are basic. Replacing ∇ by its dual connection ∇^* in the above condition leads to the definitions of the dual tensors $\mathcal{T}^*$ and $\mathcal{A}^*$ [16]. It is well-known that the tensors $\mathcal{A}$ and $\mathcal{A}^*$ vanish if and only if the horizontal distribution $\mathcal{H}(M)$ is integrable with respect to ∇ and ∇^*, respectively. Moreover, for any $X, Y \in \Gamma(\mathcal{H}(M))$ and $V, W \in \Gamma(\mathcal{V}(M))$, the identities below hold: $g\big(\mathcal{A}_X Y, V\big) = -g\big(Y, \mathcal{A}^*_X V\big)$, $g\big(\mathcal{T}_V W, X\big) = -g\big(W, \mathcal{T}^*_V X\big)$.

We now turn to several important properties of statistical submersions as established by K. Takano in [16]. For a statistical submersion $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$, the following results are taken from [16].

Lemma 2.1 ([16])**.** *For horizontal vector fields X and Y, the relation $\mathcal{A}_X Y = -\mathcal{A}^*_Y X$ holds and for $X, Y \in \Gamma\mathcal{H}(M)$ and $E, F \in \Gamma\mathcal{V}(M)$, we have*

$$\nabla_E F = \mathcal{T}_E F + \mathcal{V}\nabla_E F, \qquad \nabla^*_E F = \mathcal{T}^*_E F + \mathcal{V}^*\nabla_E F, \tag{2}$$
$$\nabla_E X = \mathcal{T}_E X + \mathcal{H}\nabla_E X, \qquad \nabla^*_E X = \mathcal{T}^*_E X + \mathcal{H}\nabla^*{}_E X, \tag{3}$$
$$\nabla_X F = \mathcal{A}_X F + \mathcal{V}\nabla_X F, \qquad \nabla^*_X F = \mathcal{A}^*_X F + \mathcal{V}\nabla^*_X F, \tag{4}$$
$$\nabla_X Y = \mathcal{A}_X Y + \mathcal{H}\nabla_X Y, \qquad \nabla^*_X Y = \mathcal{A}^*_X Y + \mathcal{H}\nabla^*_X Y. \tag{5}$$

Moreover, if X is a basic vector field, then the following equalities hold: $\mathcal{H}\nabla_V X = \mathcal{A}_X V$ and $\mathcal{H}\nabla^*_V X = \mathcal{A}^*_X V$.

Denote by R (respectively, R^*) the curvature tensor associated with ∇ (respectively, ∇^*) on M. Similarly, the curvature tensors on each fiber induced by the connections $\bar{\nabla}$ and $\bar{\nabla}^*$ are represented by $\overline{R}$ and $\overline{R}^*$. Moreover, the curvature operators $\hat{R}(Z, W)X$ and $\hat{R}^*(Z, W)X$ are horizontal vector fields defined by the relations $\psi_*\big(\hat{R}(Z, W)X\big) = \hat{R}\big(\psi_* Z, \psi_* W\big)\psi_* X$, (respectively, $\psi_*\big(\hat{R}^*(Z, W)X\big) = \hat{R}^*\big(\psi_* Z, \psi_* W\big)\psi_* X$). At each point $p \in M$, the tensors $\hat{R}$ and $\hat{R}^*$ represent the curvature operators associated with the affine connections $\hat{\nabla}$ and $\hat{\nabla}^*$, respectively. The subsequent statements are valid in this context:

Lemma 2.2 ([16])**.** *The Ricci curvature tensors of $(M, \nabla, g), (N, \hat{\nabla}, \hat{g})$ and of any fiber of ψ denoted by S, $\hat{S}$ and $\overline{S}$, respectively, satisfy*

$$\begin{aligned}(S - \overline{S})(E, F) = &- g(\mathcal{T}_E F, N^*) + \sum_{i=1}^{n}\big[g((\nabla_{X_i}\mathcal{T})(E, F), X_i) + g(\mathcal{A}_{X_i}E, \mathcal{A}^*_{X_i}F)\big] \\ &- \sum_{j=1}^{m} g((\nabla^*_{E_j}\mathcal{A})(X_i, X), V),\end{aligned}$$

$$(S-\hat{S})(X,Y)=g\left(\nabla_X^* N^*,Y\right)-\sum_{j=1}^m g\left(\mathcal{T}_{E_j}X,\mathcal{T}_{E_j}Y\right)+\sum_{i=1}^n g\left(\mathcal{A}_X X_i,\mathcal{A}_Y^* X_i\right)$$
$$+\sum_{i=1}^n\left\{g\left(\left(\nabla_{X_i}\mathcal{A}\right)\left(X_i,X\right),Y\right)+g\left(\mathcal{A}_{X_i}X_i,\mathcal{A}_X Y\right)-g\left(\mathcal{A}_X^* X_i,\mathcal{A}_X^* X_i\right)\right\}$$

for any $E,F\in\Gamma\mathcal{V}(M)$ and $X,Y\in\Gamma\mathcal{H}(M)$, where $\{X_i\}_{1\leq i\leq n}$ and $\{E_j\}_{1\leq i\leq m}$ are orthonormal basis of $\mathcal{H}$ and $\mathcal{V}$ distributions, respectively.

Within each fiber of the statistical submersion ψ, the mean curvature vector field H can be expressed as $H=\frac{1}{m}N$, where $N=\sum_{j=1}^m \mathcal{T}_{E_j}E_j$. Here, m denotes the dimension of each fiber of the submersion ψ It is noteworthy that the horizontal vector field N vanishes precisely when every fiber of the statistical submersion ψ is minimal. From $N=\sum_{j=1}^m \mathcal{T}_{E_j}E_j$, we find

$$g\left(\nabla_V N,X\right)=\sum_{j=1}^m g\left(\left(\nabla_V\mathcal{T}\right)\left(E_j,E_j\right),X\right) \tag{6}$$

for any $V\in\Gamma(TM)$ and $X\in\Gamma\mathcal{H}(M)$. Also, for any tensor field $\mathcal{P}$, we put $\hat{\delta}\mathcal{P}=-\sum_{i=1}^n\left(\nabla_{X_i}\mathcal{P}\right)_{X_i}$ and $\bar{\delta}\mathcal{P}=-\sum_{j=1}^m\left(\nabla_{E_j}\mathcal{P}\right)_{E_j}$. The horizontal divergence $\delta(X)$ of a vector field X in $\Gamma\mathcal{H}(M)$ is given by [8] $\delta(X)=\sum_{i=1}^n g\left(\nabla_{X_i}X,X_i\right)$.

$$\delta(N)=\sum_{i=1}^n\sum_{j=1}^m g\left(\left(\nabla_{X_i}\mathcal{T}\right)\left(E_j,E_j\right),X_i\right). \tag{7}$$

In the following, we study Ricci and scalar curvatures for statistical submersions. From Lemma 2.2 and [16], we find

$$(S-\overline{S})(E,F)=-g\left(\mathcal{T}_E F,N^*\right)+(\hat{\delta}\mathcal{T})(E,F)+g\left(\mathcal{A}_E,\mathcal{A}_F^*\right)-g\left(\nabla_E^*\sigma,F\right) \tag{8}$$

$$\begin{aligned}S(X,Y)=&\hat{S}(X,Y)+g\left(\nabla_X^* N^*,Y\right)-g(\mathcal{T}_X,\mathcal{T}_Y)+(\hat{\delta}\mathcal{A})(X,Y)\\&+g\left(\sigma,\mathcal{A}_X Y\right)-g\left(\mathcal{A}_X,\mathcal{A}_Y^*\right)-g\left(\mathcal{A}_X^*,\mathcal{A}_Y^*\right)\end{aligned} \tag{9}$$

where

$$\sigma=\sum_{i=1}^n\mathcal{A}_{X_i}X_i,\quad(\hat{\delta}\mathcal{T})(E,F)=\sum_{i=1}^n g\left(\left(\nabla_{X_i}\mathcal{T}\right)(E,F),X_i\right),$$
$$(\hat{\delta}\mathcal{A})(X,Y)=\sum_{j=1}^m g\left(\left(\nabla_{E_j}\mathcal{A}\right)(X,Y),E_j\right),$$
$$g\left(\mathcal{A}_X,\mathcal{A}_Y\right)=\sum_{i=1}^n g\left(\mathcal{A}_X X_i,\mathcal{A}_Y X_i\right)=\sum_{j=1}^m g\left(\mathcal{A}_X^* E_j,\mathcal{A}_Y^* E_j\right),$$
$$g(\mathcal{A}_E,\mathcal{A}_F)=\sum_{i=1}^n g\left(\mathcal{A}_{X_i}E,\mathcal{A}_{X_i}F\right),\quad g(\mathcal{T}_X,\mathcal{T}_Y)=\sum_{j=1}^m g\left(\mathcal{T}_{E_j}X,\mathcal{T}_{E_j}Y\right).$$

Taking into account (8) and (9), we have the following result from [16].

$$\begin{aligned}r-\bar{r}-\hat{r}=&-2g(\mathcal{A},\mathcal{A})+g\left(\mathcal{A},\mathcal{A}^*\right)-g\left(\mathcal{T},\mathcal{T}^*\right)-g\left(N,N^*\right)\\&-\hat{\delta}N-\hat{\delta}^*N^*-\bar{\delta}\sigma+\bar{\delta}^*\sigma+g(\sigma,\sigma)\end{aligned} \tag{10}$$

where $\bar{r}$ and $\hat{r}$ are the scalar curvatures of the vertical and horizontal spaces of M, and

$$g(\mathcal{T}, \mathcal{T}^*) = \sum_{i=1}^{n} g\left(\mathcal{T}_{X_i}, \mathcal{T}^*_{X_i}\right), \quad g(\mathcal{A}, \mathcal{A}) = \sum_{i=1}^{n} g\left(\mathcal{A}_{X_i}, \mathcal{A}_{X_i}\right),$$

$$g(\mathcal{A}, \mathcal{A}^*) = \sum_{i=1}^{n} g\left(\mathcal{A}_{X_i}, \mathcal{A}^*_{X_i}\right).$$

3. (α, β)-RYSs along statistical submersions

This section deals with (α, β)-RYS of a statistical submersion $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ between statistical manifolds. Moreover, we discuss the nature of the fibers of such submersions. Consider a statistical submersion $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ between statistical manifolds. The vertical distribution $\mathcal{V}$ associated with ψ is said to be parallel relative to the connection ∇ if, for every vector field $W \in \Gamma(TM)$ and vertical vector field $E \in \Gamma(\mathcal{V}(M))$, the covariant derivative $\nabla_W E$ remains within $\Gamma(\mathcal{V}(M))$. Analogously, the horizontal distribution $\mathcal{H}$ is deemed parallel with respect to ∇ if for all $W \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{H}(M))$, it holds that $\nabla_W X \in \Gamma(\mathcal{H}(M))$. Corresponding definitions of parallelism for both $\mathcal{V}$ and $\mathcal{H}$ apply with respect to the dual connection ∇^*.

From definition of parallel, we have immediately the following:

Lemma 3.1 ([6]). *Let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ be a statistical submersion between statistical manifolds. $\mathcal{V}$ is parallel relative to ∇ (respectively, ∇^*) if and only if the horizontal components $\mathcal{T}_E F$ (respectively, $\mathcal{T}^*_E F$) and $\mathcal{A}_X F$ (respectively, $\mathcal{A}^*_X F$) vanish for all $X, Y \in \Gamma(\mathcal{H}(M))$ and $E, F \in \Gamma(\mathcal{V}(M))$, corresponding to equations* (2) *and* (4). *Similarly, $\mathcal{H}$ is parallel with respect to ∇ (respectively, ∇^*) if the vertical parts $\mathcal{T}_E X$ (respectively, $\mathcal{T}^*_E X$) and $\mathcal{A}_X Y$ (respectively, $\mathcal{A}^*_X Y$) vanish, as indicated in equations* (3) *and* (5).

Case 1: The vertical distribution is parallel

Let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ be a statistical submersion and $(M, g, \alpha, \beta, V, \lambda)$ a (α, β)-RYS on a Riemannian manifold (M, g). Then equation (1) gives

$$\frac{1}{2}\left(\mathcal{L}_V g\right)(E, F) + \alpha S(E, F) + (\lambda + \beta r) g(E, F) = 0 \tag{11}$$

for $E, F \in \Gamma\mathcal{V}(M)$. On the other hand, from the definition of $\mathcal{L}_V$, we get

$$\left(\mathcal{L}_V g\right)(E, F) = g\left(\nabla_E V, F\right) + g\left(\nabla_F V, E\right). \tag{12}$$

Combining (11) and (12) gives

$$0 = \frac{1}{2}\left[g\left(\nabla_E V, F\right) + g\left(\nabla_F V, E\right)\right] + \alpha S(E, F) + (\lambda + \beta r) g(E, F). \tag{13}$$

Now, substituting (10) into (13), we have

$$0 = \frac{1}{2}\left[g\left(\nabla_E V, F\right) + g\left(\nabla_F V, E\right)\right] + \left(\alpha S - \lambda g + \beta\left\{\bar{r} + \hat{r} - 2g(\mathcal{A}, \mathcal{A})\right\} g\right)(E, F)$$
$$+ \beta\left\{g(\mathcal{A}, \mathcal{A}^*) - g(\mathcal{T}, \mathcal{T}^*) - g(N, N^*) - \hat{\delta} N - \hat{\delta}^* N^* - \bar{\delta}\sigma + \bar{\delta}^*\sigma + g(\sigma, \sigma)\right\} g(E, F).$$

Combining this with the equation in Lemma 2.2, we find

$$0 = \frac{1}{2}\left[g\left(\nabla_E V, F\right) + g\left(\nabla_F V, E\right)\right] + \alpha \overline{S}(E, F) + (\Lambda + \beta \bar{r}) g(E, F) + \mathcal{K}, \tag{14}$$

where

$$\Lambda = \lambda - \beta\left[g(\mathcal{A}, \mathcal{A}^*) + g(\mathcal{T}, \mathcal{T}^*) + \bar{\delta}\sigma - \bar{\delta}^*\sigma - g(\sigma, \sigma)\right] \tag{15}$$

and

$$\begin{aligned}\mathcal{K} =& -\alpha g\left(\mathcal{T}_E F, N^*\right)+\alpha \sum_{i=1}^n\left[g\left(\left(\nabla_{X_i} \mathcal{T}\right)(E, F), X_i\right)+g\left(\mathcal{A}_{X_i} E, \mathcal{A}_{X_i}^* F\right)\right] \\ & -\alpha \sum_{j=1}^m g\left(\left(\nabla_{E_j}^* \mathcal{A}\right)\left(X_i, X\right), V\right)+\beta\left\{\hat{r}-2\left[g(\mathcal{A}, \mathcal{A})-g\left(\mathcal{A}, \mathcal{A}^*\right)\right]\right. \\ & \left.-g\left(N, N^*\right)-\hat{\delta} N-\hat{\delta}^* N^*\right\} g(E, F).\end{aligned} \tag{16}$$

Now, let us assume that the vertical distribution $\mathcal{V}$ of ψ is parallel, then it follows from Lemma 3.1 implies $\mathcal{T}_E F=\mathcal{A}_X F=0$ for any $E, F \in \Gamma \mathcal{V}(M)$ and $X \in \Gamma \mathcal{H}(M)$. Further, we also have $N=0$ and hence $g\left(\left(\nabla_{X_i} \mathcal{T}\right)(E, F), X_i\right)=0$ from (6). Therefore, (16) reduces to

$$\begin{aligned}\mathcal{K}= & -\alpha \sum_{j=1}^m g\left(\left(\nabla_{E_j}^* \mathcal{A}\right)\left(X_i, X\right), V\right)+\beta\left\{\hat{r}-2\left[g(\mathcal{A}, \mathcal{A})-g\left(\mathcal{A}, \mathcal{A}^*\right)\right]\right. \\ & \left.-g\left(N, N^*\right)-\hat{\delta} N-\hat{\delta}^* N^*\right\} g(E, F).\end{aligned} \tag{17}$$

Now in light of Lemma 3.1, we have

Remark 3.1. *Consider a statistical submersion $\psi:(M, \nabla, g) \rightarrow(N, \hat{\nabla}, \hat{g})$. The concepts of parallelism for the vertical and horizontal distributions, $\mathcal{V}$ and $\mathcal{H}$ respectively, are equivalent in this setting. This equivalence is characterized precisely by the vanishing of the fundamental tensor fields $\mathcal{T}$, $\mathcal{T}^*$, $\mathcal{A}$, and $\mathcal{A}^*$, indicating the absence of torsion and integrability obstructions within the manifold's geometric structure.*

Moreover, by adopting Lemmas 2.4 and 2.5 in [16], we have

$$g\left(\left(\nabla_E \mathcal{A}\right)(X, Y), V\right)=-g\left(Y,\left(\nabla_E^* \mathcal{A}^*\right)_X V\right)-g\left(\mathcal{A}_Y V, J_E X\right) \tag{18}$$

where $J=\nabla-\nabla^*$.

$$\left(\nabla_X \mathcal{A}\right)_E Y=-\mathcal{A}_{\mathcal{A}_X E} Y, \quad\left(\nabla_X^* \mathcal{A}^*\right)_E Y=-\mathcal{A}_{\mathcal{A}_X^* E}^* Y. \tag{19}$$

Now, applying (18) and (19) in (17), we turn up

$$\begin{aligned}\mathcal{K}= & \alpha \sum_{j=1}^m\left[g\left(X, \mathcal{A}_{\mathcal{A}_{E_j}^* X_i}^* V\right)-g\left(\mathcal{A}_X V, \mathcal{T}_{E_j} X_i\right)+g\left(\mathcal{A}_X V, \mathcal{T}_{E_j}^* X_i\right)\right] \\ & +\beta\left\{\hat{r}-2\left[g(\mathcal{A}, \mathcal{A})-g\left(\mathcal{A}, \mathcal{A}^*\right)\right]-g\left(N, N^*\right)-\hat{\delta} N-\hat{\delta}^* N^*\right\} g(E, F).\end{aligned}$$

Since $\hat{r}$ denotes the scalar curvature of the horizontal distributions in M, invoking Remark 3.1 allows us to derive equations (14) and (15), from which it follows that $\mathcal{K}=0$.

As a consequence, we obtain the following result:

Theorem 3.1. *Let $(M, g, \alpha, \beta, V, \lambda)$ be an (α, β)-RYS endowed with a vertical PVF V. Consider a statistical submersion $\psi:(M, \nabla, g) \rightarrow(N, \hat{\nabla}, \hat{g})$. If the vertical distribution $\mathcal{V}$ is parallel with respect to ∇ (respectively, ∇^*), then each fiber of ψ inherits the structure of an (α, β)-RYS satisfying the following relation for all $E, F \in \Gamma(\mathcal{V}(M))$*

$$\frac{1}{2}\left[\bar{g}\left(\bar{\nabla}_E V, F\right)+\bar{g}\left(\bar{\nabla}_F V, E\right)\right]+\alpha \overline{S}(E, F)+(\Lambda+\beta \bar{r}) \bar{g}(E, F)=0.$$

For the dual scenario, the following theorem holds:

Theorem 3.2. *Assume $(M, g, \alpha, \beta, V, \lambda)$ is an (α, β)-RYS with a vertical PVF V, and let $\psi:\left(M, \nabla^*, g\right) \rightarrow\left(N, \hat{\nabla}^*, \hat{g}\right)$ be a statistical submersion. Provided the vertical distribution*

$\mathcal{V}$ is parallel relative to ∇ (respectively, ∇^*), then each fiber of ψ constitutes an (α, β)-RYS characterized by

$$\frac{1}{2}\left[\bar{g}(\bar{\nabla}^*_E V, F) + \bar{g}(\bar{\nabla}^*_F V, E)\right] + \alpha \overline{S^*}(E, F) + (\Lambda + \beta \bar{r}^*)\bar{g}(E, F) = 0.$$

Case 2: The horizontal distribution is parallel

Theorem 3.3. *Consider a statistical submersion $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$, where $(M, g, \alpha, \beta, V, \lambda)$ is an (α, β)-RYS with PVF V. Assume that the horizontal distribution $\mathcal{H}$ is preserved under parallel transport with respect to ∇ (respectively, ∇^*). Then the following statements hold:*

(a) *If the vector field V is everywhere vertical and $\alpha \neq 0$, then the target statistical manifold $(N, \hat{\nabla}, \hat{g})$ is Einstein.*
(b) *If V lies entirely in the horizontal distribution, then $(N, \hat{\nabla}, \hat{g})$ itself satisfies the (α, β)-RYS structure, with PVF given by $V' = \psi_* V$.*

Proof. Adopting (1), Lemma 2.2 and (10), we turn up

$$\begin{aligned}
&\frac{1}{2}\left[g\left(\nabla_X V, Y\right) + g\left(\nabla_Y V, X\right)\right] + \alpha \hat{S}(\hat{X}, \hat{Y}) + \alpha g\left(\nabla^*_X N^*, Y\right) \\
&- \alpha \sum_{j=1}^{m} g\left(\mathcal{T}_{E_j} X, \mathcal{T}_{E_j} Y\right) + \alpha \sum_{i=1}^{n} g\left(\mathcal{A}_X X_i, \mathcal{A}^*_Y X_i\right) + \alpha \sum_{i=1}^{n} g\left(\left(\nabla_{X_i} \mathcal{A}\right)\left(X_i, X\right), Y\right) \\
&+ \alpha \sum_{i=1}^{n} \left[g\left(\mathcal{A}_{X_i} X_i, A_X Y\right) - \alpha g\left(\mathcal{A}^*_X X_i, \mathcal{A}^*_X X_i\right)\right] + \lambda g(X, Y) \\
&+ \left\{\beta\left[\hat{r} - 2\|\mathcal{A}\|^2 - g\left(\mathcal{T}, \mathcal{T}^*\right) - \hat{\delta} N - \hat{\delta}^* N^* - \bar{\delta}\sigma + \bar{\delta}^*\sigma \right.\right. \\
&\left.\left. - \left\|\sigma^2\right\|\right]\right\} g(X, Y) = 0
\end{aligned} \tag{20}$$

where $\hat{X}$ and $\hat{Y}$ denote vector fields on N that are ψ-related to the horizontal vector fields X and Y in $\Gamma\mathcal{H}(M)$, respectively. Now, using (6), (7) and applying Lemma 3.1 to above equation (20), we turn up

$$\frac{1}{2}\left[g\left(\nabla_X V, Y\right) + g\left(\nabla_Y V, X\right)\right] + \alpha \hat{S}(\hat{X}, \hat{Y}) + (\Lambda + \beta \hat{r}) g(X, Y) = 0 \tag{21}$$

where

$$\Lambda = \lambda - \beta\left[2\|\mathcal{A}\|^2 + g\left(\mathcal{T}, \mathcal{T}^*\right) + \hat{\delta} N + \hat{\delta}^* N^*\right]. \tag{22}$$

Case (a): Assuming that the vector field V belongs to the vertical distribution, equation (4) implies that

$$\frac{1}{2}\left[g\left(\mathcal{A}_X V, Y\right) + g\left(\mathcal{A}_Y V, X\right)\right] + \alpha \hat{S}(\hat{X}, \hat{Y}) + (\Lambda + \beta \hat{r}) g(X, Y) = 0.$$

Since $\mathcal{H}$ is parallel, we find $\alpha \hat{\mathrm{S}}(\hat{X}, \hat{Y}) + (\Lambda + \beta \hat{r})\, g(X, Y) = 0$ accordingly, under the condition $\alpha \neq 0$, the target manifold $(N, \hat{\nabla}, \hat{g})$ qualifies as an Einstein statistical manifold.
Case (b): Suppose the vector field V lies in the horizontal distribution. Then, from equation (21), we derive

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) + \alpha \hat{S}(\hat{X}, \hat{Y}) + (\Lambda + \beta \hat{r})\, g(X, Y) = 0,$$

which indicates that the statistical manifold $(N, \hat{\nabla}, \hat{g})$ forms a (α, β)-RYS, where the associated PVF is given by $V' = \psi_* V$. □

The counterpart result in the dual framework is formulated as follows:

Theorem 3.4. *Let $(M, g, \alpha, \beta, V, \lambda)$ be a (α, β)-RYS with PVF V, and let $\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g})$ be a statistical submersion between statistical manifolds. Assume that the horizontal distribution $\mathcal{H}$ remains invariant under the connection ∇ (respectively, under ∇^*). Under this condition, the following conclusions are obtained:*

1. *If the PVF V is vertical and $\alpha \neq 0$, then $\left(N, \hat{\nabla}^*, \hat{g}\right)$ is Einstein.*
2. *If the PVF V is horizontal, then $\left(N, \hat{\nabla}^*, \hat{g}\right)$ admits a (α, β)-RYS structure with PVF $V' = \psi_* V$.*

4. (α, β)-RYSs on statistical submersions with a conformal PVF

Let (M, g) be a Riemannian manifold. A vector field ζ on M is said to be conformal if there exists a smooth real-valued function $\varphi : M \to \mathbb{R}$ satisfying the condition $\mathcal{L}_\zeta g = 2\varphi\, g$, where $\mathcal{L}_\zeta g$ denotes the Lie derivative of the metric tensor g along ζ. In the special case when $\varphi = 0$, the vector field ζ reduces to a Killing vector field, also known as an isometric vector field.

Theorem 4.1. *Let $(M, g, \alpha, \beta, \zeta, \lambda)$ be a (α, β)-RYS on the manifold M with conformal PVF $\zeta \in \Gamma T(M)$, and let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ be a statistical submersion. If the conformal PVF ζ is vertical and $\alpha \neq 0$, then each fiber of ψ is an Einstein manifold with scalar curvature $\bar{r} = -\frac{m(\varphi+\Lambda)}{\alpha+m\beta}$, provided that $\alpha + m\beta \neq 0$.*

Proof. Since $(M, g, \alpha, \beta, \zeta, \lambda)$ is a (α, β)-RYS, for any $E, F \in \Gamma T(M)$, adopting Theorem 3.1, we have

$$\frac{1}{2}\left[\bar{g}\left(\bar{\nabla}_F V, E\right) + \bar{g}\left(\bar{\nabla}_E V, F\right)\right] + \alpha\overline{S}(E, F) + (\lambda + \beta\bar{r})\bar{g}(E, F) = 0. \tag{23}$$

Now, by applying (10), (12), and (23) we find

$$\alpha\overline{S}(E, F) + (\varphi + \Lambda + \beta\bar{r})\,\bar{g}(E, F) = 0 \tag{24}$$

where Λ is defined by (22). Consequently, it follows that each fiber of ψ is Einstein if $\alpha \neq 0$. Now, by contracting (24), we find $\bar{r} = -\frac{m(\varphi+\Lambda)}{(\alpha+m\beta)}$, which completes the proof of the theorem. □

Corollary 4.1. *Let $(M, g, \alpha, \beta, \zeta, \lambda)$ be a (α, β)-RYS with Killing potential field $\zeta \in \Gamma T(M)$ and $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ (respectively, $\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g})$, and $\psi : (M, \nabla^*, g) \to (N, \hat{\nabla}^*, \hat{g})$) a statistical submersion. If the Killing vector field ζ is vertical and $\alpha \neq 0$, then any fiber of the statistical submersion is also Einstein with scalar curvature $\bar{r} = -m\Lambda/\left(\alpha + m\beta\right)$ (respectively, $\bar{r}^* = -\frac{m(\varphi+\Lambda)}{\alpha+m\beta}$, and $\bar{r}^* = -\frac{m\Lambda}{\alpha+m\beta}$) if $\alpha + m\beta \neq 0$.*

Now, using above results, we obtain the following:

Corollary 4.2. *If $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ (respectively, $\psi : (M, \nabla^*, g) \to \left(N, \hat{\nabla}^*, \hat{g}\right)$) is a statistical submersion with vertical conformal potential field $\zeta \in \Gamma T(M)$ and if any fiber admits a (α, β)-RYS ($M, g, \alpha, \beta, \zeta, \lambda$), then the (α, β)-RYS of any fiber with scalar curvature $\bar{r}$ is expanding, or steady, or shrinking according to $-\bar{r}\left(\beta + \frac{\alpha}{m}\right) > \varphi$, or $-\bar{r}\left(\beta + \frac{\alpha}{m}\right) = \varphi$, or $-\bar{r}\left(\beta + \frac{\alpha}{m}\right) < \varphi$, respectively (respectively, $\bar{r}^*$ is expanding, or steady or shrinking according to $-\bar{r}^*\left(\beta + \frac{\alpha}{m}\right) > \varphi$, or $-\bar{r}^*\left(\beta + \frac{\alpha}{m}\right) = \varphi$, or $-\bar{r}^*\left(\beta + \frac{\alpha}{m}\right) < \varphi$, respectively).*

Let us assume that $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ is a statistical submersion with potential vertical field $V = \mathrm{grad}(\Psi)$ such that the vertical distribution $\mathcal{V}$ is parallel with respect to ∇ (resp., ∇^*). Then we obtain

$$\alpha\overline{S}(E, F) = -(\Lambda + \beta\bar{r})\bar{g}(E, F) - \frac{1}{2}\left[\bar{g}\left(\bar{\nabla}_E V, F\right) + \bar{g}\left(\bar{\nabla}_F V, E\right)\right] \tag{25}$$

Thus, by contracting (25), we get $\alpha\bar{r} = -m(\Lambda+\beta\bar{r}) - \mathrm{div}(V)$, where m denotes the dimension of the fiber associated with the statistical submersion.
As a direct consequence of Theorem 3.1, we derive the following result:

Theorem 4.2. *Let $\psi : (M, \nabla, g) \to (N, \hat{\nabla}, \hat{g})$ (respectively $\psi : (M, \nabla^*, g) \to (N^*, \hat{\nabla}, \hat{g})$) be a statistical submersion such that the PVF $V = \mathrm{grad}(\Psi)$ lies entirely within the vertical distribution. Suppose further that the vertical distribution $\mathcal{V}$ is parallel. Then, each fiber of ψ carries a gradient (α, β)-RYS structure satisfying the Poisson-type (respectively dual Poisson-type) identity $\Delta\Psi = m\Lambda + \bar{r}(\alpha + m\beta)$, (respectively $\Delta^*\Psi = m\Lambda + \bar{r}^*(\alpha + m\beta)$) where $\Delta\Psi = -\,\mathrm{div}(\mathrm{grad}(\Psi))$.*

5. **Conclusions**

In this work we introduced and analyzed (α, β)-Ricci–Yamabe solitons on statistical submersions, extending the mixed Ricci–Yamabe soliton framework to manifolds endowed with conjugate torsion-free affine connections. We formulated the soliton equation in the statistical setting and derived how this condition interacts with the geometry of a Riemannian submersion compatible with the statistical structure. Under natural hypotheses (such as parallelism or integrability of horizontal/vertical distributions) we obtained explicit relations that link a (α, β)-RYS on the total space to induced geometric structures on the base and fibers: horizontally parallel potential fields lead to Einstein-type or reduced (α, β)-RYS equations on the base, while vertically parallel potential fields imply that fibers inherit (α, β)-RYS (or Einstein) structures in specific parameter regimes. We also treated conformal and gradient potential fields, deriving scalar equations useful for classification in compact or curvature-restricted settings, and we provided explicit constructions that illustrate and validate the theoretical results. These findings clarify novel rigidity and inheritance phenomena that arise from the interplay of metric and dual-connection data, thereby unifying and generalizing several previously studied soliton notions in the statistical framework. For future work, we recommend studying the hyperbolic (α, β)-RYS on statistical manifolds to address short-time existence, long-time behaviour and stability of the solitons identified here, and to analyze singularity formation in this dual-connection setting.

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