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A NEW DATKO TYPE RESULT CONCERNING UNIFORM H-DICHOTOMY IN MEAN OF REVERSIBLE STOCHASTIC SKEW-EVOLUTION SEMIFLOWS

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The objective of this paper is to provide a novel Datko type characterization for the uniform h-dichotomy in mean with respect to reversible stochastic skew-evolution semiflows in Banach spaces. We delineate a necessary and sufficient condition for the uniform h-dichotomy, utilizing a critical set of growth rates. As a particular case, we also offer a Datko type result for the uniform polynomial dichotomy in mean.

Keywords: reversible stochastic skew-evolution semiflows, Datko type characterization, uniform *h*-dichotomy in mean, uniform polynomial dichotomy in mean.
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1. Introduction

The asymptotic behavior of dynamical systems in Banach spaces has witnessed significant advancements in recent decades, particularly in areas such as boundedness, (in)stability, and dichotomy. These developments have been meticulously chronicled in a plethora of studies (see [1-11] and the references contained therein), spanning both deterministic and stochastic frameworks. The concept of skew-evolution semiflow, an extension of classical concepts such as C_0-semigroup, evolution operator and linear skew-product semiflow, provides a fundamental and highly versatile framework for the qualitative theory of dynamical systems. In the deterministic case, the notion of skew-evolution semiflow was initially introduced by Megan, Stoica and Buliga in [12]. Commencing with the seminal work [12], a series of notably significant papers have been published that delve into the asymptotic properties of skew-evolution semiflows (see [13-18]).

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As is well known, exponential and polynomial behaviors are the two most important behaviors in the theory of dynamical systems. In recent years, a significant amount of research has been dedicated to the study of exponential/polynomial (in)stability and dichotomy of stochastic skew-evolution semiflows. For example, Stoica and Megan [19] studied the Datko type characterization for nonuniform exponential dichotomy in mean square of stochastic skew-evolution semiflows in Hilbert spaces, using certain techniques under deterministic cases. Hai [20] utilized Banach function spaces to formulate both discrete and continuous versions of a Datko type theorem for uniform polynomial stability in mean, as well as uniform polynomial instability in mean, of stochastic skew-evolution semiflows in Banach spaces. In [21], Személy Fülöp, Megan and Borlea examined a type of asymptotic behavior that does not necessarily follow an exponential or polynomial pattern, the so-called uniform *h*-stability in mean. In their study, they discerned several contractive attributes, logarithmic criteria, and majorization criteria for the uniform *h*-stability in mean of stochastic skew-evolution semiflows. Furthermore, Személy Fülöp discussed in [22] the uniform *h*-dichotomy in mean for reversible stochastic skew-evolution semiflows, and successfully derived two Datko type characterizations that meet the criteria of uniform *h*-dichotomy in mean by creating a special set of growth rates. As particular cases of the main results, the author also obtained two integral characterizations of the uniform exponential dichotomy in mean. However, the primary limitation of study [22] lies in its construction of a set of growth rates that does not incorporate the polynomial function $h(t)=t+1$. Consequently, this omission makes it unfeasible to conclude that reversible stochastic skew-evolution semiflows satisfy the uniform polynomial dichotomy in mean.

Inspired by [22], our main objective is to derive a novel Datko type condition for the uniform *h*-dichotomy in mean of reversible stochastic skew-evolution semiflows in Banach spaces, utilizing a critical set of growth rates. Additionally, as a particular case, we present a Datko type result for the uniform polynomial dichotomy in mean.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminary facts that will be employed in the sequel. We denote by $\mathbb{R}_+=[0,+\infty)$, by $\Delta=\{(t,s)\in\mathbb{R}_+^2 : t\geq s\}$ and by $T=\{(t,s,t_0)\in\mathbb{R}_+^3 : t\geq s\geq t_0\}$. Let $\Omega=(\Omega,\mathcal{B},\mu)$ be a probability space, X a Banach space, $\mathcal{L}(X)$ the Banach algebra of bounded linear operators from X to itself. Moreover, we denote by $L(\Omega,X,\mu)$ the Banach space of all Bochner measurable functions $z:\Omega\to X$ such that

$$\|z\|_1 := \int_\Omega \|z(\omega)\|\, d\mu(\omega) < \infty,$$

identified if they are equal μ–a.e.

Definition 2.1. (see [20]) *A measurable random field* $\varphi:\Delta\times\Omega\to\Omega$ *is called a stochastic evolution semiflow if for all* $(t,s,t_0)\in T$, $\omega\in\Omega$, *the following conditions hold:*

(i) $\varphi(t,t,\omega)=\omega$;

(ii) $\varphi(t,t_0,\omega)=\varphi(t,s,\varphi(s,t_0,\omega))$.

Definition 2.2. (see [20]) *Let* φ *be a stochastic evolution semiflow. A measurable map* $\Phi:\Delta\times\Omega\to\mathcal{L}(X)$ *is called a stochastic evolution cocycle associated to* φ *if for all* $(t,s,t_0)\in T$, $\omega\in\Omega$, *the following conditions hold:*

(i) $\Phi(t,t,\omega)=I$, *(*I *stands for the identity operator);*

(ii) $\Phi(t,t_0,\omega)=\Phi(t,s,\varphi(s,t_0,\omega))\Phi(s,t_0,\omega)$.

In this case, the pair $C=(\varphi,\Phi)$ *is called a stochastic skew-evolution semiflow.*

If the stochastic evolution cocycle Φ is bijective for all $(t,s,\omega)\in\Delta\times\Omega$, then we say that Φ is reversible.

Definition 2.3. (see [22]) *A stochastic skew-evolution semiflow* $C=(\varphi,\Phi)$ *is called strongly measurable if the mapping* $s\mapsto\int_\Omega\|\Phi(s,t_0,\omega)x(\omega)\|d\mu(\omega)$ *is measurable on* $[t_0,\infty)$, *for all* $(t_0,\omega)\in\mathbb{R}_+\times\Omega$.

Definition 2.4. (see [22]) *A map* $P:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ *is called a family of projectors if* $P^2(s,\omega)=P(s,\omega)$ *for all* $(s,\omega)\in\mathbb{R}_+\times\Omega$.

Obviously, if $P:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ is a family of projectors, then the map $Q:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$, $Q(s,\omega)=I-P(s,\omega)$ is also a family of projectors, which is called the complementary family of projectors of P.

Definition 2.5. (see [22]) *A family of projectors* $P:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ *is called invariant to the stochastic skew-evolution semiflow* $C=(\varphi,\Phi)$ *if*

$$\Phi(t,s,\omega)P(s,\omega)=P(t,\varphi(t,s,\omega))\Phi(t,s,\omega),$$

for all $(t,s,\omega)\in\Delta\times\Omega$.

In what follows, if $P:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ is invariant to the stochastic skew-evolution semiflow C, then we say that (C,P) is a dichotomic pair.

Definition 2.6. *A map* $h:\mathbb{R}_+\to[1,\infty)$ *is called a growth rate if it is nondecreasing and bijective.*

Definition 2.7. (see [22]) *The dichotomic pair* (C,P) *is called uniformly h-dichotomic in mean if there are two constants* $N>1$ *and* $\nu>0$ *such that*

(uhd₁m) $h(t)^\nu\int_\Omega\|\Phi(t,t_0,\omega)P(t_0,\omega)x_0(\omega)\|d\mu(\omega)$

$$\leq Nh(s)^{v}\int_{\Omega}\left\|\Phi(s,t_0,\omega)P(t_0,\omega)x_0(\omega)\right\|d\mu(\omega);$$

(uhd$_2$m) $h(t)^{v}\int_{\Omega}\left\|\Phi(s,t_0,\omega)Q(t_0,\omega)x_0(\omega)\right\|d\mu(\omega)$

$$\leq Nh(s)^{v}\int_{\Omega}\left\|\Phi(t,t_0,\omega)Q(t_0,\omega)x_0(\omega)\right\|d\mu(\omega),$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$.

Example 2.1. *Let* Ω *be the probability space defined in Example 2.1 of [10],* $X=\mathbb{R}^2$ *with Euclidean norm and* $\varphi:\Delta\times\Omega\to\Omega$ *be a stochastic evolution semiflow on* Ω. *Then,* $\Phi:\Delta\times\Omega\to\mathcal{L}(X)$ *defined by*

$$\Phi(t,s,\omega)(x_1,x_2)=\left(\frac{h(s)}{h(t)}x_1,\frac{h(t)}{h(s)}x_2\right)$$

is a stochastic evolution cocycle associated to φ.

Let $P:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ *be the family of projectors defined by*

$$P(s,\omega)(x_1,x_2)=(x_1,0)$$

for all $(s,\omega)\in\mathbb{R}_+\times\Omega$ *and* $(x_1,x_2)\in\mathbb{R}^2$. *The complementary family of projectors* $Q:\mathbb{R}_+\times\Omega\to\mathcal{L}(X)$ *is defined by*

$$Q(s,\omega)(x_1,x_2)=(0,x_2)$$

for all $(s,\omega)\in\mathbb{R}_+\times\Omega$ *and* $(x_1,x_2)\in\mathbb{R}^2$.

Then, the dichotomic pair (C,P) *satisfies Definition 2.7 for* $v=1$ *and for all* $N>1$. *It results that* (C,P) *is uniformly h-dichotomic in mean.*

Remark 2.1. *In Definition 2.7, if we consider*

(i) $h(t)=e^{t}$, *then the concept of uniform exponential dichotomy in mean is obtained.*

(ii) $h(t)=t+1$, *then we have the uniform polynomial dichotomy in mean concept.*

(iii) $Q=0$, *then the concept of uniform h-stability in mean is obtained (see [21]).*

(iv) $P=0$, *then we have the uniform h-instability in mean concept (see [12]).*

Definition 2.8. (see [22]) *We say that the dichotomic pair* (C,P) *has uniform h-growth in mean if there are two constants* $M>1$ *and* $\alpha>0$ *such that*

(uhg$_1$m) $h(s)^{\alpha}\int_{\Omega}\left\|\Phi(t,t_0,\omega)P(t_0,\omega)x_0(\omega)\right\|d\mu(\omega)$

$$\leq Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi(s,t_0,\omega)P(t_0,\omega)x_0(\omega)\right\|d\mu(\omega);$$

(uhg$_2$m) $h(s)^{\alpha}\int_{\Omega}\left\|\Phi(s,t_0,\omega)Q(t_0,\omega)x_0(\omega)\right\|d\mu(\omega)$

$$\leq Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi(t,t_0,\omega)Q(t_0,\omega)x_0(\omega)\right\|d\mu(\omega),$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$.

Remark 2.2. *In Definition 2.8, if we consider*

(i) $h(t)=e^t$, *then the concept of uniform exponential growth in mean is obtained.*

(ii) $h(t)=t+1$, *then we have the uniform polynomial growth in mean concept.*

Proposition 2.1. (see [22]) *Let* Φ *be a reversible stochastic evolution cocycle. Then the dichotomic pair* (C,P) *is uniformly h-dichotomic in mean if and only if there are* $N>1$ *and* $\nu>0$ *such that*

$$h(t)^{\nu}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$
$$\leq Nh(s)^{\nu}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega); \tag{2.1}$$

$$h(t)^{\nu}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$
$$\leq Nh(s)^{\nu}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega), \tag{2.2}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$.

Proposition 2.2. (see [22]) *Let* Φ *be a reversible stochastic evolution cocycle. Then the dichotomic pair* (C,P) *has uniform h-growth in mean if and only if there are* $M>1$ *and* $\alpha>0$ *such that*

$$h(s)^{\alpha}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$
$$\leq Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega); \tag{2.3}$$

$$h(s)^{\alpha}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$
$$\leq Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega), \tag{2.4}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$.

3. Main results

This section is dedicated to formulating a novel Datko type characterization theorem for the concept of uniform *h*-dichotomy in mean. In order to do this, we consider an important class of functions:

$\mathcal{H}$ the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with the properties:

- $h(t)\geq t+1$, for all $t\geq 0$;

- for all $\beta > 0$, there exists $\xi > 1$ such that $\int_0^t h(s)^{\beta-1} ds \le \xi h(t)^{\beta}$, for all $t \ge 0$;
- there exists $\eta \ge 2$ such that $h(\eta h(t)) \le \eta^2 h(t)$, for all $t \ge 0$.

Throughout this section, we suppose that Φ is reversible and $C = (\varphi, \Phi)$ is a strongly measurable stochastic skew-evolution semiflow.

Theorem 3.1. *Let $h \in \mathrm{H}$ and the dichotomic pair (C, P) has uniform h-growth in mean. Then (C, P) is uniformly h-dichotomic in mean if and only if there exist $D > 1$ and $d > 0$ such that*

$$\int_{t_0}^{t} \frac{\int_\Omega \left\| \Phi^{-1}(s,t_0,\omega) P(s,\varphi(s,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(s)^{d+1}} ds \le \frac{D \int_\Omega \left\| \Phi^{-1}(t,t_0,\omega) P(t,\varphi(t,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(t)^d}, \tag{3.1}$$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$;

$$\int_{t_0}^{t} \frac{ds}{h(s)^{d+1} \int_\Omega \left\| \Phi^{-1}(s,t_0,\omega) Q(s,\varphi(s,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)} \le \frac{D}{h(t)^d \int_\Omega \left\| \Phi^{-1}(t,t_0,\omega) Q(t,\varphi(t,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}, \tag{3.2}$$

for all $(t,s,t_0,\omega) \in T \times \Omega$ and $x_0 \in L(\Omega, X, \mu)$ with $Q(t,\varphi(t,t_0,\omega)) x_0(\omega) \ne 0$.

***Proof*.** Necessity. If (C, P) is uniformly h-dichotomic in mean, then by Proposition 2.1, there are $N > 1$ *and* $\nu > 0$ such that the conditions (2.1) and (2.2) are satisfied. Let $d \in (0, \nu)$.

By (2.1) we have

$$\begin{aligned}
&\int_{t_0}^{t} \frac{\int_\Omega \left\| \Phi^{-1}(s,t_0,\omega) P(s,\varphi(s,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(s)^{d+1}} ds \\
&\le N \int_{t_0}^{t} \left(\frac{h(s)}{h(t)} \right)^{\nu} \frac{\int_\Omega \left\| \Phi^{-1}(t,t_0,\omega) P(t,\varphi(t,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(s)^{d+1}} ds \\
&= \frac{N \int_\Omega \left\| \Phi^{-1}(t,t_0,\omega) P(t,\varphi(t,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(t)^{\nu}} \int_{t_0}^{t} h(s)^{\nu-d-1} ds \\
&\le \frac{N \int_\Omega \left\| \Phi^{-1}(t,t_0,\omega) P(t,\varphi(t,t_0,\omega)) x_0(\omega) \right\| d\mu(\omega)}{h(t)^{\nu}} \int_{0}^{t} h(s)^{\nu-d-1} ds
\end{aligned}$$

$$\le \frac{N\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(t)^{v}}\cdot\xi h(t)^{v-d}$$
$$=\frac{D\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(t)^{d}},$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x_0\in L(\Omega,X,\mu)$, where $D=N\xi$.

Analogously, by (2.2) we have

$$\int_{t_0}^{t}\frac{ds}{h(s)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}$$
$$\le N\int_{t_0}^{t}\left(\frac{h(s)}{h(t)}\right)^{v}\frac{1}{h(s)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}ds$$
$$=\frac{N}{h(t)^{v}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}\int_{t_0}^{t}h(s)^{v-d-1}ds$$
$$\le\frac{N}{h(t)^{v}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}\int_{0}^{t}h(s)^{v-d-1}ds$$
$$\le\frac{N\xi h(t)^{v-d}}{h(t)^{v}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}$$
$$=\frac{D}{h(t)^{d}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)},$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x_0\in L(\Omega,X,\mu)$ with $Q(t,\varphi(t,t_0,\omega))x_0(\omega)\neq 0$.

Sufficiency. Let $(t,s,t_0,\omega)\in T\times\Omega$. Firstly, we prove that (3.1) implies (2.1).

Take $t\ge\eta h(s)$. By (2.3) and (3.1), we obtain that

$$\frac{\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(s)^{d}}$$
$$=\frac{1}{(\eta-1)h(s)}\int_{h(s)}^{\eta h(s)}\frac{\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(s)^{d}}d\tau$$
$$\le\frac{M}{(\eta-1)h(s)}\int_{h(s)}^{\eta h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\alpha}\frac{\int_{\Omega}\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(s)^{d}}d\tau$$

$$=\frac{M}{(\eta-1)h(s)}\int_{h(s)}^{\eta h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\alpha+d+1}\frac{\int_\Omega\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(\tau)^{d+1}}d\tau$$

$$\le\frac{M}{\eta-1}\int_{h(s)}^{\eta h(s)}\left(\frac{h(\eta h(s))}{h(s)}\right)^{\alpha+d+1}\frac{\int_\Omega\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(\tau)^{d+1}}d\tau$$

$$\le\frac{M\eta^{2(\alpha+d+1)}}{\eta-1}\int_{h(s)}^{\eta h(s)}\frac{\int_\Omega\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(\tau)^{d+1}}d\tau$$

$$\le\frac{M\eta^{2(\alpha+d+1)}}{\eta-1}\int_{s}^{t}\frac{\int_\Omega\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(\tau)^{d+1}}d\tau$$

$$\le\frac{M\eta^{2(\alpha+d+1)}}{\eta-1}\int_{t_0}^{t}\frac{\int_\Omega\left\|\Phi^{-1}(\tau,t_0,\omega)P(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{h(\tau)^{d+1}}d\tau$$

$$\le\frac{MD\eta^{2(\alpha+d+1)}}{(\eta-1)h(t)^d}\int_\Omega\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega).$$

On the other hand, using (2.3), we have, if $t\in[s,\eta h(s))$,

$$h(t)^d\int_\Omega\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\le M\left(\frac{h(t)}{h(s)}\right)^{\alpha+d}h(s)^d\int_\Omega\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\le M\left(\frac{h(\eta h(s))}{h(s)}\right)^{\alpha+d}h(s)^d\int_\Omega\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\le M\eta^{2(\alpha+d)}h(s)^d\int_\Omega\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega).$$

Based on the above two cases, we conclude that there exist

$$N=\max\{MD\eta^{2(\alpha+d+1)}(\eta-1)^{-1},M\eta^{2(\alpha+d)}\}$$

and $\nu=d$ such that (2.1) holds for all $(t,s,t_0,\omega)\in T\times\Omega$ and $x_0\in L(\Omega,X,\mu)$.

Secondly, we prove that (3.2) implies (2.2). Take $t\ge\eta h(s)$. By (2.4) and (3.2), we obtain that

$$\frac{1}{h(s)^d\int_\Omega\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}$$

$$=\frac{1}{(\eta-1)h(s)}\int_{h(s)}^{\eta h(s)}\frac{d\tau}{h(s)^d\int_\Omega\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}$$

$$\leq \frac{M}{(\eta-1)h(s)}\int_{h(s)}^{\eta h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\alpha}\frac{1}{h(s)^{d}\int_{\Omega}\left\|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}d\tau$$

$$=\frac{M}{\eta-1}\int_{h(s)}^{\eta h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\alpha+d+1}\frac{1}{h(\tau)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}d\tau$$

$$\leq \frac{M\eta^{2(\alpha+d+1)}}{\eta-1}\int_{h(s)}^{\eta h(s)}\frac{1}{h(\tau)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}d\tau$$

$$\leq \frac{M\eta^{2(\alpha+d+1)}}{\eta-1}\int_{t_0}^{t}\frac{1}{h(\tau)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(\tau,t_0,\omega)Q(\tau,\varphi(\tau,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}d\tau$$

$$\leq \frac{MD\eta^{2(\alpha+d+1)}}{(\eta-1)h(t)^{d}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}.$$

Additionally, if $t\in[s,\eta h(s))$, by (2.4) we have

$$h(t)^{d}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\leq M\left(\frac{h(t)}{h(s)}\right)^{\alpha+d}h(s)^{d}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\leq M\left(\frac{h(\eta h(s))}{h(s)}\right)^{\alpha+d}h(s)^{d}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)$$

$$\leq M\eta^{2(\alpha+d)}h(s)^{d}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega).$$

Hence, we obtain that (2.2) holds with

$$N=\max\{MD\eta^{2(\alpha+d+1)}(\eta-1)^{-1},M\eta^{2(\alpha+d)}\}\text{ and }\nu=d.$$

Finally, according to Proposition 2.1, we can conclude that (C,P) is uniformly h-dichotomic in mean.

Corollary 3.1. *We suppose that* $h\in\mathrm{H}$ *and there are two constants* $M>1$ *and* $\alpha>0$ *such that*

$$h(s)^{\alpha}\int_{\Omega}\left\|\Phi(t,t_0,\omega)x_0(\omega)\right\|d\mu(\omega)\leq Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi(s,t_0,\omega)x_0(\omega)\right\|d\mu(\omega),$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$. *Then* (C,P) *is uniformly h-stable in mean if and only if there exist* $D>1$ *and* $d>0$ *such that*

$$\int_{t_0}^{t}\frac{\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)x_0(\omega)\right\|d\mu(\omega)}{h(s)^{d+1}}ds\leq\frac{D\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)x_0(\omega)\right\|d\mu(\omega)}{h(t)^{d}},\tag{3.3}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0\in L(\Omega,X,\mu)$.

Proof. It is a particular case of Theorem 3.1 for $Q=0$.

Corollary 3.2. *We suppose that* $h \in \mathrm{H}$ *and there are two constants* $M > 1$ *and* $\alpha > 0$ *such that*

$$h(s)^{\alpha}\int_{\Omega}\left\|\Phi(s,t_0,\omega)x_0(\omega)\right\|d\mu(\omega) \le Mh(t)^{\alpha}\int_{\Omega}\left\|\Phi(t,t_0,\omega)x_0(\omega)\right\|d\mu(\omega),$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0 \in L(\Omega,X,\mu)$. *Then* (C,P) *is uniformly h-unstable in mean if and only if there exist* $D>1$ *and* $d>0$ *such that*

$$\int_{t_0}^{t}\frac{ds}{h(s)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)x_0(\omega)\right\|d\mu(\omega)} \le \frac{D}{h(t)^{d}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)x_0(\omega)\right\|d\mu(\omega)}, \tag{3.4}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0 \in L(\Omega,X,\mu)$ *with* $x_0(\omega)\neq 0$.

Proof. It is a particular case of Theorem 3.1 for $P=0$.

Corollary 3.3. *Let the dichotomic pair* (C,P) *has uniform polynomial growth in mean. Then* (C,P) *is uniformly polynomially dichotomic in mean if and only if there exist* $D>1$ *and* $d>0$ *such that*

$$\begin{aligned}&\int_{t_0}^{t}\frac{\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)P(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{(s+1)^{d+1}}ds\\ &\qquad\le\frac{D\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)P(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}{(t+1)^{d}},\end{aligned} \tag{3.5}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0 \in L(\Omega,X,\mu)$;

$$\begin{aligned}&\int_{t_0}^{t}\frac{ds}{(s+1)^{d+1}\int_{\Omega}\left\|\Phi^{-1}(s,t_0,\omega)Q(s,\varphi(s,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)}\\ &\qquad\le\frac{D}{(t+1)^{d}\int_{\Omega}\left\|\Phi^{-1}(t,t_0,\omega)Q(t,\varphi(t,t_0,\omega))x_0(\omega)\right\|d\mu(\omega)},\end{aligned} \tag{3.6}$$

for all $(t,s,t_0,\omega)\in T\times\Omega$ *and* $x_0 \in L(\Omega,X,\mu)$ *with* $Q(t,\varphi(t,t_0,\omega))x_0(\omega)\neq 0$.

Proof. It is a particular case of Theorem 3.1 for $h(t)=t+1$.

4. Conclusions

In this paper, we have investigated the Datko type characterization for the uniform *h*-dichotomy in mean of reversible stochastic skew-evolution semiflows in Banach spaces, using an important class of growth rates. More precisely, we gave a new theorem of Datko type for uniform *h*-dichotomy in mean (see Theorem 3.1). As particular cases, a Datko type characterization for uniform *h*-stability in mean and a necessary and sufficient condition for the uniform *h*-instability in mean are obtained (see Corollaries 3.1 and 3.2). Additionally, we also obtained a Datko type result for the uniform polynomial dichotomy in mean (see Corollary 3.3). In the future, we will continue to study the variants of these results in the discrete time case and generalizations for the (non)uniform *h*-trichotomies behaviors.

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