SOME RESULTS ON R-CONNECTED SETS

Ali Ebrahimi Meymand[1*] and Azadeh Alijani[2]

One of the most important theories across all branches of mathematics is the theory of metric spaces. Recently, this theory has been extended to R-metric spaces. In this paper, we introduce the notion of R-connectivity in R-metric spaces and study some of their topological properties. Another goal of this manuscript is the extension of concepts of R-preserving maps and R-continuous maps, and also some conditions for maps that preserve R-connectivity are investigated. The last part of this paper is devoted to studying some other applications of this theory.

Keywords: R-connected component, R-connected set, R-convex set, (R, S)-continuous map, (R, S)-preserving map.

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1. Introduction and preliminaries

The notion of orthogonal metric spaces and the extension of the Banach fixed point theorem in these spaces as well as some of their applications in differential equations were studied by some researchers [4, 5]. Also, the notion of R-metric spaces has been introduced recently in [9], and a real generalization of the Banach fixed-point theorem and that of the Brouwer fixed-point theorem were given there. Also, the existence of a solution for a fractional integral equation, as an application of this theory, was proved there. In [10], the authors introduced the notions of R-topological spaces and SR-topological spaces. The properties of these spaces, such as their relationship with the initial topological spaces, were given there. Also, see [13] for some fixed point results for multivalued mappings in R-metric spaces. On the other hand, the subjects of connectivity and connected sets have been studied and applied extensively. For example, see [3, 6, 7, 11, 16].

The structure of the paper is as follows. In the remaining part of the current section, the definitions and preliminaries used in this study are given. In Section 2, we define the notion of R-connected sets in R-metric spaces and consider some topological properties of them in R-metric spaces and in the product spaces. Section 3 introduces R-connected components and studies

[1],[2] Department of Mathematics, Faculty of Mathematical Sciences, $Vali{-}e{-}Asr$ University of Rafsanjan, P.O. Box 7719758457, Rafsanjan, Iran. e-mail: a.ebrahimi@vru.ac.ir, alijani@vru.ac.ir

their properties. The notions of R-preserving maps and R-continuous maps and some of their results are extended to (R, S)-preserving maps and (R, S)-continuous maps in Section 4. To continue, some relations between two R-connected spaces are presented by R or (R, S)-continuous maps. Finally, some other applications of this theory are considered in the last section of this paper.

Afterward, we review some important definitions that are needed in this study. The used references are [1, 2, 9, 10, 14, 15]; see the references therein for more details.

The triple (M, d, R) is an R-metric space if (M, d) is a metric space, and R is an arbitrary relation on M (not necessarily reflexive, nor transitive, etc). An R-sequence is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n R x_m$ for all $m, n \in \mathbb{N}$, where $n \leq m$. Every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the R-sequence $\{x_n\}_{n=1}^{\infty}$ is itself an R-sequence which is called an R-subsequence of $\{x_n\}_{n=1}^{\infty}$. We denote $x_n \xrightarrow{R} x$ if the R-sequence $\{x_n\}_{n=1}^{\infty}$ converges to x. In this case, if $x_n \neq x$ for all n, then x is called an R-limit point. The set of all R-limit points of E is denoted by E'^R. As in the classical analysis, a subset E is called R-closed if $E'^R \subseteq E$, and E is called R-open if E has an R-closed complement.
It has proved that E is R-closed if and only if $E = \overline{E}^R$, where $\overline{E}^R = E \cup E'^R$ [9, Theorem 2.5]. Moreover, x is said to be an R-interior point of E in an R-metric space (M, d), if for every R-sequence $\{x_n\}_{n=1}^{\infty}$ of M converging to x, there exists an integer $K \in \mathbb{N}$ such that $x_n \in E$ for all $n \geq K$, and the set of all R-interior points of E is denoted by $R - int(E)$. Also, a set E in R-vector spaces is called R-convex, if it is closed under the R-convex combinations of elements of E, where R-convex combination of two elements x_1, x_2 of E, is $\alpha x_1 + \beta x_2$, where $x_1 R x_2$, $\alpha, \beta \in (0, 1)$, and $\alpha + \beta = 1$.

2. R-connected sets

In this section, firstly, the notions of R-separated and R-connected sets in R-metric spaces are introduced, and then some properties and the relation between the R-connectivity and connectivity in the classical sense are considered. Also, the R-connectivity property of the union and product of R-connected sets is investigated.

Definition 2.1. *Let (M, d, R) be an R-metric space. The subsets A and B of M are called R-separated if $A \cap \overline{B}^R = B \cap \overline{A}^R = \varnothing$. Also, $E \subseteq M$ is said to be an R-connected set, if E cannot be written as a union of any two nonempty R-separated sets. else, E is R-disconnected.*

It is trivial that if $R := M \times M$, then the concepts of connectedness and R-connectedness are the same

In an R-metric space (M, d, R), it is clear that every R-limit point is also a limit point and so every R-connected set is also a connected set in the usual sense. But, the converse of this fact is not true in general. To see this, consider the following set E in the Euclidean metric space $\mathbb{R}^2$ with $R := \mathbb{R}^{>0} \times \mathbb{R}^{>0}$.

$$E := \{(x, y) : \ (x-1)^2 + (y-1)^2 \leq 1\} \cup \{(x, y) : \ (x+1)^2 + (y-1)^2 \leq 1\}.$$

Also, consider the following example.

Example 2.1. *Every finite set E with the cardinal number more than one, $|E| > 1$, is not R-connected, and the empty set and singletons are R-connected sets for every relation R.*

In different studies on R-spaces, these spaces and classical spaces have been compared. Hereafter, we can see some differences.

- It is well known that a subset of $\mathbb{R}$ is connected if and only if it is an interval, but this fact is not valid for R-connected sets in the R-metric space $(\mathbb{R}, |\cdot|, \geq)$, where $\geq$ is the usual greater than or equal relation in real numbers. The interval $[a, b] = [a, c) \cup [c, b]$, for some $a < c < b$, is not $\geq$-connected.
- It is well known that every convex set in $\mathbb{R}^n$ is connected, but this result does not hold for R-convex sets. For example, consider the R-convex set $E = [1, 2] \cup (2, 3]$ in the R-metric space $(\mathbb{R}, |\cdot|, \leq)$, where $\leq$ is the usual less than or equal relation in real numbers. The set E is not $\leq$-connected.

By the attention to the above discussion, we can see that not only the real number set equipped with the relations " $\leq$ " or " $\geq$ " is not $\leq$-connected or $\geq$-connected, but also it has no $\leq$-connected or $\geq$-connected subset with more than one element. In other words, only $\leq$-connected or $\geq$-connected subsets of $\mathbb{R}$ are singletons. To see this, by use of Example 2.1, we can assume that A is an arbitrary infinite subset of $\mathbb{R}$. Without loss of generality, suppose $R :=\leq$ and fixed $a \in A$, then $A = \{x \in A : x \leq a\} \cup \{x \in A : x > a\}$ and A is R-disconnected. Clearly, if A is bounded below and $\inf A \in A$, then $\{a\} \cup A - \{a\}$ is an R-separation for A, where $a = \inf A$. Similarly, if A is bounded above and $\sup A \in A$, then $\{a\} \cup A - \{a\}$ is an R-separation for A, where $a = \sup A$.

Question. Here, the following problems are considerable.

i. Is there a relation R on the real number space such that $\mathbb{R}$ and some of its infinite subsets be R-connected?

ii. If the answer of i is positive, then what are the properties of R?

Here, we present an answer to the above questions. To see this answer, assume that E is a dense subset of $\mathbb{R}$ and that $R := E \times E$. Then $\mathbb{R}$ and every interval of $\mathbb{R}$ are R-connected.

Generally, the closure of a connected set is connected, too. The following proposition investigates this fact for R-connected sets.

Proposition 2.1. *If E is an R-connected set in (M, d, R), then $\overline{E}^R$ is also an R-connected set. Moreover, E and $\overline{E}^S$ are also S-connected sets, for every relation S on M such that $R \subseteq S$.*

Proof. Assume that A and B are R-separated sets and that $\overline{E}^R = A \cup B$. Then $E = A_0 \cup B_0$ in which $A_0 = A \cap E$ and $B_0 = B \cap E$. The sets A_0 and B_0 are R-separated. It contradicts with the R-connectivity of E, and hence $\overline{E}^R$ is R-connected. To see the final part, we use the fact that every convergent R-sequence is also a convergent S-sequence and that $\overline{A}^R \subseteq \overline{A}^S$ for $A \subseteq M$. □

In continuation, investigating the R-connectivity of the set of R-interior points of an R-connected set is our goal. Generally, it is trivial that the set of interior points of a connected set is not necessarily connected. This result is also valid for R-connected sets. See the following example.

Example 2.2. *Let $E = \{(x, y) :\ (x+1)^2 + y^2 \leq 1$ or $(x-1)^2 + y^2 \leq 1\}$ as a subset of $\mathbb{R}^2$, and let $R = (E \times E) \cup (A \times A)$, where $A = \{(0, y) :\ y \in \mathbb{R}\}$. Then E is an R-connected set, but the set of R-interior points of E, $R - int(E)$, is not R-connected, because*

$$\begin{aligned} R - int(E) &= \{(x, y) :\ (x+1)^2 + y^2 < 1 \text{ or } (x-1)^2 + y^2 < 1\}, \\ &= \{(x, y) : (x+1)^2 + y^2 < 1\} \cup \{(x, y) : (x-1)^2 + y^2 < 1\}. \end{aligned}$$

The purpose of the following theorem is to consider some equivalent conditions for R-connectivity in R-metric spaces.

Theorem 2.1. *Let (M, d, R) be an R-metric space. Then the following statements are equivalent:*

i. M is an R-connected set.
ii. M cannot be written as a union of two disjoint nonempty R-open sets.
iii. M cannot be written as a union of two disjoint nonempty R-closed sets.
iv. $\emptyset$ and M are the only subsets of M that are both R-closed and R-open.

Proof. The implications $i \Longrightarrow ii$ and $ii \Longleftrightarrow iii$ are straightforward.

$iii \Longrightarrow iv$. Let A be a nonempty proper subset of M that is both R-closed and R-open. Then $M = A \cup A^c$ is a representation of M as a union of two nonempty R-closed sets, which is in contrast with iii.

$iv \Longrightarrow i$. Let $M = A \cup B$ be a representation of M as a union of nonempty R-separated sets A and B. Then $B \neq \emptyset$ and $B \neq M$ because $A \neq \emptyset$. Also, $A \cap \overline{B}^R = \emptyset$ implies that $\overline{B}^R \subseteq B$, and hence $B = \overline{B}^R$. More precisely, B is R-closed. Similarly, A is R-closed, and hence $B = A^c$ is R-open. Thus B is a proper subset of M that is both R-open and R-closed, which is a contradiction and so M is R-connected. □

The following results present the correctness of extending facts of connected sets to R-connected sets and are given by similar classical techniques, and hence the proofs are omitted.

Proposition 2.2. *Let (A, B) be an R-separation pair for an R-metric space M, and let X be an R-connected subset of M. Then $X \subseteq A$ or $X \subseteq B$.*

Similar to the classical form, we can obtain R-connected sets that the union of them is an R-connected set, too. The following proposition illustrates this subject.

Proposition 2.3. *Let X and Y be R-connected subsets of an R-metric space M such that $X \cap \overline{Y}^R \neq \emptyset$ or $Y \cap \overline{X}^R \neq \emptyset$. Then $X \cup Y$ is an R-connected set.*

Corollary 2.1. *If X and Y are R-connected subsets of an R-metric space M such that they have a common point, then $X \cup Y$ is also an R-connected set.*

Proposition 2.3 can be extended for a family of R-connected subsets. In the following proposition, this result is considered.

Proposition 2.4. *Let X and $\{Y_i\}_{i\in I}$ be R-connected subsets of an R-metric space M such that for each $i \in I$, $X \cap \overline{Y_i}^R \neq \emptyset$ or $Y_i \cap \overline{X}^R \neq \emptyset$. Then $X \cup (\bigcup_{i\in I} Y_i)$ is R-connected.*

Corollary 2.2. *If X and $\{Y_i\}_{i\in I}$ are R-connected subsets of R-metric space M such that $X \cap Y_i \neq \emptyset$, for all $i \in I$, then $X \cup (\bigcup_{i\in I} Y_i)$ is R-connected.*

Notation. Suppose that (X, R, d_1) and (Y, S, d_2) are R-metric and S-metric spaces, respectively. We define the relation $T := R \times S$ on $X \times Y$ as follows, for every $(x, y), (u, v) \in X \times Y$:

$$(x, y)T(u, v) \iff xRu \text{ and } ySv.$$

In this case, $(X \times Y, R \times S, d_1 \times d_2)$ is an $R \times S$-metric space, where $d_1 \times d_2$ is any product metric on $X \times Y$. For example, we can consider the supremum metric $d_1 \times d_2((x, y), (u, v)) = \max\{d_1(x, u), d_2(y, v)\}$, or the metric defined by $d_1 \times d_2((x, y), (u, v)) = \sqrt{(d_1(x, u))^2 + (d_2(y, v))^2}$, for all (x, y) and (u, v) in $X \times Y$.

Theorem 2.2. *Suppose that (X, d_1) and (Y, d_2) are R-metric and S-metric spaces, respectively. Then the following statements hold:*

i. *If $X \times Y$ is an $R \times S$-connected set, then X is an R-connected set and also Y is an S-connected set.*
ii. *If R is reflexive and there is an element $y_0 \in Y$ such that $y_0 S y_0$, then the R-connectivity of X and the S-connectivity of Y, imply that $X \times Y$ is an $R \times S$-connected set.*

3. R-connected components

Here, this section is devoted to obtaining the largest R-connected set corresponding to every point in an R-metric space. To see this, assume that x is an arbitrary element of R-metric space M. Set

$$\mathcal{F} = \{A \subseteq M : \ A \text{ is an } R-\text{connected set and } x \in A.\}.$$

The family $\mathcal{F}$ is nonempty by Example 2.1. Now, consider an arbitrary chain $\{A_i\} \subseteq \mathcal{F}$. If $A = \cup A_i$, then Proposition 2.4 concludes that A is R-connected and is also an upper bound for $\{A_i\}$ in $\mathcal{F}$, and by the Zorn's lemma $\mathcal{F}$ has a unique maximal element, $\mathcal{C}_x^R$. For $x \in M$, the presented R-connected set $\mathcal{C}_x^R$ is called to be R-connected component of x.

We present some properties of R-connected component $\mathcal{C}_x^R$ for a member x in an R-metric space M as follows.

Property 1. Every R-connected set is contained in a unique R-connected component.

Property 2. Every R-connected component is R-closed but not necessarily R-open. This fact is obtained by using Proposition 2.1 and the maximality of R-connected components.

Property 3. In the classical case, it has been seen that for a given topological space M, define an equivalence relation S on M by setting xSy if there exists a connected subset of M containing both x and y. The equivalency of S is obtained obviously by the properties of connected sets. The equivalence classes of the relation S are S-connected components. It means that $[x] = \mathcal{C}_x^S$.

Property 4. Every point of M is contained in a unique R-connected component. Moreover, $M = \dot{\bigcup\limits_{x\in M}} \mathcal{C}_x^R$.

Property 5. If R and S are two relations on M such that $R \subseteq S$, then $\mathcal{C}_x^R \subseteq \mathcal{C}_x^S$, for all $x \in M$. Moreover, by Property 3, M is covered by a smaller family of S-connected components with respect to R-connected components.

The R-connected components for a point depend on the relation R. It means that if R and S are two different relations on M, then for $x \in M$, $\mathcal{C}_x^R$ and $\mathcal{C}_x^S$ are not necessarily the same. The following example illustrates this fact.

Example 3.1. *The Euclidean metric space* $\mathbb{R}^2$ *with the following three relations is considered:*

$$R := \mathbb{R}^{>0}\times\mathbb{R}^{>0}, S := (\mathbb{R}^{>0}\times\mathbb{R}^{>0})\cup(\mathbb{R}^{<0}\times\mathbb{R}^{<0}), T := (\mathbb{R}^{>0}\times\mathbb{R}^{>1})\cup(\mathbb{R}^{<0}\times\mathbb{R}^{<-1}).$$

Then R-connected components, S-connected components, and T-connected components for some points of $\mathbb{R}^2$ *are given as follows:*

$$\mathcal{C}_{(0,0)}^R = \{(x, y) :\ x \geq 0,\ y \geq 0\},$$

$$\mathcal{C}_{(0,0)}^S = \{(x, y) :\ x \geq 0,\ y \geq 0\} \cup \{(x, y) :\ x \leq 0,\ y \leq 0\},$$

$$\mathcal{C}_{(0,0)}^T = \{(0, 0)\},$$

and

$$\mathcal{C}_{(-1,-1)}^R = \{(-1, -1)\},$$

$$\mathcal{C}_{(-1,-1)}^S = \{(x, y) :\ x \leq 0,\ y \leq 0\},$$

$$\mathcal{C}_{(-1,-1)}^T = \{(x, y) :\ x \leq 0,\ y \leq -1\},$$

and also, the connected component for $(2, -3) \in \mathbb{R}^2$ *with respect to every relation R, S, and T is the singleton* $\mathcal{C}_{(2,-3)}^R = \mathcal{C}_{(2,-3)}^S = \mathcal{C}_{(2,-3)}^T = \{(2, -3)\}$.

For the rest, we can use $\mathcal{C}_x$ instead of $\mathcal{C}_x^R$, where there is only one relation on the space.

Other properties of R-connected components are investigated in the following proposition.

Proposition 3.1. *Let (M, d, R) be an R-metric space. Then the following statements hold:*

i. For any two distinct elements x and y of M, either $\mathcal{C}_x \cap \mathcal{C}_y = \emptyset$ or $\mathcal{C}_x = \mathcal{C}_y$.

ii. Assume that M is an R-compact space, that A and B are R-closed subsets of M where $A \cap B = \emptyset$, and that for each $x \in M$, exactly one of the sets $A \cap \mathcal{C}_x$ or $B \cap \mathcal{C}_x$ is empty. Then there exist disjoint R-compact subsets M_A and M_B such that $A \subseteq M_A$, $B \subseteq M_B$, and $M = M_A \cup M_B$.

Proof. *i.* This result is easily obtained by Proposition 2.3.
ii. Set $M_A := \bigcup_{x \in A} \mathcal{C}_x$ and $M_B := \bigcup_{x \in B} \mathcal{C}_x$. By *i*, we have $M_A \cap M_B = \emptyset$ and also $M = M_A \cup M_B$. Now, let $\{x_n\}_{n \in \mathbb{N}}$ be an R-sequence of M_A. Then there exists a convergent R-subsequence $x_{n_k} \xrightarrow{R} x$ in M becuase M is R-compact. If $x \in M_B$, then $x \in \mathcal{C}_u$, for some $u \in B$, and $x_{n_k} \in \mathcal{C}_u$, for infinite n_k by $\mathcal{C}_u$ is R-closed. It shows that $M_A \cap M_B \neq \emptyset$, and it is a contradiction. So $x \in M_A$, and M_A is R-compact. Similarly, M_B is R-compact. ☐

4. Functions on R-connected sets

In this section, we introduce the notions of (R, S)-continuous maps and (R, S)-preserving maps and afterward study maps that transfer R-connected sets to S-connected sets. For this, we need to extend two notions R-continuous maps and R-preserving maps, which are defined in [2] and [9].

In the last papers, the different definitions for continuous maps on R-metric spaces were presented. First, we review these definitions and then introduce (R, S)-continuous maps. Afterward, these concepts are compared with each other.

Eshaghi-R-continuous map. A map f on an R-metric space M is said to be Eshaghi-R-continuous (or briefly R-continuous) at $x \in M$, if $f(x_n) \to f(x)$, for every R-sequence $\{x_n\}_{n=1}^{\infty}$ converging to x in M, and also f is an Eshaghi-R-continuous (or briefly R-continuous) map on M if f is R-continuous at every point of M.

E-A-R-continuous map. A map f on an R-metric space M is said to be E-A-R-continuous at $x \in M$, if $f(x_n) \xrightarrow{R} f(x)$, for every R-sequence $\{x_n\}_{n=1}^{\infty}$ converging to x in M, and also f is an E-A-R-continuous map on M if f is E-A-R-continuous at every point of M.

R-preserving map. A map $f : M \to M$ is called to be an R-preserving map if, for every $x, y \in M$, xRy implies that $f(x)Rf(y)$.

Now, we extend these concepts for functions between R-metric space M and S-metric space N as follows.

Definition 4.1. *Let $f : (M, d_M, R) \to (N, d_N, S)$ be a map.*

i. *The map f is said to be an (R, S)-continuous map at $x \in M$, if for every R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M with $x_n \xrightarrow{R} x$, we have $f(x_n) \xrightarrow{S} f(x)$. Also, f is (R, S)-continuous on M if f is (R, S)-continuous at each $x \in M$.*
ii. *The map f is called to be an (R, S)-preserving map if, for every $x, y \in M$, xRy implies that $f(x)Sf(y)$.*

Obviously, an (R, S)-continuous map is a directed extension of an E-A-R-continuous map. Here, obtaining a relation between definitions of (R, S)-continuous map and Eshaghi-R-continuous map is a goal.

Clearly, every (R, S)-continuous map is an Eshaghi-R-continuous map, but the reverse of this relation is not true. The following example illustrates this fact.

Example 4.1. *Suppose that $f : (\mathbb{R}, R, |\cdot|) \longrightarrow (\mathbb{R}, S, |\cdot|)$ is defined by $f(x) = -|x|$, where the relations R and S are as follows:*

$$R = \mathbb{R}^{<0} \times \mathbb{R}^{<0}, \qquad S = [1, \infty) \times [1, \infty).$$

Clearly, f is an Eshaghi-R-continuous map but it is not (R, S)-continuous. To see this, for $n \in \mathbb{N}$, set $x_n = -1 + \frac{1}{n}$. Then $x_n \xrightarrow{R} -1$, but $\{f(x_n)\}$ is not an S-sequence.

There is another note in studying functions on R-metric spaces; two properties "R-continuous and R-preserving" are needed for holding some results. While we can see that if f is an (R, S)-continuous map, then these results are satisfying and f does not necessarily need to be an R-preserving map or (R, S)-preserving map. Before considering some properties of maps on these spaces, we see the following relations:

$$(R, S) - continuous \Longrightarrow Eshaghi - R - continuous,$$

but

$$Eshaghi - R - continuous \nRightarrow (R, S) - continuous.$$

$$Eshaghi - R - continuous + (R, S) - preserving \Longrightarrow (R, S) - continuous.$$

The first statement is clear, and Example 4.1 concludes the second statement. To see the third statement, let $x_n \xrightarrow{R} x$. Then $\{f(x_n)\}$ is an S-sequence because f is (R, S)-preserving and $f(x_n) \xrightarrow{S} f(x)$ because f is Eshaghi-R-continuous. Moreover,

$$(R, S) - preserving \nRightarrow (R, S) - continuous.$$

Set $M = N = A_n \cup \{0, 2, 4\}$, where $A_n := \{\frac{1}{n} : n \in \mathbb{N}\}$, and

$$R := A_n \times A_n \cup \{(0, 2)\}, \quad S = A_n \times A_n,$$

$$f : (M, R, |\cdot|) \longrightarrow (N, S, |\cdot|), \quad f(x) = x^2.$$

There is only one R-sequence such that

$$\frac{1}{n} \xrightarrow{R} 0 \implies f(\frac{1}{n}) \xrightarrow{S} f(0).$$

So, f is (R, S)-continuous but it is not (R, S)-preserving because $0R2$ but $f(0) = 0 \not S\, 4 = f(2)$.

Hereafter, we consider some properties for functions on R-metric spaces. It will be seen that the condition "(R, S)-preserving Eshaghi-R-continuous map" can be replaced by (R, S)-continuous map. The following theorem presents an equivalent definition for (R, S)-continuous maps, analogous to the usual case. Since this theorem extends Theorem 4 in [10] and its proof follows a similar argument, we omit the details.

Theorem 4.1. *A map $f : (M, d_M, R) \longrightarrow (N, d_N, S)$ is (R, S)-continuous if and only if $f^{-1}(F)$ is an R-closed subset of M for every S-closed subset F of N.*

Now, preserving of (R, S)-connected sets by (R, S)-continuous maps is the goal of the following theorem. The proof of it is again similar to the result in the classical case, but we present it as a sample and an application of the obtained results for R-metric spaces in this paper.

Theorem 4.2. *Suppose that $f : (M, d_M, R) \longrightarrow (N, d_N, S)$ is an (R, S)-continuous map. Then f preserves (R, S)-connected sets; that is, $f(E)$ is an S-connected subset of N, for every R-connected subset E of M.*

Proof. (Reductio ad absurdum) Suppose that E is an R-connected subset of M, and that A and B are two nonempty S-separated sets in N such that $f(E) = A \cup B$. Put $C = E \cap f^{-1}(A)$, and $D = E \cap f^{-1}(B)$. Then C and D are nonempty sets and $E = C \cup D$. On the other hand, by using Theorem 4.1, the map f is (R, S)-continuous if and only if $f^{-1}(F)$ is an R-closed set for every S-closed subsets F of N. Therefore, the set $f^{-1}(\overline{A}^S)$ is R-closed, and hence $C \subseteq f^{-1}(\overline{A}^S)$ implies that $\overline{C}^R \subseteq f^{-1}(\overline{A}^S)$. So, $f(\overline{C}^R) \subseteq \overline{A}^S$. Moreover, $D \subseteq f^{-1}(B)$ implies that $f(D) \subseteq B$. By the relations $f(\overline{C}^R) \subseteq \overline{A}^S$, $f(D) \subseteq B$, and $\overline{A}^S \cap B = \varnothing$, we conclude that $f(\overline{C}^R) \cap f(D) = \varnothing$, and hence $\overline{C}^R \cap D = \varnothing$. Similarly, $C \cap \overline{D}^R = \varnothing$. It is in contrast with the R-connectivity of E, and the proof is completed. □

We saw that some functions could preserve (R, S)-connected sets, but this result is not valid for (R, S)-connected components. To see this, consider a constant function $f(x) = a$ on the $\leq$-metric space $(\mathbb{R}, |\cdot|, \leq)$.

Now, we obtain an interesting result for evaluating R-connectivity of an R-metric space by using two-valued (R, R)-continuous maps on that space. The following theorem illustrates this fact.

Theorem 4.3. *Let (M, d, R) be an R-metric space and let $a, b \in M$ such that $a \neq b$. If $\{(a,a),(a,b),(b,a),(b,b)\} \subseteq R$, then the following statements are equivalent:*

i. M is an R-disconnected set.

ii. There is a nonconstant (R,R)-continuous map $f : M \to \{a,b\}$.

Proof. $i \Longrightarrow ii$. Let (A, B) be an R-separation pair of nonempty sets for M; that is, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap \overline{B}^R = B \cap \overline{A}^R = \emptyset$, and $M = A \cup B$. Define f on M as follows:

$$f(x) = \begin{cases} a, & x \in A, \\ b, & x \in B. \end{cases}$$

The map f is well-defined by $A \cap B = \emptyset$ and since A and B are nonempty, f is nonconstant. To see that f is R-continuous at x_0 (without loss of generality, assume $x_0 \in A$), suppose that $\{x_n\}_{n=1}^{\infty}$ is an R-sequence such that $x_n \xrightarrow{R} x_0$. By using $\overline{B}^R \cap A = \emptyset$, there exists $N \in \mathbb{N}$ such that $x_n \in A$, for every $n \geq N$, and $f(x_0) = f(x_n) = a$. Then $f(x_n) \xrightarrow{R} f(x_0)$ and it shows that f is (R,R)-continuous at every point of A. Hence, f is (R,R)-continuous.

$ii \Longrightarrow i$. Put $A = f^{-1}(\{a\})$ and $B = f^{-1}(\{b\})$. The sets A and B are nonempty subsets of M because f is not constant, and also we have $M = A \cup B$. Now, we want to show that A and B are R-separated sets. To see this, suppose $x_0 \in A \cap \overline{B}^R$. Then $f(x_0) = a$, and there is an R-sequence $\{x_n\}_{n=1}^{\infty} \subseteq B$ such that $x_n \xrightarrow{R} x_0$ and $f(x_n) = b$, for each $n \in \mathbb{N}$. On the other hand, f is (R,R)-continuous, so $b = f(x_n) \to f(x_0) = a$. It is a contradiction. Thus $A \cap \overline{B}^R = \emptyset$, and similarly $B \cap \overline{A}^R = \emptyset$. Therefore, M is R-connected. □

Similar results to the classical cases are obtained by the presented properties for functions on R-spaces and R-connected sets as well as the properties of R-connected components of these spaces. The following proposition demonstrates an interesting result. The proof is similar to the classical case, but we review it to see some applications of studied properties in the manuscript.

Proposition 4.1. *Let $f : M \longrightarrow N$ be a surjective function such that the set $f^{-1}(b)$ is R-connected, for every $b \in N$.*

Then f induces a bijection between the set of all R-connected components of M, $\mathcal{C}_{\mathbf{M}}^{\mathbf{R}}$, and the set of all S-connected components of N, $\mathcal{C}_{\mathbf{N}}^{\mathbf{S}}$.

Proof. Let $\mathcal{C}_a^R \in \mathcal{C}_{\mathbf{M}}^{\mathbf{R}}$. Then $f(a) \in N$ and there is a unique S-connected component of $\mathcal{C}_{\mathbf{N}}^{\mathbf{S}}$ containing $f(a)$. Therefore, corresponding to an element of $\mathcal{C}_{\mathbf{M}}^{\mathbf{R}}$, a unique component of $\mathcal{C}_{\mathbf{N}}^{\mathbf{S}}$is found .

Now, suppose that $\mathcal{C}_b^S \in \mathcal{C}_{\mathbf{N}}^{\mathbf{S}}$ is an arbitrary S-connected component. Since f is surjective, $f(a) = b$ for some $a \in M$. Then $f^{-1}(b)$ is nonempty and is R-connected by the assumption. Hence, by using Property 1 again, there exists a unique R-connected component $\mathcal{C}_a^R \in \mathcal{C}_{\mathbf{M}}^{\mathbf{R}}$ such that $f^{-1}(b) \subseteq \mathcal{C}_a^R$. More precisely, for an element of $\mathcal{C}_{\mathbf{N}}^{\mathbf{S}}$, we obtain a unique component of $\mathcal{C}_{\mathbf{M}}^{\mathbf{R}}$.

□

5. **Other applications**

As mentioned in the introduction, some of the applications of R-topological and SR-topological spaces and specially R-metric spaces, in the context of fixed point theory, functional analysis, C*-algebras, etc., has investigated in several papers such as [4], [8], [9], [10], [12], [13].

We extended the notion of convexity to R-convexity in [2], and also the notions of R-extreme points, R-convex functions, and R-continuous functions have been investigated there. Some applications of this theory in optimization theory have been presented in [2]. For example, Theorem 5.1 of [2] as follows.

Theorem 5.1. *Suppose that (M, R) is an R-metric vector space, K is a subset of M where $R - ext(K)$ is R-closed and $R - ext(K) \times R - ext(K) \subset R$, and B is an R-compact subset of $\overline{co}^R(R - ext(K))$ such that $R - ext(K) \subset B$. Then every R-affine and R-continuous map $f : M \longrightarrow \mathbb{R}$ attains its maximum and minimum on B at R-extreme points of K. Moreover, the maximum and minimum of f on B is equal with its maximum and minimum on $R - ext(K)$, respectively.*

Also, some conclusions about the local and global minimum points of R-convex maps on R-convex sets, have been considered there. As an example, see the following theorem, Theorem 5.3 of [2]:

Theorem 5.2. *Let (M, R) be an R-vector metric space, $K \subset M$ be an R-convex set, and $f : K \longrightarrow \mathbb{R}$ be an R-convex function which has a local minimum at x_0, then x_0 is also a global minimum of f on $[x_0]_R \cap K$, where $[x_0]_R = \{x \in M; x_0R\ x\}$. Specially, x_0 is a global minimum on K, if $x_0R\ x$, for all $x \in K$.*

See also the Corollary 5.4, and Theorem 5.5 of [2]. In the present paper, we extend the well-known results on the subject of connectivity to R-connectivity and the notion of R-connected component has defined, which are applicable in different branches of mathematics and other sciences. For example, they can be used frequently in graph theory, and sciences in which graph theory is used.

Notes on contributors

The authors approved the final manuscript. Also, the authors have contributed equally to this work and share the firs authorship.

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