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DIGAMMA- AND TRIGAMMA-BASED CONVERGENCE TO THE EULER–MASCHERONI CONSTANT

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We introduce a new sequence convergent to the Euler-Mascheroni constant. It is related to the harmonic sum and its extension - the digamma function. Moreover, we replace the logarithm term appearing in the classical sequence convergent to the Euler-Mascheroni constant by the trigamma function. Finally, some inequalities are given.

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1. **Introduction and motivation**

In electrical engineering, many problems involve mathematical models that go beyond simple algebraic or differential equations. As systems become more complex - especially in areas like signal processing, control theory, and electromagnetic analysis - engineers increasingly rely on advanced mathematical tools. One such tool is the Gamma function, a generalization of the factorial function that extends naturally to real and complex numbers.

Although it originates from pure mathematics, the Gamma function has proven to be extremely useful in engineering contexts. It appears in the solutions of differential equations with non-integer orders, in the analysis of system

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responses using Laplace transforms, and in the modeling of probability distributions relevant to noise, reliability, and queuing systems. Its flexibility and deep connections to other special functions make it a powerful component in both theoretical and applied electrical engineering.

The Gamma function is one of the most important and versatile functions in mathematics. It extends the idea of factorials - normally defined only for positive integers - to real and complex numbers. While the factorial of a number n is given by

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n,$$

the Gamma function generalizes this through an integral definition that works even when n is not an integer. Mathematically, the Gamma function is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \ , \quad x > 0.$$

One of the reasons the Gamma function is so powerful is its deep connections to other special functions, such as the Beta function, the Digamma function, and the Gamma distribution. These connections allow it to model a wide range of real-world phenomena, from signal behavior in electrical circuits to the distribution of lifetimes in reliability engineering.

The Digamma function ψ is defined as the logarithmic derivative of the Gamma function:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The derivatives ψ', ψ'', ψ''' are called respectively the Trigamma-, Tetragamma- and Pentagamma function. In general, $\psi^{(m)}(x)$ are called the polygamma function.

2. **The results**

The Euler-Mascheroni constant is defined as the limit of the sequence

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

It is denoted by $\gamma = 0.577...$. Although these concepts originate in pure mathematics, they have surprising and valuable applications in engineering. In particular, harmonic sums and the Euler-Mascheroni constant appear in the analysis of algorithms, signal processing, and the behavior of electrical systems involving logarithmic or asymptotic growth. For example, in the evaluation of certain integrals in circuit theory or in the modeling of noise and attenuation in communication systems, expressions involving harmonic numbers or γ can arise.

In the recent past, many researchers gave new convergence to the Euler-Mascheroni constant. See, e.g., [2]-[4] and all reference therein.

The harmonic sum

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

is closely related to the Digamma function, since

$$H_n = \psi(n+1) + \gamma, \quad n = 1, 2, 3, \dots . \tag{2.1}$$

The Digamma function admits the following asymptotic expansion:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}} , \quad \text{as } x \to \infty,$$

where $B_j's$ are the Bernoulli's numbers. The first terms are

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \dots .$$

In general, the following asymptotic expansion holds true, as $x \to \infty$:

$$\psi^{(m)}(x) \sim (-1)^{m+1} \sum_{k=0}^{\infty} \frac{(k+m)! B_{k+m}}{k! x^{k+m+1}} , \quad m \in \mathbb{N}.$$

In particular:

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \dots, \tag{2.2}$$

$$\psi''(x) \sim -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6} - \frac{1}{6x^8} + \dots . \tag{2.3}$$

For further properties, see, e.g., [1].

If we look carefully at the previous asymptotics, we can see that

$$\psi(x) \approx \ln x \quad \text{and} \quad \psi'(x) \approx \frac{1}{x}.$$

In consequence, if we replace in γ_n the harmonic sum by $\psi(n)$ (see (2.1)) and the logarithm term $\ln n$ by $-\ln \psi'(n)$, then we are entitled to introduce the following new sequence:

$$v_n = \gamma + \psi(n) + \ln \psi'(n)$$

convergent to γ.

The associated function to the sequence v_n satisfies the following property:

Theorem 2.1. *The function*

$$v(x) = \gamma + \psi(x) + \ln \psi'(x)$$

is strictly increasing on $[1, \infty)$.

Proof. We have to prove that the derivative:

$$v'(x) = \psi'(x) + \frac{\psi''(x)}{\psi'(x)}$$

is positive on $[1, \infty)$. In this sense, we use the following well-known inequalities:

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}$$

and

$$\psi''(x) > -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4},$$

obtained by truncation the asymptotic series (2.2)-(2.3). Thus

$$\begin{aligned} \psi'(x)\, v'(x) &= (\psi'(x))^2 + \psi''(x) \\ &> \left(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}\right)^2 + \left(-\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4}\right) \\ &= \frac{P(x-1)}{900x^{10}}, \end{aligned}$$

where

$$P(x) = 75x^6 + 600x^5 + 1840x^4 + 2830x^3 + 2315x^2 + 950x + 151.$$

As $P(x) > 0$, it follows that $v'(x) > 0$, for all $x \geq 1$.
The proof is completed. □

Corollary 2.1. *The sequence v_n is strictly increasing to γ.*

Proof. We have

$$v_{n+1} - v_n = v(n+1) - v(n) > 0,$$

since the function v is strictly increasing. □

Corollary 2.2. *The following inequalities hold true, for all integers $n \geq 1$:*

$$\gamma - \ln \frac{\pi^2}{6} \leq v_n < \gamma.$$

Proof. Having in mind that the sequence v_n is strictly increasing, we deduce that:

$$v_1 \leq v_n < \lim_{n\to\infty} v_n = \gamma.$$

Here, $v_1 = \gamma - \ln \frac{\pi^2}{6}$, since

$$\psi(1) = -\gamma \quad \text{and} \quad \psi'(1) = \frac{\pi^2}{6}.$$

□

By using the asymptotic series of the digamma and trigamma functions, and the Maple software for symbolic computation, we obtain the following asymptotic series for the sequence

$$v_n \sim \gamma - \frac{1}{24n^2} - \frac{1}{24n^3} - \frac{37}{2880n^4} + \frac{23}{1440n^5} + \frac{5807}{362\,880n^6} + O\left(\frac{1}{n^7}\right), \quad \text{as } n \to \infty.$$

Inspired by this asymptotic series, we give the following:

Theorem 2.2. *The following inequalities hold true:*

$$\gamma - \frac{1}{24n^2} - \frac{1}{24n^3} - \frac{37}{2880n^4} < v_n < \gamma - \frac{1}{24n^2} - \frac{1}{24n^3} - \frac{37}{2880n^4} + \frac{23}{1440n^5}.$$

The proof is similar to the proof of the previous Theorem. We omit it for sake of simplicity.

The Gamma function serves as a powerful mathematical tool in engineering, especially in fields that involve complex modeling, signal analysis, and system dynamics. Its ability to generalize factorials and connect with other special functions makes it essential in solving differential equations, evaluating integrals, and working with probability distributions. In electrical engineering, the Gamma function appears in areas such as control theory, communication systems, and reliability analysis, offering elegant solutions to problems that involve non-integer behavior or asymptotic analysis. Its continued relevance highlights the deep connection between advanced mathematics and practical engineering applications.

We are convinced that our ideas presented in this paper are useful to compute expressions involving the gamma function and related functions.

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