

A NEW STRONG CONVERGENT TWO-STEP INERTIAL ALGORITHM FOR SPLIT COMMON FIXED POINT PROBLEMS

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The purpose of this work is to investigate a new approach for finding a solution to the split common fixed point problem for the class of demicontractive operators. One advantage of the proposed algorithm is that the strong convergence result of the proposed algorithm is proven without calculating norms of the bounded linear operators. Our results extend and improve results studied in the literature.

Keywords: Split common fixed point; demicontractive operator; two-step inertial; self-adaptive algorithm.

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1. Introduction

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 , respectively. The split feasibility problem (SFP) is formulated as finding a point x satisfying the property

$$x \in C \text{ such that } Ax \in Q, \quad (1)$$

where $A: H_1 \rightarrow H_2$ is a bounded linear operator. Recently, the SFP has been widely studied by many authors (see [1, 4, 10, 19, 22, 23, 24]), due to its application in signal processing [5]. In particular, Byrne [4] introduced the so-called CQ algorithm. For $x_0 \in H_1$, define $\{x_n\}$ as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad (2)$$

where $0 < \gamma < \frac{2}{\rho(A^*A)}$ and where P_C denotes the projector onto C and $\rho(A^*A)$ is the spectral radius of the operator A^*A . It is known that the CQ algorithm converges weakly to a solution of the SFP if such a solution exists.

In the case where both C and Q consist of fixed point sets of some nonlinear operators, the SFP is known as the split common fixed point problem (SCFP). More specifically, the

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SCFP is to find

$$x \in \text{Fix}(S) \text{ such that } Ax \in \text{Fix}(T),$$

where $\text{Fix}(S)$ and $\text{Fix}(T)$ are the fixed point sets of $S: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$, respectively. We denote the solution set of the SCFP by

$$\Omega := \{x \in H_1 : x \in \text{Fix}(S) \text{ and } Ax \in \text{Fix}(T)\}. \quad (3)$$

In the case S and T are directed operators, Censor and Segal [6] proposed and proved the convergence of the following algorithm in the setting of finite dimensional spaces:

$$x_{n+1} = S(I - \gamma A^*(I - T)A)x_n.$$

It is important to know that the class of directed operators recovers the metric projection. So, the algorithm of Censor and Segal extends Byrne's CQ algorithm.

After that, Moudafi [12] proposed the algorithm for solving the SCFP (3) when S and T are demicontractive mappings with constants $k_1 \in [0, 1)$, $k_2 \in [0, 1)$ respectively, as follows:

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \end{cases} \quad (4)$$

where $\alpha_n \in (\delta, 1 - k_1 - \delta)$ for a small enough $\delta > 0$ and $\gamma \in \left(0, \frac{1 - k_2}{\rho}\right)$ with ρ being the spectral radius of A^*A . It is essential to know that, to solve the SCFP (3) for demicontractive operators, the algorithm (4) must calculate the norm of the linear mapping A . This creates a barrier in practice. To overcome this barrier, some authors have introduced algorithms with variable step sizes.

In [7], Cui and Wang proposed the following algorithm for solving the SCFP (3). The algorithm is of the form:

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \lambda)u_n + \lambda S u_n, \end{cases} \quad (5)$$

where $\lambda \in (0, 1 - k_1)$ and

$$\tau_n = \begin{cases} \frac{(1 - k_2)\|(I - T)Ax_n\|^2}{\|A^*(I - T)Ax_n\|^2}, & \text{if } Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

The weak convergence result of the sequence generated by (5) was proved for demicontractive operators provided that $\Omega \neq \emptyset$. Some other results using the self-adaptive step sizes for solving the SCFP (3) with demicontractive operators have been studied; the reader refers to [20, 21, 25].

To obtain strong convergence results for solving SCFP (3), in [3], based on the algorithm of Cui and Wang, Boikanyo introduced the Halpern-type algorithm for demicontractive

operators with step sizes τ_n chosen as in (6) as follows:

$$\begin{cases} x_0, u \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ y_n = (1 - \lambda)u_n + \lambda Su_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n. \end{cases}$$

In optimization theory, to speed up the convergence rate, Polyak [14] firstly proposed the so-called heavy ball method for solving smooth convex minimization problem. After that, Nesterov [13] introduced a modified heavy ball method as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = y_n - \lambda_n \nabla f(y_n), \end{cases} \quad (7)$$

where $\theta_n \in [0, 1)$ is an extrapolation factor and λ_n is a step-size parameter (sufficiently small) and ∇f is the gradient of a smooth convex function f . Let us recall that the term $\theta_n(x_n - x_{n-1})$ in (7) is known as the inertial step and it can be regarded as a procedure of speeding up the convergence properties (see, e.g., [2, 14]).

To improve the performance of the CQ algorithm (2) for solving SFP (1), Dang et al. [9] recently applied the inertial technique to the CQ algorithm. They proposed the following algorithm:

$$\begin{cases} x_0, x_1 \in H_1, \\ q_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_C(q_n - \tau_n A^*(I - P_Q)Aq_n), \end{cases} \quad (8)$$

where $0 \leq \theta_n < \theta < 1$ and $0 < \tau_n < \frac{2}{\|A\|^2}$. They proved that the inertial CQ algorithm (8) converges weakly to a solution of the SFP provided that

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < +\infty.$$

In [8], Cui *et al.* proposed the inertial Censor-Segal algorithm with the self-adaptive step-sizes that can improve the performance of the original algorithm. The algorithm is of the form:

$$\begin{cases} x_0, x_1 \in H_1, \\ q_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(q_n - \tau_n A^*(I - T)Aq_n), \end{cases} \quad (9)$$

where $0 \leq \theta_n < \theta < 1$, $0 < a \leq \rho_n \leq b < 2$ and

$$\tau_n = \begin{cases} \rho_n \frac{\|(I - T)Ax_n\|^2}{\|A^*(I - T)Aq_n\|^2}, & \text{if } \|A^*(I - T)Aq_n\| \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

with $S: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ firmly quasi-nonexpansive mappings such that $I - S$ and $I - T$ are demiclosed at 0. They proved that if the sequence $\{\theta_n\}$ is nondecreasing and converges to $\theta \in [0, \sqrt{5} - 2)$, then the sequence $\{x_n\}$ generated by algorithm (9) weakly converges to a solution of SCFP (3).

Recently, to obtain strong convergence results, Suparatulatorn *et al.* [18] introduced self-adaptive algorithms with inertial effects for solving the SCFP (3) when S and T are demicontractive mappings with constants $k_1 \in [0, 1)$, $k_2 \in [0, 1)$ respectively:

$$\begin{cases} x_0, x_1 \in H_1, \\ q_n = (1 - \beta_n)[x_n + \theta_n(x_n - x_{n-1})], \\ y_n = q_n - \tau_n A^*(I - T)Aq_n, \\ x_{n+1} = (1 - \lambda)y_n + \lambda S y_n, \end{cases} \quad (10)$$

$$\tau_n = \begin{cases} (1 - k_2)\rho \frac{\|(I - T)Ax_n\|^2}{\|q_n - Sq_n\|^2 + \|A^*(I - T)Aq_n\|^2}, & \text{if } \|A^*(I - T)Aq_n\| \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leq \theta_n < \theta < 1$, $\rho \in (0, 1)$, $\lambda \in (0, 1 - k_1)$, $\{\beta_n\} \subset (0, 1)$, $\lim_{n \rightarrow +\infty} \beta_n = 0$, $\sum_{n=1}^{+\infty} \beta_n = +\infty$. They proved that the sequence generated by (10) converges strongly to $x^* = P_\Omega(0)$, provided that $\lim_{n \rightarrow +\infty} \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| = 0$.

Recently, the two-step inertial technique has been introduced by Liang in [11], Poon and Liang [15, 16] for minimization problems. Two-step inertial algorithms incorporate two inertial parameters, resulting in improved convergence and stability. The basic idea is to update iterates using a combination of current and previous step information, formalized as:

$$t_n = v_n + \theta(v_n - v_{n-1}) + \beta(v_{n-1} - v_{n-2}),$$

where $\theta > 0$ and $\beta < 0$ are inertial parameters that introduce momentum from earlier iterates.

Following the above research going in this direction, we propose a new scheme for solving the SCFP (3) for demicontractive mappings. We further investigate the two-step inertial technique to construct an algorithm with self-adaptive step sizes and without calculating the norm of the bounded linear mapping A , and provide strong convergence results for solving SCFP (3) so that it can improve the convergence rate of the algorithm.

This paper is organized as follows. In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with analyzing the convergence of the proposed algorithms. Finally, in Section 4, conclusions are given.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H . The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \rightarrow \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \rightarrow x$ as $n \rightarrow \infty$. We denote $\omega_w(x_n)$ the set of weak cluster points of a sequence $\{x_n\}$. For each $x, y \in H$, we have the following:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (11)$$

Definition 2.1. Assume that $T: H \rightarrow H$ is a nonlinear operator with $\text{Fix}(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H , the following implication holds:

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \rightarrow 0 \implies x \in \text{Fix}(T).$$

Definition 2.2. Let $T: H \rightarrow H$ be an operator with $\text{Fix}(T) \neq \emptyset$. Then

- $T: H \rightarrow H$ is called directed if

$$\langle z - Tx, x - Tx \rangle \leq 0, \quad \forall z \in \text{Fix}(T), x \in H,$$

or equivalently

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2, \quad \forall z \in \text{Fix}(T), x \in H;$$

- $T: H \rightarrow H$ is called quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall z \in \text{Fix}(T), x \in H;$$

- $T: H \rightarrow H$ is called β -demicontractive with $0 \leq \beta < 1$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|(I - T)x\|^2, \quad \forall z \in \text{Fix}(T), x \in H,$$

or equivalently

$$\langle Tx - x, x - z \rangle \leq \frac{\beta - 1}{2} \|x - Tx\|^2, \quad \forall z \in \text{Fix}(T), x \in H, \quad (12)$$

or equivalently

$$\langle Tx - z, x - z \rangle \leq \|x - z\|^2 + \frac{\beta - 1}{2} \|x - Tx\|^2, \quad \forall z \in \text{Fix}(T), x \in H.$$

Lemma 2.1. Let $U: H \rightarrow H$ be a β -demicontractive operator, with $\text{Fix}(U) \neq \emptyset$, and set $U_\lambda = (1 - \lambda)I + \lambda U$, with $\lambda \in (0, 1 - \beta)$. Then,

$$\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2, \quad \forall x \in H, z \in \text{Fix}(U).$$

Proof. By (12), we have

$$\begin{aligned} \|U_\lambda x - z\|^2 &= \|(1 - \lambda)x + \lambda Ux - z\|^2 \\ &= \|(x - z) + \lambda(Ux - x)\|^2 \\ &= \|x - z\|^2 + 2\lambda\langle x - z, Ux - x \rangle + \lambda^2\|Ux - x\|^2 \\ &\leq \|x - z\|^2 + \lambda(\beta - 1)\|Ux - x\|^2 + \lambda^2\|Ux - x\|^2 \\ &= \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2, \end{aligned}$$

hence the proof. \square

Lemma 2.2 ([17]). Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ holds for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Strong convergence results

Our strong convergence theorem is established under the following conditions:

Condition 3.1. *The solution set $\Omega \neq \emptyset$.*

Condition 3.2. *$S: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ are two demicontractive operators with constants $k_1 \in [0, 1)$ and $k_2 \in [0, 1)$, respectively, such that $I - S$ and $I - T$ are demiclosed at zero.*

Condition 3.3. *$A: H_1 \rightarrow H_2$ is a bounded linear operator with its adjoint operator A^* .*

Now, we introduce the proposed algorithm. This algorithm allows the approximation of a solution to problem (3) that has a minimum norm. The algorithm is of the form:

Algorithm 3.1.

Initialization: *Let $\xi_1 > 0$, $\xi_2 > 0$, $\alpha \in (0, 1]$, $0 < \gamma < 1$, $\lambda \in (0, 1 - k_1)$ and $x_{-1}, x_0, x_1 \in H_1$ be arbitrary. We assume that sequences $\{\beta_n\} \subset \left(0, \frac{1}{1 + \alpha}\right)$, $\{\epsilon_{1,n}\}$, $\{\epsilon_{2,n}\}$ are all positive sequences such that:*

$$\lim_{n \rightarrow \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = +\infty, \quad \lim_{n \rightarrow \infty} \frac{\epsilon_{1,n}}{\beta_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{\epsilon_{2,n}}{\beta_n} = 0.$$

Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the current iterates x_{n-1} and x_n ($n \geq 1$), compute*

$$\begin{cases} t_n = x_n + \xi_{1,n}(x_n - x_{n-1}) + \xi_{2,n}(x_{n-1} - x_{n-2}), \\ v_n = \beta_n(1 - \alpha)x_n + (1 - \beta_n)t_n, \\ z_n = v_n - \tau_n A^*(I - T)Av_n, \end{cases}$$

where

$$\tau_n = (1 - k_2)\gamma \frac{\|(I - T)Av_n\|^2}{\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2},$$

and

$$\xi_{1,n} = \begin{cases} \min\{\xi_1, \frac{\epsilon_{1,n}}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \xi_1, & \text{otherwise,} \end{cases} \quad (13)$$

and

$$\xi_{2,n} = \begin{cases} \min\{\xi_2, \frac{\epsilon_{2,n}}{\|x_{n-1} - x_{n-2}\|}\}, & \text{if } x_{n-1} \neq x_{n-2}, \\ \xi_2, & \text{otherwise.} \end{cases} \quad (14)$$

If $\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2 = 0$, then stop and v_n is a solution to problem (3). Otherwise, go to **Step 2**.

Step 2. *Compute*

$$x_{n+1} = (1 - \lambda)z_n + \lambda Sz_n.$$

Let $n := n + 1$ and return to **Step 1**.

Remark 3.1. 1. From (13), (14) and $\lim_{n \rightarrow \infty} \frac{\epsilon_{1,n}}{\beta_n} = 0$, $\lim_{n \rightarrow \infty} \frac{\epsilon_{2,n}}{\beta_n} = 0$, it is easy to

show that $\lim_{n \rightarrow +\infty} \frac{\xi_{1,n}}{\beta_n} \|x_n - x_{n-1}\| = 0$ and $\lim_{n \rightarrow +\infty} \frac{\xi_{2,n}}{\beta_n} \|x_{n-1} - x_{n-2}\| = 0$.

2. It is easy to see that if $\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2 = 0$, then $v_n \in \Omega$. Indeed, we have $Sv_n = v_n$ and $A^*(I-T)Av_n = 0$. Thus $v_n \in \text{Fix}(S)$. On the other hand, by $\Omega \neq \emptyset$, there exists a point $z \in \text{Fix}(S)$ such that $Az \in \text{Fix}(T)$. Since the operator T is k_2 -demicontractive, we have

$$\begin{aligned} \frac{1-k_2}{2} \|(I-T)Av_n\|^2 &\leq \langle (I-T)Av_n, Av_n - Az \rangle \\ &= \langle A^*(I-T)Av_n, v_n - z \rangle = 0. \end{aligned}$$

This implies that $(I-T)Av_n = 0$. Thus $Av_n = T(Av_n)$. Therefore $v_n \in \Omega$, as asserted.

Theorem 3.1. *Assume that the Conditions 3.1 - 3.3 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $x^* \in \Omega$, where $x^* = P_\Omega(0)$.*

Proof. The proof of the theorem is divided into some steps as follows:

Step 1.

$$\|x_{n+1} - x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|v_n - x^*\|^2. \quad (15)$$

Indeed, let $S_\lambda := (1-\lambda)I + \lambda S$. We have $x_{n+1} = S_\lambda z_n$. By Lemma 2.1 we have

$$\|x_{n+1} - x^*\|^2 = \|S_\lambda z_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \lambda(1-\lambda-k_1)\|z_n - Sz_n\|^2. \quad (16)$$

We also obtain

$$\|x_{n+1} - x^*\|^2 \leq \|z_n - x^*\|^2. \quad (17)$$

Now, we prove that

$$\|z_n - x^*\|^2 \leq \|v_n - x^*\|^2 - (1-k_2)^2 \gamma (1-\gamma) \frac{\|(I-T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2}. \quad (18)$$

Indeed, using inequality (12) we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|v_n - \tau_n A^*(I-T)Av_n - x^*\|^2 \\ &= \|v_n - x^*\|^2 + \tau_n^2 \|A^*(I-T)Av_n\|^2 - 2\tau_n \langle A^*(I-T)Av_n, v_n - x^* \rangle \\ &= \|v_n - x^*\|^2 + \tau_n^2 \|A^*(I-T)Av_n\|^2 - 2\tau_n \langle (I-T)Av_n, Av_n - Ax^* \rangle \\ &\leq \|v_n - x^*\|^2 + \tau_n^2 \|A^*(I-T)Av_n\|^2 - (1-k_2)\tau_n \|(I-T)Av_n\|^2 \\ &= \|v_n - x^*\|^2 + (1-k_2)^2 \gamma^2 \frac{\|(I-T)Av_n\|^4}{(\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2)^2} \|A^*(I-T)Av_n\|^2 \\ &\quad - (1-k_2)^2 \gamma \frac{\|(I-T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2} \\ &\leq \|v_n - x^*\|^2 + (1-k_2)^2 \gamma^2 \frac{\|(I-T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2} \\ &\quad - (1-k_2)^2 \gamma \frac{\|(I-T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2} \\ &= \|v_n - x^*\|^2 - (1-k_2)^2 \gamma (1-\gamma) \frac{\|(I-T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I-T)Av_n\|^2}. \end{aligned}$$

This implies that

$$\|z_n - x^*\|^2 \leq \|v_n - x^*\|^2. \quad (19)$$

Combining inequalities (17) and (19) we obtain Step 1.

Step 2. The sequence $\{x_n\}$ is bounded. Indeed, from the definition of v_n , we get

$$\begin{aligned}
\|v_n - x^*\| &= \|\beta_n(1 - \alpha)x_n + (1 - \beta_n)t_n - x^*\| \\
&= \|\beta_n[(1 - \alpha)x_n - x^*] + (1 - \beta_n)[t_n - x^*]\| \\
&\leq \beta_n\|(1 - \alpha)x_n - x^*\| + (1 - \beta_n)\|t_n - x^*\| \\
&= \beta_n\|(1 - \alpha)(x_n - x^*) - \alpha x^*\| + (1 - \beta_n)\|t_n - x^*\| \\
&\leq \beta_n(1 - \alpha)\|x_n - x^*\| + \beta_n\alpha\|x^*\| + (1 - \beta_n)\|t_n - x^*\| \\
&= \beta_n(1 - \alpha)\|x_n - x^*\| + \beta_n\alpha\|x^*\| \\
&\quad + (1 - \beta_n)\|x_n + \xi_{1,n}(x_n - x_{n-1}) + \xi_{2,n}(x_{n-1} - x_{n-2}) - x^*\| \\
&\leq \beta_n(1 - \alpha)\|x_n - x^*\| + \beta_n\alpha\|x^*\| + (1 - \beta_n)\|x_n - x^*\| \\
&\quad + (1 - \beta_n)\xi_{1,n}\|x_n - x_{n-1}\| + (1 - \beta_n)\xi_{2,n}\|x_{n-1} - x_{n-2}\| \\
&= (1 - \alpha\beta_n)\|x_n - x^*\| + \beta_n\alpha\left[\|x^*\| + \frac{1 - \beta_n}{\alpha}\frac{\xi_{1,n}}{\beta_n}\|x_n - x_{n-1}\| \right. \\
&\quad \left. + \frac{1 - \beta_n}{\alpha}\frac{\xi_{2,n}}{\beta_n}\|x_{n-1} - x_{n-2}\|\right].
\end{aligned}$$

From Remark 3.1, there exists $M_{x^*} > 0$ such that

$$\|v_n - x^*\| \leq (1 - \alpha\beta_n)\|x_n - x^*\| + \beta_n\alpha M_{x^*} \quad (20)$$

Combining (15) and (20) we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \alpha\beta_n)\|x_n - x^*\| + \beta_n\alpha M_{x^*} \\
&\leq \max\{\|x_n - x^*\|, M_{x^*}\} \\
&\leq \dots \leq \max\{\|x_0 - x^*\|, M_{x^*}\}.
\end{aligned}$$

So, $\{x_n\}$ is bounded.

Step 3.

$$\begin{aligned}
(1 - k_2)^2\gamma(1 - \gamma)\frac{\|(I - T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2} + \lambda(1 - \lambda - k_1)\|z_n - Sz_n\|^2 \\
\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n A_{x^*},
\end{aligned}$$

for some $A_{x^*} > 0$. Indeed, it implied from (16) and (18) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|v_n - x^*\|^2 - (1 - k_2)^2\gamma(1 - \gamma)\frac{\|(I - T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2} \\
&\quad - \lambda(1 - \lambda - k_1)\|z_n - Sz_n\|^2.
\end{aligned} \quad (21)$$

On the other hand, from (20) we have

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq [(1 - \alpha\beta_n)\|x_n - x^*\| + \beta_n\alpha M_{x^*}]^2 \\
&= (1 - \alpha\beta_n)^2\|x_n - x^*\|^2 + 2(1 - \alpha\beta_n)\beta_n\alpha\|x_n - x^*\|M_{x^*} + \beta_n^2\alpha^2 M_{x^*}^2 \\
&\leq \|x_n - x^*\|^2 + \beta_n[2(1 - \alpha\beta_n)\alpha\|x_n - x^*\|M_{x^*} + \beta_n\alpha^2 M_{x^*}^2].
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists $A_{x^*} > 0$ such that

$$\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \beta_n A_{x^*}. \quad (22)$$

Substituting (22) into (21) we get

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \beta_n A_{x^*} - (1 - k_2)^2 \gamma (1 - \gamma) \frac{\|(I - T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2} - \lambda(1 - \lambda - k_1)\|z_n - Sz_n\|^2.$$

This implies that

$$(1 - k_2)^2 \gamma (1 - \gamma) \frac{\|(I - T)Av_n\|^4}{\|v_n - Sv_n\|^2 + \|A^*(I - T)Av_n\|^2} + \lambda(1 - \lambda - k_1)\|z_n - Sz_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n A_{x^*}.$$

Step 4.

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0. \quad (23)$$

Indeed, we have

$$\begin{aligned} \|v_n - x_n\| &= \|\beta_n(1 - \alpha)x_n + (1 - \beta_n)t_n - x_n\| \\ &\leq \beta_n(1 - \alpha)\|x_n\| + \|(1 - \beta_n)t_n - x_n\| \\ &= \beta_n(1 - \alpha)\|x_n\| + \|(1 - \beta_n)[x_n + \xi_{1,n}(x_n - x_{n-1}) + \xi_{2,n}(x_{n-1} - x_{n-2})] - x_n\| \\ &\leq \beta_n(1 - \alpha)\|x_n\| + (1 - \beta_n)\xi_{1,n}\|x_n - x_{n-1}\| \\ &\quad + (1 - \beta_n)\xi_{2,n}\|x_{n-1} - x_{n-2}\| + \beta_n\|x_n\| \\ &= \beta_n(2 - \alpha)\|x_n\| + \beta_n(1 - \beta_n)\frac{\xi_{1,n}}{\beta_n}\|x_n - x_{n-1}\| + \beta_n(1 - \beta_n)\frac{\xi_{2,n}}{\beta_n}\|x_{n-1} - x_{n-2}\| \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \beta_n = 0$, $\lim_{n \rightarrow \infty} \frac{\xi_{1,n}}{\beta_n}\|x_n - x_{n-1}\| = 0$ and $\lim_{n \rightarrow \infty} \frac{\xi_{2,n}}{\beta_n}\|x_{n-1} - x_{n-2}\| = 0$, we get

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0.$$

Step 5.

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \rho_n)\|x_n - x^*\|^2 + \rho_n \left(\frac{\beta_n}{2\alpha}\|x_n - x^*\|^2 + \frac{(1 - \beta_n)^2}{2\alpha\beta_n} [\xi_{1,n}^2\|x_n - x_{n-1}\|^2 \right. \\ &\quad + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\ &\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\ &\quad \left. + \langle x^*, x_n - v_n \rangle + \langle x^*, x^* - x_n \rangle \right), \end{aligned}$$

where $\rho_n = \frac{2\alpha\beta_n}{1 - \beta_n + \alpha\beta_n}$.

Indeed, we have

$$\begin{aligned} \|t_n - x^*\|^2 &= \|x_n + \xi_{1,n}(x_n - x_{n-1}) + \xi_{2,n}(x_{n-1} - x_{n-2}) - x^*\|^2 \\ &= \|x_n - x^*\|^2 + \xi_{1,n}^2\|x_n - x_{n-1}\|^2 + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n}\langle x_n - x_{n-1}, x_n - x^* \rangle \\ &\quad + 2\xi_{2,n}\langle x_{n-1} - x_{n-2}, x_n - x^* \rangle + 2\xi_{1,n}\xi_{2,n}\langle x_n - x_{n-1}, x_{n-1} - x_{n-2} \rangle \\ &\leq \|x_n - x^*\|^2 + \xi_{1,n}^2\|x_n - x_{n-1}\|^2 + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\ &\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|. \end{aligned} \quad (24)$$

On the other hand, using inequality (11) we also have

$$\begin{aligned}
\|v_n - x^*\|^2 &= \|\beta_n(1 - \alpha)x_n + (1 - \beta_n)t_n - x^*\|^2 \\
&= \|\beta_n[(1 - \alpha)x_n - x^*] + (1 - \beta_n)(t_n - x^*)\|^2 \\
&\leq (1 - \beta_n)^2\|t_n - x^*\|^2 + 2\beta_n\langle(1 - \alpha)x_n - x^*, v_n - x^*\rangle \\
&= (1 - \beta_n)^2\|t_n - x^*\|^2 + 2\beta_n\langle(1 - \alpha)(x_n - x^*) - \alpha x^*, v_n - x^*\rangle \\
&= (1 - \beta_n)^2\|t_n - x^*\|^2 + 2\beta_n(1 - \alpha)\langle x_n - x^*, v_n - x^*\rangle + 2\beta_n\alpha\langle x^*, x^* - v_n\rangle \\
&\leq (1 - \beta_n)^2\|t_n - x^*\|^2 + 2\beta_n(1 - \alpha)\|x_n - x^*\|\|v_n - x^*\| + 2\beta_n\alpha\langle x^*, x^* - v_n\rangle \\
&\leq (1 - \beta_n)^2\|t_n - x^*\|^2 + \beta_n(1 - \alpha)\|x_n - x^*\|^2 + \beta_n(1 - \alpha)\|v_n - x^*\|^2 \\
&\quad + 2\beta_n\alpha\langle x^*, x^* - v_n\rangle. \tag{25}
\end{aligned}$$

Combining (24) and (25) we deduce

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq (1 - \beta_n)^2[\|x_n - x^*\|^2 + \xi_{1,n}^2\|x_n - x_{n-1}\|^2 + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\
&\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\
&\quad + \beta_n(1 - \alpha)\|x_n - x^*\|^2 + \beta_n(1 - \alpha)\|v_n - x^*\|^2 + 2\beta_n\alpha\langle x^*, x^* - v_n\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - \beta_n + \alpha\beta_n)\|v_n - x^*\|^2 &\leq (1 - \beta_n + \beta_n^2 - \alpha\beta_n)\|x_n - x^*\|^2 + (1 - \beta_n)^2[\xi_{1,n}^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 \\
&\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\
&\quad + 2\beta_n\alpha\langle x^*, x^* - v_n\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \left(1 - \frac{2\alpha\beta_n}{1 - \beta_n + \alpha\beta_n}\right)\|x_n - x^*\|^2 + \frac{\beta_n^2}{1 - \beta_n + \alpha\beta_n}\|x_n - x^*\|^2 \\
&\quad + \frac{(1 - \beta_n)^2}{1 - \beta_n + \alpha\beta_n}[\xi_{1,n}^2\|x_n - x_{n-1}\|^2 + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\
&\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\
&\quad + \frac{2\beta_n\alpha}{1 - \beta_n + \alpha\beta_n}\langle x^*, x_n - v_n\rangle + \frac{2\beta_n\alpha}{1 - \beta_n + \alpha\beta_n}\langle x^*, x^* - x_n\rangle.
\end{aligned}$$

Let $\rho_n = \frac{2\alpha\beta_n}{1 - \beta_n + \alpha\beta_n}$. We get

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq (1 - \rho_n)\|x_n - x^*\|^2 + \rho_n\left(\frac{\beta_n}{2\alpha}\|x_n - x^*\|^2 \right. \\
&\quad + \frac{(1 - \beta_n)^2}{2\alpha\beta_n}[\xi_{1,n}^2\|x_n - x_{n-1}\|^2 + \xi_{2,n}^2\|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n}\|x_n - x_{n-1}\|\|x_n - x^*\| \\
&\quad + 2\xi_{2,n}\|x_{n-1} - x_{n-2}\|\|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|] \\
&\quad \left. + \langle x^*, x_n - v_n\rangle + \langle x^*, x^* - x_n\rangle\right). \tag{26}
\end{aligned}$$

Combining (15) and (26) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \rho_n)\|x_n - x^*\|^2 + \rho_n \left(\frac{\beta_n}{2\alpha} \|x_n - x^*\|^2 \right. \\ &\quad + \frac{(1 - \beta_n)^2}{2\alpha\beta_n} [\xi_{1,n}^2 \|x_n - x_{n-1}\|^2 + \xi_{2,n}^2 \|x_{n-1} - x_{n-2}\|^2 + 2\xi_{1,n} \|x_n - x_{n-1}\| \|x_n - x^*\| \\ &\quad + 2\xi_{2,n} \|x_{n-1} - x_{n-2}\| \|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n} \|x_n - x_{n-1}\| \|x_{n-1} - x_{n-2}\|] \\ &\quad \left. + \langle x^*, x_n - v_n \rangle + \langle x^*, x^* - x_n \rangle \right). \end{aligned}$$

Step 6. The sequence $\{\|x_n - x^*\|\}$, $n \geq 0$, converges to zero.

To see this, set

$$a_n := \|x_n - x^*\|^2,$$

and

$$\begin{aligned} b_n &:= \frac{\beta_n}{2\alpha} \|x_n - x^*\|^2 + \frac{(1 - \beta_n)^2}{2\alpha\beta_n} \left[\xi_{1,n}^2 \|x_n - x_{n-1}\|^2 + \xi_{2,n}^2 \|x_{n-1} - x_{n-2}\|^2 \right. \\ &\quad + 2\xi_{1,n} \|x_n - x_{n-1}\| \|x_n - x^*\| \\ &\quad \left. + 2\xi_{2,n} \|x_{n-1} - x_{n-2}\| \|x_n - x^*\| + 2\xi_{1,n}\xi_{2,n} \|x_n - x_{n-1}\| \|x_{n-1} - x_{n-2}\| \right] \\ &\quad + \langle x^*, x_n - v_n \rangle + \langle x^*, x^* - x_n \rangle. \end{aligned}$$

Then **Step 5** can be rewritten as follows:

$$a_{n+1} \leq (1 - \rho_n)a_n + \rho_n b_n.$$

Note that since $\alpha \in (0, 1]$, it follows that $\rho_n = \frac{2\alpha\beta_n}{1 - \beta_n + \alpha\beta_n} \geq 2\alpha\beta_n$. So, $\sum_{n=1}^{\infty} \rho_n = \infty$, and since $\beta_n \in \left(0, \frac{1}{1 + \alpha}\right)$ we obtain $\rho_n = \frac{2\alpha\beta_n}{1 - \beta_n + \alpha\beta_n} \in (0, 1)$. Using Lemma 2.2, to prove $\lim_{n \rightarrow +\infty} \|x_n - x^*\| = 0$, it is sufficient to show that $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0.$$

Since $\{x_n\}$ is a bounded sequence, $\lim_{n \rightarrow +\infty} \beta_n = 0$, $\lim_{n \rightarrow +\infty} \frac{\xi_{1,n}}{\beta_n} \|x_n - x_{n-1}\| = 0$, $\lim_{n \rightarrow +\infty} \frac{\xi_{2,n}}{\beta_n} \|x_{n-1} - x_{n-2}\| = 0$, and by (23), we obtain $\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0$, hence to prove $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ we only need to show that

$$\limsup_{k \rightarrow \infty} \langle x^*, x^* - x_{n_k} \rangle \leq 0 \tag{27}$$

for every subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k \rightarrow \infty} (\|x_{n_{k+1}} - x^*\| - \|x_{n_k} - x^*\|) \geq 0.$$

Suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that

$$\liminf_{k \rightarrow \infty} (\|x_{n_{k+1}} - x^*\| - \|x_{n_k} - x^*\|) \geq 0.$$

Then

$$\begin{aligned} & \liminf_{k \rightarrow \infty} (\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2) \\ &= \liminf_{k \rightarrow \infty} [(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)] \geq 0. \end{aligned}$$

Using **Step 3**, we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left((1 - k_2)^2 \frac{\gamma(1 - \gamma)\|(I - T)Av_{n_k}\|^4}{\|v_{n_k} - Sv_{n_k}\|^2 + \|A^*(I - T)Av_{n_k}\|^2} + \lambda(1 - \lambda - k_1)\|z_{n_k} - Sz_{n_k}\|^2 \right) \\ & \leq \limsup_{k \rightarrow \infty} [\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 + \beta_{n_k}A_{x^*}] \\ & \leq \limsup_{k \rightarrow \infty} [\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2] + \limsup_{k \rightarrow \infty} \beta_{n_k}A_{x^*} \\ & = - \liminf_{k \rightarrow \infty} [\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2] \\ & \leq 0. \end{aligned}$$

This implies that

$$\lim_{k \rightarrow \infty} \frac{\|(I - T)Av_{n_k}\|^4}{\|v_{n_k} - Sv_{n_k}\|^2 + \|A^*(I - T)Av_{n_k}\|^2} = 0 \text{ and } \lim_{k \rightarrow \infty} \|z_{n_k} - Sz_{n_k}\| = 0.$$

Hence

$$\lim_{k \rightarrow \infty} \|(I - T)Av_{n_k}\| = 0 \text{ and } \lim_{k \rightarrow \infty} \|z_{n_k} - Sz_{n_k}\| = 0 \quad (28)$$

On the other hand, using the definition of $\{z_n\}$, see that

$$\begin{aligned} \|z_{n_k} - v_{n_k}\| &= \tau_{n_k} \|A^*(I - T)Av_{n_k}\| \\ &\leq \frac{\|(I - T)Av_{n_k}\|^2}{\|v_{n_k} - Sv_{n_k}\|^2 + \|A^*(I - T)Av_{n_k}\|^2} \|A^*\| \|(I - T)Av_{n_k}\| \rightarrow 0, \end{aligned} \quad (29)$$

as $k \rightarrow \infty$.

Since the sequence $\{x_{n_k}\}$ is bounded, without any loss of generality we may assume that $\{x_{n_k}\}$ converges weakly to some $z^* \in H$ so that

$$\limsup_{k \rightarrow \infty} \langle x^*, x^* - x_{n_k} \rangle = \langle x^*, x^* - z^* \rangle. \quad (30)$$

Using **Step 4**, we get

$$v_{n_k} \rightharpoonup z^* \text{ as } k \rightarrow \infty.$$

Using (29), we obtain

$$z_{n_k} \rightharpoonup z^* \text{ as } k \rightarrow \infty.$$

Now, using the demiclosedness of $I - S$ and $I - T$ and (28), we get $z^* \in \Omega$.

From (30) and the definition of $x^* = P_\Omega(0)$, and $z^* \in \Omega$ we have

$$\limsup_{k \rightarrow \infty} \langle x^*, x^* - x_{n_k} \rangle = \langle x^*, x^* - z^* \rangle \leq 0.$$

So, the inequality (27) holds. Therefore, we prove that $\limsup_{k \rightarrow \infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k \rightarrow \infty} (a_{n_k+1} - a_{n_k}) \geq 0.$$

Apply Lemma 2.2 to **Step 5**, we obtain $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$. The proof is completed. \square

Remark that in the case $\alpha = 1$ and $\xi_{2,n} = 0$, Algorithm 3.1 reduces to (10).

4. Conclusions

In this paper, we have focused on the two-step inertial Censor-Segal algorithm, with the goal of obtaining strong convergence results for solving the split common fixed point problem for demicontractive operators. We proposed a new algorithm with self-adaptive step sizes. Under suitable conditions, we analyzed and proved the strong convergence theorem of the proposed algorithm, such that its proof does not require calculating the norm of the operator.

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