SYMMETRICALLY PSEUDO-AMENABLE BANACH ALGEBRAS

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We introduce and study a new notion of amenability called symmetric pseudo-amenability. We obtain some properties of symmetrically pseudo-amenable Banach algebras and with examples, we compare this type of amenability with some other types of amenability. We also provide some special classes of symmetrically pseudo-amenable Banach algebras. Finally, Jordan derivations and Lie derivations from a class of Banach algebras into appropriate Banach bimodules are investigated using the notion of symmetric pseudo-amenability.

Keywords: symmetrically pseudo-amenable, pseudo-amenable, amenable, Jordan derivation, Lie derivation.

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1. Introduction

B. E. Johnson studied cohomology of Banach algebras in [13] and defined the concept of amenable Banach algebra which was based on the amenability of locally compact groups and proved in [14] that a Banach algebra $\mathfrak{U}$ is amenable if and only if $\mathfrak{U}$ has a bounded approximate diagonal. In the following, many studies were conducted on amenability of Banach algebras, and various other types of amenability have been introduced and studied, see [5, 6, 16, 23] for a comprehensive survey of results of this type. In [15], Johnson introduced symmetric amenable Banach algebras as a special class of amenable Banach algebras. He called a Banach algebra $\mathfrak{U}$ is symmetrically amenable if it has a bounded approximate diagonal consisting of symmetric tensors, and then applied this concept to the study of Jordan derivations and Lie derivations. The study of Jordan derivations and Lie derivations also has a long history. The common problem regarding these mappings is when Jordan or Lie derivations can be characterized in terms of derivations? Many studies have been carried out in line with the this problem raised, and here we only refer to Johnson's results. Johnson showed in [15, Theorem 6.2] that any continuous Jordan derivation from a symmetrically amenable Banach algebra $\mathfrak{U}$ into a Banach $\mathfrak{U}$-bimodule is a derivation. As a consequence he obtained the same result for continuous Jordan derivations on arbitrary C^*-algebras, although not every C^*-algebra is symmetrically amenable. Also, in [15, Theorem 9.2] Johnson showed that any continuous Lie derivation from a symmetrically amenable Banach algebra $\mathfrak{U}$ into a Banach $\mathfrak{U}$-bimodule decomposed into the sum of a continuous derivation and a continuous center-valued trace, and as a consequence he obtained the same result for continuous Lie derivations on arbitrary C^*-algebras. To see the historical course and other results in

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this study path, we refer to [1, 2, 3, 4, 9, 10, 11, 17, 18, 19, 20, 22, 24] and the references therein.

As a generalization of amenability, F. Ghahramani and Y. Zhang in [7] introduced and studied the notion of pseudo-amenability, which is based on existence of an approximate diagonal for Banach algebras. Precisely, a Banach algebra $\mathfrak{U}$ is called pseudo-amenable if it has an approximate diagonal which is not necessarily bounded. According to this definition and with the idea of the definition of symmetric amenability, in the continuation of studies related to amenability, in this article we introduce the concept of symmetric pseudo-amenability. We say Banach algebra $\mathfrak{U}$ is *symmetrically pseudo-amenable* if it has an approximate diagonal consisting of symmetric tensors which is not necessarily bounded. This concept is a generalization of symmetrically amenable Banach algebras, which is a special class of pseudo-amenable Banach algebras. In this article, we study symmetrically pseudo-amenable Banach algebras and identify some of their properties and compare this type of amenability with some other types of amenability with examples, and we also present some special classes of Banach algebras that are symmetrically pseudo-amenable. According to [15, Theorem 6.2] and [15, Theorem 9.2], the question arises whether it is possible to obtain Johnson's results for Jordan derivations and Lie derivations of other suitable Banach algebras into suitable Banach modules? We give answers to this question using the concept of symmetric pseudo-amenability and obtain generalizations of [15, Theorem 6.2] and [15, Theorem 9.2].

This article is organized as follows. In section 2, definitions, notions and required tools are introduced. Section 3 is devoted to the study of properties of symmetrically pseudo-amenable Banach algebras and examples. In section 4, we present some special classes of symmetrically pseudo-amenable Banach algebras. In sections 5 and 6, respectively, Jordan and Lie derivations from a class of Banach algebras into appropriate Banach bimodules are investigated using the concept of symmetric pseudo-amenability.

2. **Preliminaries**

In this section we fix the notation, and give some basic definitions and points which will be used in the next sections. Let $\mathfrak{U}$ be an algebra and X be a $\mathfrak{U}$-bimodule. Note that $\mathcal{Z}_{\mathfrak{U}}(X)$ represents the center of X, which is defined as

$$\mathcal{Z}_{\mathfrak{U}}(X) = \{x \in X \mid ax = xa \text{ for all } a \in \mathfrak{U}\}.$$

A mapping $f : \mathfrak{U} \to X$ is central if $f(\mathfrak{U}) \subseteq \mathcal{Z}_{\mathfrak{U}}(X)$.

Recall that a linear mapping $\delta : \mathfrak{U} \to X$ is called a derivation if

$$\delta(ab) = \delta(a)b + a\delta(b);$$

a Jordan derivation if

$$\delta(ab + ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a);$$

and a Lie derivation if

$$\delta(ab - ba) = \delta(a)b + a\delta(b) - \delta(b)a - b\delta(a),$$

for all $a, b \in \mathfrak{U}$. A linear mapping $\tau : \mathfrak{U} \to X$ is a trace, if $\tau([a, b]) = 0$ for all $a, b \in \mathfrak{U}$, where $[a, b] = ab - ba$ (Lie product). Note that a Lie derivation is central if and only if it is a central trace.

Assume that $\mathfrak{U}$ is a Banach algebra and X is a Banach $\mathfrak{U}$-bimodule. The space of all bounded derivations from $\mathfrak{U}$ into X is denoted by $Z^1(\mathfrak{U}, X)$. A derivation δ is called inner derivation, if there is $x \in X$ such that $\delta(a) = ax - xa$ for all $a \in \mathfrak{U}$. Each inner derivation is bounded and $N^1(\mathfrak{U}, X)$ is the space of all inner derivations from $\mathfrak{U}$ into X. The first cohomology group of $\mathfrak{U}$ with coefficient in X is the quotient space $H^1(\mathfrak{U}, X) = Z^1(\mathfrak{U}, X)/N^1(\mathfrak{U}, X)$. We observe that X^* is a Banach $\mathfrak{U}$-bimodule with the following module operations

$$\langle af, x\rangle = \langle f, xa\rangle, \quad \langle fa, x\rangle = \langle f, ax\rangle$$

for $a \in \mathfrak{U}$, $x \in X$ and $f \in X^*$. We call X^* the dual bimodule of X. Recall that a Banach algebra $\mathfrak{U}$ is said to be amenable if for every Banach $\mathfrak{U}$-bimodule X, we have $H^1(\mathfrak{U}, X^*) = \{0\}$. The notion of an amenable Banach algebra was introduced by Johnson in 1972 [13]. A net $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ in the projective tensor product $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ is called approximate diagonal if satisfies the following two conditions

(1) $a\mathbf{t}_\lambda - \mathbf{t}_\lambda a \longrightarrow 0$;
(2) $\pi(\mathbf{t}_\lambda)a \longrightarrow a$

for all $a \in \mathfrak{U}$, where the operations on $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ are defined through

$$a(b \otimes c) = ab \otimes c, \quad (b \otimes c)a = b \otimes ca$$

for all $a, b, c \in \mathfrak{U}$. Here and in the sequel π always denotes the product morphism from $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ into $\mathfrak{U}$, specified by $\pi(a \otimes b) = ab$ for all $a, b \in \mathfrak{U}$. In [14] (see also [5, 2.9.65]), it is proved that a Banach algebra $\mathfrak{U}$ is amenable if and only if $\mathfrak{U}$ has a bounded approximate diagonal. The flip map on $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ is defined by

$$(a \otimes b)^\circ = b \otimes a$$

for $a, b \in \mathfrak{U}$ and an element $\mathbf{t} \in \mathfrak{U}\widehat{\otimes}\mathfrak{U}$ is called symmetric if $\mathbf{t}^\circ = \mathbf{t}$. Johnson in [15] is introduced symmetric amenability of Banach algebras. He called a Banach algebra $\mathfrak{U}$ is symmetrically amenable if it has a bounded approximate diagonal consisting of symmetric tensors. The opposite algebra $\mathfrak{U}^\circ$ is the Banach space $\mathfrak{U}$ with product $a \circ b = ba$. An approximate diagonal in $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ for $\mathfrak{U}^\circ$ is a net $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ if it meets the following two conditions

$(1)^\circ$ $a \circ \mathbf{t}_\lambda - \mathbf{t}_\lambda \circ a \to 0$;
$(2)^\circ$ $a\pi^\circ(\mathbf{t}_\lambda) - a \to 0$

for all $a \in \mathfrak{U}$, where

$$a \circ (b \otimes c) = b \otimes ac, \quad (b \otimes c) \circ a = ba \otimes c \quad \text{and} \quad \pi^\circ(b \otimes c) = cb$$

for all $a, b, c \in \mathfrak{U}$. There are a number of obvious relationships between these operations, for example $a \circ \mathbf{t}^\circ = (a\mathbf{t})^\circ$ $(a \in \mathfrak{U}, \mathbf{t} \in \mathfrak{U}\widehat{\otimes}\mathfrak{U})$. The Banach algebra $\mathfrak{U}$ is symmetrically amenable if and only if there is a bounded net $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ such that satisfies (1), (2), $(1)^\circ$ and $(2)^\circ$ ([15, Proposition 2.2]). The properties and examples of symmetrically amenable Banach algebras can be found in [15].

In [6], the authors have introduced and studied a generalization of amenability called approximate amenability, which is based on a property of derivations from the algebra. Precisely, a Banach algebra $\mathfrak{U}$ is approximately amenable if, for every Banach $\mathfrak{U}$-bimodule X, every bounded derivation d from $\mathfrak{U}$ into the dual bimodule X^* is approximately inner, which means that there is a net $\{x_\lambda\}_{\lambda\in\Lambda} \subset X^*$ such that $d(a) = \lim_\lambda(ax_\lambda - x_\lambda a)$ for all $a \in \mathfrak{U}$. In [7] pseudo-amenability is presented as another generalization of amenability, which is based on approximate diagonals. Precisely, a Banach algebra $\mathfrak{U}$ is pseudo-amenable if it has an approximate diagonal which is not necessarily bounded. In [7, Theorem 3.1] it is proved that a Banach algebra $\mathfrak{U}$ is approximately amenable if and only if the unitization $\mathfrak{U}^\sharp$ of $\mathfrak{U}$ is pseudo-amenable (also, see [8]). The properties and examples of these kinds of amenabilities can be found in [6, 7, 8].

3. Basic properties and comparisons

In this section, we present definition and some basic properties of symmetric pseudo-amenability, and with examples, we compare this type of amenability with some other types. Throughout this section, $\mathfrak{U}$ is a Banach algebra.

Definition 3.1. The Banach algebra $\mathfrak{U}$ is *symmetrically pseudo-amenable* if it has an approximate diagonal consisting of symmetric elements.

In view of this definition, it is clear that a symmetrically amenable Banach algebras is symmetrically pseudo-amenable and a symmetrically pseudo-amenable Banach algebra is pseudo-amenable. In the next proposition, a condition equivalent to the symmetric pseudo-amenablity is presented, the proof of which is similar to [15, Proposition 2.2], and its proof is omitted.

Proposition 3.2. $\mathfrak{U}$ *is symmetrically pseudo-amenable if and only if there exists a net* $\{\boldsymbol{t}_\lambda\}_{\lambda\in\Lambda}$ *in* $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ *which satisfies*

(i) $a\boldsymbol{t}_\lambda - \boldsymbol{t}_\lambda a \longrightarrow 0$;
(ii) $\pi(\boldsymbol{t}_\lambda)a \longrightarrow a$;
(iii) $a \circ \boldsymbol{t}_\lambda - \boldsymbol{t}_\lambda \circ a \longrightarrow 0$;
(iv) $a\pi^\circ(\boldsymbol{t}_\lambda) \longrightarrow a$.

for all $a \in \mathfrak{U}$.

It should be noted that if $\mathbf{t}_\lambda$ satisfies in conditions (i) to (iv) of the Proposition 3.2, then for each $a \in \mathfrak{U}$, $a\pi(\mathbf{t}_\lambda) \longrightarrow a$ can be concluded from (i) and (ii), and $\pi^\circ(\mathbf{t}_\lambda)a \longrightarrow a$ can be concluded from (iii) and (iv).

We know that in order to show that the Banach algebra $\mathfrak{U}$ is pseudo-amenable, it is sufficient to show that for each finite subset F of $\mathfrak{U}$ and each $\varepsilon > 0$ there is an element $\mathbf{t}$ of $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ with $\|a\mathbf{t} - \mathbf{t}a\| < \varepsilon$ and $\|\pi(\mathbf{t})a - a\| < \varepsilon$ for all $a \in F$. Now according to the definition of symmetric pseudo-amenability to show that $\mathfrak{U}$ is symmetrically pseudo-amenable, it is sufficient to show that for each finite subset F of $\mathfrak{U}$ and each $\varepsilon > 0$ there is a symmetric element $\mathbf{t}$ of $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ such that $\|a\mathbf{t} - \mathbf{t}a\| < \varepsilon$ and $\|\pi(\mathbf{t})a - a\| < \varepsilon$ for all $a \in F$. Also, according to Proposition 3.2, in order to prove that $\mathfrak{U}$ is symmetrically pseudo-amenable, it is sufficient to show that for each finite subset F of $\mathfrak{U}$ and each $\varepsilon > 0$ there is an element $\mathbf{t}$ of $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ with $\|a\mathbf{t} - \mathbf{t}a\| < \varepsilon$, $\|a \circ \mathbf{t} - \mathbf{t} \circ a\| < \varepsilon$, $\|\pi(\mathbf{t})a - a\| < \varepsilon$ and $\|a\pi^\circ(\mathbf{t}) - a\| < \varepsilon$ for all $a \in F$.

As an application of Proposition 3.2, we have the following result.

Corollary 3.3. *If* $\mathfrak{U}$ *is a commutative pseudo-amenable Banach algebra, then* $\mathfrak{U}$ *is symmetrically pseudo-amenable.*

Proof. For commutative Banach algebras, in Proposition 3.2, conditions (i) and (iii), and conditions (ii) and (iv) are the same. □

In the previous corollary, we saw that pseudo-amenability is equivalent to symmetric pseudo-amenability on commutative Banach algebras. By using the next proposition, we can obtain an example of Banach algebras that are pseudo-amenable, but not symmetrically pseudo-amenable, and therefore, these classes of Banach algebras are different in general. The idea of the next result comes from [15, Proposition 2.4].

Proposition 3.4. *Let $\mathfrak{U}$ be a symmetrically pseudo-amenable Banach algebra, and $z \neq 0$ be an element of $\mathcal{Z}(\mathfrak{U})$. Then there is a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{U}^*$ such that $f_\lambda(ab - ba) \longrightarrow 0$ for every $a, b \in \mathfrak{U}$ and $f_\lambda(z) \longrightarrow 1$. Especially, if $\mathfrak{U}$ is unital, then for $z = 1$ there exists a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{U}^*$ such that $f_\lambda(ab - ba) \longrightarrow 0$ for every $a, b \in \mathfrak{U}$ and $f_\lambda(1) \longrightarrow 1$.*

Proof. Suppose that $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ is a net in $\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ that satisfies to conditions (i) to (iv) of Proposition 3.2, and $g \in \mathfrak{U}^*$ is such that $g(z) = 1$. For each $\lambda \in \Lambda$ define

$$f_\lambda(a) = g(\pi^\circ(a\mathbf{t}_\lambda)) \quad (a \in \mathfrak{U}).$$

Each f_λ is a bounded linear functional on $\mathfrak{U}$. For every $\lambda \in \Lambda$ and $a, b \in \mathfrak{U}$ we have $\pi^\circ(ab\mathbf{t}_\lambda) = \pi^\circ(b\mathbf{t}_\lambda a)$, and hence

$$\begin{aligned} f_\lambda(ab - ba) &= g(\pi^\circ(ab\mathbf{t}_\lambda - ba\mathbf{t}_\lambda)) \\ &= g(\pi^\circ(b\mathbf{t}_\lambda a - ba\mathbf{t}_\lambda)) \\ &= g(\pi^\circ(b(\mathbf{t}_\lambda a - a\mathbf{t}_\lambda))) \longrightarrow 0 \end{aligned}$$

for each $a, b \in \mathfrak{U}$. Also, since $z \in \mathcal{Z}(\mathfrak{U})$, we have

$$\pi^\circ(z\mathbf{t}_\lambda) = \pi^\circ(\mathbf{t}_\lambda \circ z) = \pi^\circ(\mathbf{t}_\lambda)z \longrightarrow z$$

Now, it follows from $g \in \mathfrak{U}^*$ that

$$f_\lambda(z) \longrightarrow g(z) = 1.$$

□

In the next example, we present a Banach algebra, which is pseudo-amenable but not symmetrically pseudo-amenable.

Example 3.5. Let O_n be the Cuntz algebra, which is a unital amenable C^*-algebra, and hence it is pseudo-amenable. There are members $T_1, ...T_n$ in O_n such that $T_i^*T_i = I$ and $\sum_{i=1}^n T_iT_i^* = I$. Suppose that O_n for $n > 1$ is symmetrically pseudo-amenable. According to Proposition 3.4, there exists a net $\{f_\lambda\}_{\lambda\in\Lambda}$ of bounded linear functionals on O_n such that

$$f_\lambda(AB - BA) \longrightarrow 0$$

for each $A, B \in O_n$ and

$$f_\lambda(I) \longrightarrow 1.$$

Therefore

$$1 = \lim_\lambda f_\lambda(I) = \lim_\lambda f_\lambda(\sum_{i=1}^n T_iT_i^*) = \lim_\lambda \sum_{i=1}^n f_\lambda(T_iT_i^*)$$

On the other hand

$$\lim_\lambda \sum_{i=1}^n f_\lambda(T_iT_i^*) - \lim_\lambda \sum_{i=1}^n f_\lambda(T_i^*T_i) = \lim_\lambda \sum_{i=1}^n f_\lambda(T_iT_i^* - T_i^*T_i) \longrightarrow 0$$

So $\lim_\lambda \sum_{i=1}^n f_\lambda(T_i^*T_i) = 1$. but

$$\lim_\lambda \sum_{i=1}^n f_\lambda(T_i^*T_i) = \sum_{i=1}^n \lim_\lambda f_\lambda(T_i^*T_i) = \sum_{i=1}^n \lim_\lambda f_\lambda(I) = n$$

So $n = 1$, which contradicts $n > 1$. Therefore, O_n for $n > 1$ cannot be a symmetrically pseudo-amenable Banach algebra.

In fact, this example presents a Banach algebra that is amenable but not symmetrically pseudo-amenable. By using the next proposition, we can give an example of a symmetrically pseudo-amenable Banach algebra that is not amenable. To express this proposition, we first introduce some concepts.

Let $\{X_i : i \in I\}$ be a collection of Banach spaces. Denote by $\prod_{i\in I} X_i$ the product space of the collection. This is the space consisting of all mappings $x : I \longrightarrow \bigcup_{i\in I} X_i$ for which $x(i) \in X_i$, the linear operations being given coordinatewise. For $1 \leq p < \infty$, we recall that the ℓ_p-direct sum of the collection is

$$\bigoplus_{i\in I}^{p} X_i = \left\{ x \in \prod_{i\in I} X_i \,:\, \|x\|_p = \left(\sum_{i\in I} \|x(i)\|^p\right)^{\frac{1}{p}} < \infty \right\},$$

and the c_0-direct sum of the collection is

$$\bigoplus_{i\in I}^{0} X_i = \left\{ x \in \prod_{i\in I} X_i \,:\, \|x\|_\infty = max\|x(i)\| < \infty \,\text{and}\, \lim_i x(i) = 0 \right\}.$$

For a collection $\{\mathfrak{U}_i : i \in I\}$ of Banach algebras, the sum $\bigoplus_{i\in I}^{p} \mathfrak{U}_i$, $p \geqslant 1$ or $p = 0$, is also a Banach algebra with the multiplication being defined coordinatewise. If $J \subset I$, then $\bigoplus_{i\in J}^{p} \mathfrak{U}_i$ can be identified with the complemented closed ideal of $\bigoplus_{i\in I}^{p} \mathfrak{U}_i$ consisting of all x with $x(i) = 0$ for $i \notin J$. We let P_J denote the associated projection from $\bigoplus_{i\in I}^{p} \mathfrak{U}_i$ onto $\bigoplus_{i\in J}^{p} \mathfrak{U}_i$. It should be noted that for $i_0 \in I$, if $\rho_{i_0} : \mathfrak{U}_{i_0} \longrightarrow \bigoplus_{i\in I}^{p} \mathfrak{U}_i$ is the natural embedding map, then $\tilde{\rho}_{i_0} : \mathfrak{U}_{i_0} \widehat{\otimes} \mathfrak{U}_{i_0} \longrightarrow (\bigoplus_{i\in I}^{p} \mathfrak{U}_i) \widehat{\otimes} (\bigoplus_{i\in I}^{p} \mathfrak{U}_i)$ is given by $\tilde{\rho}_{i_0}(\sum_{k=1}^{\infty} a_k^{i_0} \otimes b_k^{i_0}) = \sum_{k=1}^{\infty} \rho_{i_0}(a_k^{i_0}) \otimes \rho_{i_0}(b_k^{i_0})$, where $\sum_{k=1}^{\infty} a_k^{i_0} \otimes b_k^{i_0} \in \mathfrak{U}_{i_0} \widehat{\otimes} \mathfrak{U}_{i_0}$ is a bounded linear embedding with $\|\tilde{\rho}\| \leq 1$. Now we have the following proposition, the idea of proof of which is taken from [7, Proposition 2.1].

Proposition 3.6. *Suppose that for each $i \in I$, $\mathfrak{U}_i$ is a symmetrically pseudo-amenable Banach algebra. Then so is $\bigoplus_{i\in I}^{p} \mathfrak{U}_i$ for $p \geq 1$ or $p = 0$.*

Proof. Let $\mathfrak{U} = \bigoplus_{i\in I}^{p} \mathfrak{U}_i$. Given $\varepsilon > 0$ and a finite set $F \subset \mathfrak{U}$, we can choose a finite set $J_{F,\varepsilon} \subset I$ for which $\|P_{J_{F,\varepsilon}}(a) - a\| < \dfrac{\varepsilon}{2}$ for each $a \in F$. Since each $\mathfrak{U}_i$ is symmetrically pseudo-amenable, by Proposition 3.2, for every $i \in J_{F,\varepsilon}$ there is $\mathbf{t}_i \in \mathfrak{U}_i \widehat{\otimes} \mathfrak{U}_i$ such that

$$\|P_i(a)\mathbf{t}_i - \mathbf{t}_i P_i(a)\| < \frac{\varepsilon}{\mid J_{F,\varepsilon} \mid};$$

$$\|\pi(\mathbf{t}_i)P_i(a) - P_i(a)\| < \frac{\varepsilon}{2 \mid J_{F,\varepsilon} \mid};$$

$$\|P_i(a) \circ \mathbf{t}_i - \mathbf{t}_i \circ P_i(a)\| < \frac{\varepsilon}{\mid J_{F,\varepsilon} \mid}$$

and

$$\|P_i(a)\pi^\circ(\mathbf{t}_i) - P_i(a)\| < \frac{\varepsilon}{2 \mid J_{F,\varepsilon} \mid}$$

for each $a \in F$, where $\mid J_{F,\varepsilon} \mid = card J_{F,\epsilon}$ and P_i denotes the projection $P_{\{i\}}$. Now we consider the embedding $\tilde{\rho}_i : \mathfrak{U}_i \widehat{\otimes} \mathfrak{U}_i \longrightarrow \mathfrak{U} \widehat{\otimes} \mathfrak{U}$ and choose the element $\mathbf{t}_{F,\varepsilon}$ in $\mathfrak{U} \widehat{\otimes} \mathfrak{U}$ as follows

$$\mathbf{t}_{F,\varepsilon} = \sum_{i\in J_{F,\varepsilon}} \tilde{\rho}_i(\mathbf{t}_i).$$

For every $a \in \mathfrak{U}$ we have $a = \sum_{i \in I} P_i(a)$, and hence, according to the definition of $\mathbf{t}_{F,\varepsilon}$ and $\tilde{\rho}_i$ we have

$$a\mathbf{t}_{F,\varepsilon} = \sum_{i \in J_{F,\varepsilon}} P_i(a)\tilde{\rho}_i(\mathbf{t}_i) = \sum_{i \in J_{F,\varepsilon}} \tilde{\rho}_i(P_i(a)\mathbf{t}_i).$$

In the same way

$$\mathbf{t}_{F,\varepsilon}a = \sum_{i \in J_{F,\varepsilon}} \tilde{\rho}_i(\mathbf{t}_i P_i(a));$$

$$a \circ \mathbf{t}_{F,\varepsilon} = \sum_{i \in J_{F,\epsilon}} \tilde{\rho}_i(P_i(a) \circ \mathbf{t}_i)$$

and

$$\mathbf{t}_{F,\varepsilon} \circ a = \sum_{i \in J_{F,\varepsilon}} \tilde{\rho}_i(\mathbf{t}_i \circ P_i(a)).$$

Therefore, for each $a \in F$ we have

$$\begin{aligned} \|a\mathbf{t}_{F,\varepsilon} - \mathbf{t}_{F,\varepsilon}a\| &= \| \sum_{i \in J_{F,\varepsilon}} \tilde{\rho}_i(P_i(a)\mathbf{t}_i - \mathbf{t}_i P_i(a))\| \\ &\leq \sum_{i \in J_{F,\varepsilon}} \|\tilde{\rho}_i(P_i(a)\mathbf{t}_i - \mathbf{t}_i P_i(a))\| \\ &\leq \sum_{i \in J_{F,\varepsilon}} \|(P_i(a)\mathbf{t}_i - \mathbf{t}_i P_i(a)\| < \varepsilon \end{aligned}$$

and

$$\begin{aligned} \|\pi(\mathbf{t}_{F,\varepsilon})a - a\| &= \|\pi(\mathbf{t}_{F,\varepsilon}a) - a\| \\ &= \| \sum_{i \in J_{F,\varepsilon}} \pi(\tilde{\rho}_i(\mathbf{t}_i P_i(a)) - a\| \\ &= \| \sum_{i \in J_{F,\varepsilon}} \pi(\mathbf{t}_i)P_i(a) - a \| \\ &\leq \| \sum_{i \in J_{F,\varepsilon}} \pi(\mathbf{t}_i)P_i(a) - \sum_{i \in J_{F,\varepsilon}} P_i(a)\| + \| \sum_{i \in J_{F,\varepsilon}} P_i(a) - a\| \\ &\leq \sum_{i \in J_{F,\varepsilon}} \|\pi(\mathbf{t}_i)P_i(a) - P_i(a)\| + \|P_{J_{F,\varepsilon}}(a) - a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Similarly, we have

$$\|a \circ \mathbf{t}_{F,\varepsilon} - \mathbf{t}_{F,\varepsilon} \circ a\| < \varepsilon$$

and

$$\|a\pi^{\circ}(\mathbf{t}_{F,\varepsilon}) - a\| < \varepsilon$$

for each $a \in F$. So from Proposition 3.2 it follows that $\mathfrak{U}$ is symmetrically pseudo-amenable. □

Now we are in a position to present the desired example.

Example 3.7. For each $n \geqslant 1$, let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. Every $M_n(\mathbb{C})$ is symmetrically amenable because

$$\mathbf{t} = \frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji},$$

where the E_{ij} are the usual matrix units, is a symmetric diagonal in $M_n(\mathbb{C})$ (see [5, Proposition 1.9.20]). According to Proposition 3.6, $\bigoplus_{n\in\mathbb{N}}^{0} M_n(\mathbb{C})$ is a symmetrically pseudo-amenable C^*-algebra that is not amenable.

We know that every symmetrically amenable Banach algebra is amenable and symmetrically pseudo-amenable. According to this point and the above examples, the following question arises:

Question. What is the relationship between the class of symmetrically amenable Banach algebras and the class of amenable and symmetrically pseudo-amenable Banach algebras? Is the Banach algebra that is amenable and symmetrically pseudo-amenable, is the symmetrically amenable Banach algebra?

We continue this section by studying some hereditary properties.

Proposition 3.8. *Let $\mathfrak{U}$ and $\mathcal{B}$ be two Banach algebras. If $\mathfrak{U}$ is symmetrically pseudo-amenable and there is a continuous epimorphism θ from $\mathfrak{U}$ onto $\mathcal{B}$, then $\mathcal{B}$ is symmetrically pseudo-amenable. In particular, the quotient algebra $\mathfrak{U}/I$ is symmetrically pseudo-amenable for any two-sided closed ideal I of $\mathfrak{U}$.*

Proof. Consider the continuous linear mapping $\theta\otimes\theta : \mathfrak{U}\widehat{\otimes}\mathfrak{U} \longrightarrow \mathcal{B}\widehat{\otimes}\mathcal{B}$. For each $\mathbf{t}\in\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ and $a\in\mathfrak{U}$ we have the followings

$$\theta\otimes\theta(a\mathbf{t}) = \theta(a)\theta\otimes\theta(\mathbf{t}) \quad \text{and} \quad \theta\otimes\theta(\mathbf{t}a) = \theta\otimes\theta(\mathbf{t})a;$$
$$\theta\otimes\theta(a\circ\mathbf{t}) = \theta(a)\circ\theta\otimes\theta(\mathbf{t}) \quad \text{and} \quad \theta\otimes\theta(\mathbf{t}\circ a) = \theta\otimes\theta(\mathbf{t})\circ\theta(a);$$
$$\pi(\theta\otimes\theta(\mathbf{t})) = \theta(\pi(\mathbf{t})) \quad \text{and} \quad \pi^\circ(\theta\otimes\theta(\mathbf{t})) = \theta(\pi^\circ(\mathbf{t})).$$

Considering these relationships and that θ is epimorphism, it follows that $\theta\otimes\theta$ maps any symmetric approximate diagonal for $\mathfrak{U}$ to a symmetric approximate diagonal for $\mathcal{B}$. □

Proposition 3.9. *Let $\mathfrak{U}$ be a symmetrically pseudo-amenable Banach algebra, and let J be a two-sided closed ideal of $\mathfrak{U}$. If J has an approximate identity $\{e_i\}_{i\in I}$ such that the associated left and right multiplication operators $L_i : a \to e_i a$ and $R_i : a \to a e_i$ from $\mathfrak{U}$ into J are uniformly bounded, then J is symmetrically pseudo-amenable.*

Proof. Under the condition on e_i, there is a constant $M \geq 1$ such that $\|e_i a\| \leq M\|a\|$ and $\|a e_i\| \leq M\|a\|$ for all e_i and all $a\in\mathfrak{U}$. So $\|e_i\mathbf{t}\| \leq M\|\mathbf{t}\|$, $\|\mathbf{t}e_i\| \leq M\|\mathbf{t}\|$, $\|e_i\circ\mathbf{t}\| \leq M\|\mathbf{t}\|$ and $\|\mathbf{t}\circ e_i\| \leq M\|\mathbf{t}\|$ for all e_i and all $\mathbf{t}\in\mathfrak{U}\widehat{\otimes}\mathfrak{U}$.

To prove the theorem, we first prove the following claim.

Claim. For each $a\in J$ and $\mathbf{t}\in\mathfrak{U}\widehat{\otimes}\mathfrak{U}$ we have

$$e_i\circ(\mathbf{t}a) \longrightarrow \mathbf{t}a.$$

Reason. Suppose $\mathbf{t} = \sum_{j=1}^{\infty} a_j\otimes b_j \in \mathfrak{U}\widehat{\otimes}\mathfrak{U}$. For each $k\in\mathbb{N}$, we put $\mathbf{t}_k = \sum_{j=1}^{k} a_j\otimes b_j$. Since $\mathbf{t}_k \longrightarrow \mathbf{t}$, it follows that for $a\in J$, $\mathbf{t}_k a \longrightarrow \mathbf{t}a$. So for $\varepsilon \geq 0$, there exists a k_0 such that

$$\|\mathbf{t}_{k_0}a - \mathbf{t}a\| < \frac{\varepsilon}{M+2}.$$

On the other hand, since $\{e_i\}_{i\in I}$ is an approximate identity for J and $\mathbf{t}_{k_0} = \sum_{j=1}^{k_0} a_j\otimes b_j$, we have $e_i\circ(\mathbf{t}_{k_0}a) = \sum_{j=1}^{k_0} a_j\otimes e_i b_j a$ and therefore $e_i\circ(\mathbf{t}_{k_0}a) \to \mathbf{t}_{k_0}a$. So there is an $i_0\in I$ such that for every $i \geqslant i_0$ we have

$$\|e_i\circ(\mathbf{t}_{k_0}a) - \mathbf{t}_{k_0}a\| < \frac{\varepsilon}{M+2}.$$

Thus if $i \geq i_0$, then

$$\begin{aligned}\|e_i \circ (\mathbf{t}a) - \mathbf{t}a\| &\leq \| e_i \circ (\mathbf{t}a) - e_i \circ (\mathbf{t}_{k_0}a)\| + \|e_i \circ (\mathbf{t}_{k_0}a) - \mathbf{t}_{k_0}a\| + \|\mathbf{t}_{k_0}a - \mathbf{t}a\| \\ &\leq M\|\mathbf{t}a - \mathbf{t}_{k_0}a\| + \|e_i \circ (\mathbf{t}_{k_0}a) - \mathbf{t}_{k_0}a\| + \|\mathbf{t}_{k_0}a - \mathbf{t}a\| \\ &< M\frac{\varepsilon}{M+2} + \frac{\varepsilon}{M+2} + \frac{\varepsilon}{M+2} = \varepsilon.\end{aligned}$$

Therefore, the claim is true.

Now we prove the theorem.

Let $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathfrak{U}\widehat{\otimes}\mathfrak{U}$ be a symmetric approximate diagonal for $\mathfrak{U}$. For an arbitrary $\varepsilon > 0$ and finite set $F \subseteq J$, we choose $\lambda \in \Lambda$ such that

$$\|a\mathbf{t}_\lambda - \mathbf{t}_\lambda a\|M^2 < \frac{\varepsilon}{2}$$

and

$$\|\pi(\mathbf{t}_\lambda)a - a\|M < \frac{\varepsilon}{3}$$

for each $a \in F$. Then, according to the proven claim and that $\{e_i\}_{i\in I}$ is an approximate identity for J, we choose $i \in I$ so that

$$\|ae_i - e_ia\|M\|\mathbf{t}_\lambda\| < \frac{\varepsilon}{2};$$

$$\|e_ia - a\|M\|\mathbf{t}_\lambda\| < \frac{\varepsilon}{3}$$

and

$$\|e_i \circ (\mathbf{t}_\lambda a) - \mathbf{t}_\lambda a\| < \frac{\varepsilon}{3}$$

for $a \in F$. We put

$$\mathbf{m}_{\lambda i} = (\mathbf{t}_\lambda \circ e_i)e_i,$$

where $\lambda \in \Lambda, i \in I$. We have the following relations

$$a\mathbf{m}_{\lambda i} - \mathbf{m}_{\lambda i}a = [(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) \circ e_i]e_i + (\mathbf{t}_\lambda \circ e_i)(ae_i - e_ia)$$

and

$$\pi(\mathbf{m}_{\lambda i})a = \pi(e_i \circ \mathbf{t}_\lambda)(e_ia - a) + \pi(e_i \circ (\mathbf{t}_\lambda a))$$

for $a \in J$. So, for $\lambda \in \Lambda$ and $i \in I$ chosen before, we have

$$\begin{aligned}\|a\mathbf{m}_{\lambda i} - \mathbf{m}_{\lambda i}a\| &\leq \|[(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) \circ e_i]e_i\| + \|(\mathbf{t}_\lambda \circ e_i)(ae_i - e_ia)\| \\ &\leq M^2\|a\mathbf{t}_\lambda - \mathbf{t}_\lambda a\| + M\|\mathbf{t}_\lambda\|\|ae_i - e_ia\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|\pi(\mathbf{m}_{\lambda i})a - a\| &= \|\pi(e_i \circ \mathbf{t}_\lambda)(e_ia - a) + \pi(e_i \circ (\mathbf{t}_\lambda a)) - a\| \\ &\leq \|\pi(e_i \circ \mathbf{t}_\lambda)(e_ia - a)\| + \|\pi(e_i \circ (\mathbf{t}_\lambda a)) - \pi(\mathbf{t}_\lambda a)\| + \|\pi(\mathbf{t}_\lambda a) - a\| \\ &\leq \|e_i \circ \mathbf{t}_\lambda\|\|e_ia - a\| + \|e_i \circ (\mathbf{t}_\lambda a) - \mathbf{t}_\lambda a\| + \|\pi(\mathbf{t}_\lambda)a - a\| \\ &\leq M\|\mathbf{t}_\lambda\|\|e_ia - a\| + \|e_i \circ (\mathbf{t}_\lambda a) - \mathbf{t}_\lambda a\| + \|\pi(\mathbf{t}_\lambda)a - a\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for each $a \in F$. It is also checked routinely for each $(\lambda, i) \in \Lambda \times I$ that $\mathbf{m}^\circ_{\lambda i} = \mathbf{m}_{\lambda i}$. Thus choosing an appropriate subnet of $\{\mathbf{m}_{\lambda i}\}_{\Lambda\times I} \subset J\widehat{\otimes}J$, we get a symmetric approximate diagonal for J. So J is symmetrically pseudo-amenable. □

We have the following result from the above theorem.

Corollary 3.10. *Let $\mathfrak{U}$ be a symmetrically pseudo-amenable Banach algebra and let J be a two-sided closed ideal of $\mathfrak{U}$. If J has a bounded approximate identity, then J is symmetrically pseudo-amenable.*

Proposition 3.11. *Let $\mathfrak{U}$ be a Banach algebra with a system of closed subalgebras $\{\mathfrak{U}_\gamma : \gamma \in \Gamma\}$ such that*

(i) $\overline{(\bigcup_{\gamma\in\Gamma} \mathfrak{U}_\gamma)} = \mathfrak{U}$;

(ii) *if* $\gamma_1, \gamma_2 \in \Gamma$ *then there is* $\gamma \in \Gamma$ *with* $\mathfrak{U}_{\gamma_1} \cup \mathfrak{U}_{\gamma_2} \subseteq \mathfrak{U}_\gamma$;

(iii) *for each* $\gamma \in \Gamma$, $\mathfrak{U}_\gamma$ *is a symmetrically pseudo-amenable Banach algebra.*

Then $\mathfrak{U}$ is symmetrically pseudo-amenable.

Proof. For an arbitrary $\varepsilon > 0$ and finite set $F \subseteq \bigcup_{\gamma\in\Gamma} \mathfrak{U}_\gamma$, by (ii) we choose $\gamma \in \Gamma$ with $F \subseteq \mathfrak{U}_\gamma$. Since $\mathfrak{U}_\gamma$ is symmetrically pseudo-amenable, there is a symmetric element $\mathbf{t}_{F,\varepsilon} \in \mathfrak{U}_\gamma \otimes \mathfrak{U}_\gamma \subseteq \mathfrak{U} \otimes \mathfrak{U}$ such that

$$\|a\mathbf{t}_{F,\varepsilon} - \mathbf{t}_{F,\varepsilon}a\| < \varepsilon$$

and

$$\|\pi(\mathbf{t}_{F,\varepsilon})a - a\| < \varepsilon$$

for each $a \in F$. So the result follows from (i). □

4. Some classes of symmetrically pseudo-amenable Banach algebras

In the previous section we saw some classes of symmetrically pseudo-amenable Banach algebras, especially that every commutative pseudo-amenable Banach algebra is symmetrically pseudo-amenable. In this section, we introduce some other classes of symmetrically pseudo-amenable Banach algebras.

First, we study a class of Banach algebras that belongs to the class of ℓ^1-Munn algebras. Let $\mathbb{N}$ be the set of natural numbers. We denote by $M_{\mathbb{N}}(\mathbb{C})$, the set of $\mathbb{N} \times \mathbb{N}$ matrices (a_{ij}) with entries in $\mathbb{C}$ such that

$$\|(a_{ij})\| = \sum_{i,j\in\mathbb{N}} |a_{ij}| < \infty.$$

Then $M_{\mathbb{N}}(\mathbb{C})$ with the usual matrix multiplication is a Banach algebra that belongs to the class of ℓ^1-Munn algebras. We write E_{ij} for the matrix units and aE_{ij} for the matrix with the a at the (i,j)-entry and 0 in all other entries.

Theorem 4.1. *The Banach algebra $M_{\mathbb{N}}(\mathbb{C})$ is symmetrically pseudo-amenable.*

Proof. For $n \in \mathbb{N}$ denote the finite set $\{1, 2, \ldots n\}$ by $\mathbb{N}_n$, and define

$$\mathbf{t}_n = \frac{1}{n} \sum_{i,j\in\mathbb{N}_n} E_{ij} \otimes E_{ji} \in M_{\mathbb{N}}(\mathbb{C}) \widehat{\otimes} M_{\mathbb{N}}(\mathbb{C}).$$

We will show that $\{\mathbf{t}_n\}_{n\in\mathbb{N}}$ is a symmetric approximate diagonal for $M_{\mathbb{N}}(\mathbb{C})$. Let $A = (a_{ij}) \in M_{\mathbb{N}}(\mathbb{C})$. According to the definition of $M_{\mathbb{N}}(\mathbb{C})$, for each $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\sum_{i,j\notin\mathbb{N}_n} |a_{ij}| < \varepsilon$. For $n \in \mathbb{N}$ we have

$$\pi(\mathbf{t}_n) = \frac{1}{n} \sum_{i\in\mathbb{N}_n} nE_{ii} = I_n,$$

where $I_n := \sum_{i=1}^{n} E_{ii}$. So for $n \geq n_0$

$$\|\pi(\mathbf{t}_n)A - A\| = \|I_n A - A\| \leq \sum_{i,j \notin \mathbb{N}_n} |a_{ij}| < \varepsilon.$$

Consequentially, $\pi(\mathbf{t}_n)A \longrightarrow A$ for each $A \in M_{\mathbb{N}}(\mathbb{C})$. For $n \in \mathbb{N}$ define the matrices A_r, $1 \leq r \leq 4$ as follows

$$A_1 = (x_{ij}), \text{ where } x_{ij} = a_{ij} \text{ for } 1 \leq i, j \leq n;\ x_{ij} = 0 \text{ for } i > n \text{ or } j > n;$$

$$A_2 = (x_{ij}), \text{ where } x_{ij} = a_{ij} \text{ for } 1 \leq i \leq n \text{ and } j > n;\ x_{ij} = 0 \text{ for } i > n \text{ or } 1 \leq j \leq n;$$

$$A_3 = (x_{ij}), \text{ where } x_{ij} = a_{ij} \text{ for } i > n \text{ and } 1 \leq j \leq n;\ x_{ij} = 0 \text{ for } 1 \leq i \leq n \text{ or } j > n;$$

and

$$A_4 = (x_{ij}), \text{ where } x_{ij} = a_{ij} \text{ for } i > n \text{ and } j > n;\ x_{ij} = 0 \text{ for } 1 \leq i \leq n \text{ or } 1 \leq j \leq n.$$

It can be seen by routine calculations that

$$A_1\mathbf{t}_n = \mathbf{t}_n A_1 = \frac{1}{n} \sum_{k,l,j \in \mathbb{N}_n} a_{kl} E_{kj} \otimes E_{jl};$$

$$A_2\mathbf{t}_n = 0 \quad \text{and} \quad \mathbf{t}_n A_2 = \frac{1}{n} \sum_{1 \leq j,k \leq n < l} a_{kl} E_{kj} \otimes E_{jl};$$

$$\mathbf{t}_n A_3 = 0 \quad \text{and} \quad A_3\mathbf{t}_n = \frac{1}{n} \sum_{1 \leq j,l \leq n < k} a_{kl} E_{kj} \otimes E_{jl};$$

$$A_4\mathbf{t}_n = \mathbf{t}_n A_4 = 0.$$

So for $n \geq n_0$

$$\|A\mathbf{t}_n - \mathbf{t}_n A\| = \|A_3\mathbf{t}_n - \mathbf{t}_n A_2\| \leq \sum_{1 \leq l \leq n < k} |a_{ij}| + \sum_{1 \leq k \leq n < l} |a_{ij}| \leq \sum_{i,j \notin \mathbb{N}_n} |a_{ij}| < \varepsilon.$$

Hence $A\mathbf{t}_n - \mathbf{t}_n A \longrightarrow 0$. Also, for each $n \in \mathbb{N}$, it is clear that $\mathbf{t}_n^\circ = \mathbf{t}_n$. Therefore, $M_{\mathbb{N}}(\mathbb{C})$ is symmetrically pseudo-amenable. ☐

In the following, we will discuss some algebras over locally compact groups.

Theorem 4.2. *Let G be a locally compact group. The group algebra $L^1(G)$ is symmetrically pseudo-amenable if and only if G is an amenable group.*

Proof. Suppose that $L^1(G)$ is symmetrically pseudo-amenable. So $L^1(G)$ is pseudo-amenable, and hence from [7, Proposition 4.1] it follows that G is an amenable group. Conversely, if G is amenable, by [15, Theorem 4.1] we have $L^1(G)$ is symmetrically pseudo-amenable. ☐

For any compact group G, $L^2(G)$ is non-amenable, except in the finite-dimensional cases. In the next theorem, we see that $L^2(G)$ is symmetrically pseudo-amenable.

Theorem 4.3. *For any compact group G, the Banach algebra $L^2(G)$ is symmetrically pseudo-amenable.*

Proof. By [21, § 32. Theorem 1], the group algebra $L^2(G)$ (G compact) is the ℓ_2-direct sum of its minimal two-sided ideals I_α, each of which is completely isomorphic to an algebra $M_n(\mathbb{C})$ ($n \in \mathbb{N}$, and a matrix for each ideal). We know that each $M_n(\mathbb{C})$ is symmetrically amenable (see Example 3.7). Hence from Proposition 3.6 it follows that $L^2(G)$ is symmetrically pseudo-amenable. ☐

We note that according to [15, Proposition 2.7] every strongly amenable C^*-algebra is symmetrically amenable and therefore is symmetrically pseudo-amenable.

5. **Jordan derivations**

Let $\mathfrak{U}$ be a Banach algebra. In the following, $\mathfrak{U}^\sharp$ means the unitization of $\mathfrak{U}$ with the ℓ^1-norm, which we consider in any case, whether $\mathfrak{U}$ is unital or not. The Banach algebra $\mathfrak{U}^\sharp$ is unital with unity e where $\|e\| = 1$. Let X be a Banach $\mathfrak{U}$-bimodule and we turn X into a Banach $\mathfrak{U}^\sharp$-bimodule by defining $1x = x1 = x$ for each $x \in X$, and hence $ex = xe = x$ for each $x \in X$. Let $x \in X$. The mapping $(a, b) \mapsto axb$ from $\mathfrak{U}^\sharp \times \mathfrak{U}^\sharp$ into X is bilinear and $\|axb\| \leq M_X\|a\|\|x\|\|b\|$ for all $a, b \in \mathfrak{U}^\sharp$, where $M_X = \sup\{\|ay\|, \|ya\| : a \in \mathfrak{U}, y \in X, \|a\| = \|y\| = 1\}$. Thus we can define a continuous linear operator $\psi_x : \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp \to X$ by $\psi_x(a \otimes b) = axb$ for all $a, b \in \mathfrak{U}^\sharp$. It is clear that $\|\psi_x\| \leq M_X\|x\|$. Let $T : \mathfrak{U} \to X$ be a bounded linear map, and and we extend T to $\mathfrak{U}^\sharp$ by putting $T(1) = 0$. So $T(e) = 0$. Then $\Phi_T : \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp \to X$ is the bounded linear mapping specified by $\Phi_T(a \otimes b) = aT(b)$ with $\|\Phi_T\| \leq \|T\|$

Now we are ready to state the main results of this section. In the following theorems, it is assumed that ψ_x and Φ_T are defined as above.

Theorem 5.1. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable with the a symmetric approximate diagonal $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$. Suppose X is a Banach $\mathfrak{U}$-bimodule such that*

(i) *for each $x \in X$ the net $\{\psi_x(\mathbf{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded, and*
(ii) *for each bounded Jordan derivation $D : \mathfrak{U} \to X$ the net $\{\Phi_D(\mathbf{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded.*

Then every bounded Jordan derivation from $\mathfrak{U}$ to X is a derivation.

Proof. Suppose that $D : \mathfrak{U} \to X$ is a bounded Jordan derivation. We extend D to $\mathfrak{U}^\sharp$ by putting $D(1) = 0$. So $D(e) = 0$, where e is the unity of $\mathfrak{U}^\sharp$. Then

$$\Phi_D(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) = a\Phi_D(\mathbf{t}_\lambda) + \Phi_D(a \circ \mathbf{t}_\lambda) - \pi(\mathbf{t}_\lambda)D(a) - \Phi_D(\mathbf{t}_\lambda)a - \Phi_D(\mathbf{t}_\lambda \circ a) - \psi_{D(a)}(\mathbf{t}_\lambda),$$

for each $a \in \mathfrak{U}^\sharp$. Let $x_\lambda := \Phi_D(\mathbf{t}_\lambda)$. So

$$\pi(\mathbf{t}_\lambda)D(a) = (ax_\lambda - x_\lambda a) - \Phi_D(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) + \Phi_D(a \circ \mathbf{t}_\lambda - \mathbf{t}_\lambda \circ a) - \psi_{D(a)}(\mathbf{t}_\lambda),$$

for each $a \in \mathfrak{U}^\sharp$. We have $\pi(\mathbf{t}_\lambda) \longrightarrow e$ and since X is unital over $\mathfrak{U}^\sharp$, $\pi(\mathbf{t}_\lambda)D(a) \longrightarrow D(a)$ for $a \in \mathfrak{U}^\sharp$. On the other hand

$$\|\Phi_D(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a)\| \leq \|D\|\|a\mathbf{t}_\lambda - \mathbf{t}_\lambda a\| \longrightarrow 0$$

and

$$\|\Phi_D(a \circ \mathbf{t}_\lambda - \mathbf{t}_\lambda \circ a)\| \leq \|D\|\|a \circ \mathbf{t}_\lambda - \mathbf{t}_\lambda \circ a\| \longrightarrow 0$$

for $a \in \mathfrak{U}^\sharp$. Therefore,

$$D(a) = \lim_\lambda((ax_\lambda - x_\lambda a) - \psi_{D(a)}(\mathbf{t}_\lambda)) \tag{5.1}$$

for $a \in \mathfrak{U}^\sharp$. Now, viewing X as a closed $\mathfrak{U}^\sharp$-subbimodule of X^{**}, and hence D is a bounded Jordan derivation from $\mathfrak{U}^\sharp$ to X^{**}. Since $\{x_\lambda\}_{\lambda\in\Lambda}$ is bounded, define $\Omega \in X^{**}$ as follows:

$$\langle\Omega, f\rangle = Lim_\lambda\langle x_\lambda, f\rangle,$$

where $f \in X^*$ and Lim_λ is a generalized limit on Λ. Also, by our assumption, define the bounded linear map $\Delta : \mathfrak{U}^\sharp \to X^{**}$ by

$$\langle\Delta(a), f\rangle = Lim_\lambda\langle\psi_{D(a)}(\mathbf{t}_\lambda), f\rangle,$$

where $a \in \mathfrak{U}^\sharp$ and $f \in X^*$. It follows from (5.1) that

$$\begin{aligned}\langle D(a), f\rangle &= Lim_\lambda \langle a x_\lambda - x_\lambda a, f\rangle - Lim_\lambda \langle \psi_{D(a)}(\mathbf{t}_\lambda), f\rangle \\ &= \langle a\Omega - \Omega a, f\rangle - \langle \Delta(a), f\rangle\end{aligned}$$

for any $a \in \mathfrak{U}^\sharp$ and $f \in X^*$. So

$$D(a) = a\Omega - \Omega a - \Delta(a)$$

for each $a \in \mathfrak{U}^\sharp$, and hence Δ is a bounded Jordan derivation. For each $a, b \in \mathfrak{U}^\sharp$ we have

$$\begin{aligned}\langle a\Delta(b), f\rangle &= Lim_\lambda \langle \psi_{D(b)}(\mathbf{t}_\lambda), fa\rangle \\ &= Lim_\lambda \langle a\psi_{D(b)}(\mathbf{t}_\lambda), f\rangle \\ &= Lim_\lambda \langle \psi_{D(b)}(a\mathbf{t}_\lambda), f\rangle \\ &= Lim_\lambda \langle \psi_{D(b)}(\mathbf{t}_\lambda a), f\rangle \\ &= Lim_\lambda \langle \psi_{D(b)}(\mathbf{t}_\lambda)a, f\rangle \\ &= \langle \Delta(b)a, f\rangle\end{aligned}$$

for all $f \in X^*$, because $a\mathbf{t}_\lambda - \mathbf{t}_\lambda a \longrightarrow 0$ and the nets $\{\psi_{D(b)}(a\mathbf{t}_\lambda)\}_{\lambda\in\Lambda}$, $\{\psi_{D(b)}(\mathbf{t}_\lambda a)\}_{\lambda\in\Lambda}$ are bounded. Thus

$$a\Delta(b) = \Delta(b)a \tag{5.2}$$

for each $a, b \in \mathfrak{U}^\sharp$. Now we do the same process for Δ as we did earlier for D and therefore

$$\Delta(a) = a\Omega_1 - \Omega_1 a - \Delta_1(a)$$

for $a \in \mathfrak{U}^\sharp$, where $\Omega_1 \in X^{**}$ and Δ_1 is a bounded linear map from $\mathfrak{U}$ to X^{**} defined by

$$\langle \Delta_1(a), f\rangle = Lim_\lambda \langle \psi_{\Delta(a)}(\mathbf{t}_\lambda), f\rangle$$

for $a \in \mathfrak{U}^\sharp$ and $f \in X^*$ (By condition (ii) of our assumption Δ_1 is well-defined). It follows from (5.2) that

$$\psi_{\Delta(a)}(\mathbf{t}) = \pi(\mathbf{t})\Delta(a)$$

for each $a \in \mathfrak{U}^\sharp$ and $\mathbf{t} \in \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp$. So $\psi_{\Delta(a)}(\mathbf{t}_\lambda) \longrightarrow \Delta(a)$. Consequentially, $\Delta_1(a) = \Delta(a)$ for all $a \in \mathfrak{U}^\sharp$ and

$$\Delta(a) = a(\frac{1}{2}\Omega_1) - (\frac{1}{2}\Omega_1)a$$

for $a \in \mathfrak{U}^\sharp$. According to this identity and $D(a) = a\Omega - \Omega a - \Delta(a)$ we have

$$D(a) = a(\Omega - \frac{1}{2}\Omega_1) - (\Omega - \frac{1}{2}\Omega_1)a$$

for $a \in \mathfrak{U}^\sharp$, and hence D is a derivation. □

We note that if $\mathfrak{U}$ is a commutative approximately amenable Banach algebra, then by [7, Theorem 3.1-(iv)] and Corollary 3.3, $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable. Examples of this kind of Banach algebras are given in [8].

We have the following result that is proved in [15, Theorem 6.2]. Therefore, it can be said that Theorem 5.1 is a generalization of [15, Theorem 6.2].

Corollary 5.2. *Let $\mathfrak{U}$ be a symmetrically amenable Banach algebra and X be a Banach $\mathfrak{U}$-bimodule. Then every bounded Jordan derivation from $\mathfrak{U}$ to X is a derivation.*

Proof. By [15, Theorem 3.1], $\mathfrak{U}^\sharp$ is symmetrically amenable, and so it has a bounded symmetric approximate diagonal $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$. Since $\psi_x : \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp \to X$ for each $x \in X$ and $\Phi_T : \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp \to X$ for each bounded linear map $T : \mathfrak{U}^\sharp \to X$ are bounded mappings, it follows from boundness of $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ that the conditions (i) and (ii) of Theorem 5.1 are satisfied. Therefore, the desired result is proved. □

In the following theorem, we consider a special type of Jordan derivations.

Theorem 5.3. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable and X is a Banach $\mathfrak{U}$-bimodule. Then every bounded central Jordan derivation from $\mathfrak{U}$ to X is a derivation.*

Proof. Suppose that $D : \mathfrak{U} \to X$ is a bounded central Jordan derivation and $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ is a symmetric approximate diagonal for $\mathfrak{U}^\sharp$. We extend D to $\mathfrak{U}^\sharp$ by putting $D(1) = 0$. With the same process of proving Theorem 5.1, it is proved that

$$D(a) = \lim_\lambda((ax_\lambda - x_\lambda a) - \psi_{D(a)}(\mathbf{t}_\lambda))$$

for $a \in \mathfrak{U}^\sharp$. Since D is central, it follows that

$$\psi_{D(a)}(\mathbf{t}) = \pi(\mathbf{t})D(a)$$

for each $a \in \mathfrak{U}^\sharp$ and $\mathbf{t} \in \mathfrak{U}^\sharp \otimes \mathfrak{U}^\sharp$. Thus $\psi_{D(a)}(\mathbf{t}_\lambda) \longrightarrow D(a)$ and

$$D(a) = \frac{1}{2}\lim_\lambda(ax_\lambda - x_\lambda a)$$

for $a \in \mathfrak{U}^\sharp$. □

Let $\mathfrak{U}$ be a Banach algebra. The Banach $\mathfrak{U}$-bimodule X is called *symmetric* if $ax = xa$, for all $a \in \mathfrak{U}$ and $x \in X$. According to Theorem 5.3, we have the following result which checks the bounded Jordan derivations into a certain class of Banach bimodules.

Corollary 5.4. *If $\mathfrak{U}$ is a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable, then every bounded Jordan derivation from $\mathfrak{U}$ to a symmetric Banach $\mathfrak{U}$-bimodule X is a derivation.*

6. **Lie derivations**

In this section, we consider $\mathfrak{U}^\sharp$ as in the previous section and convert a $\mathfrak{U}$-bimodule to a $\mathfrak{U}^\sharp$-bimodule. Also, Φ_T is defined as in the previous section. The following lemma is about central derivations.

Lemma 6.1. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable and X be a Banach $\mathfrak{U}$-bimodule. Then every bounded central derivation from $\mathfrak{U}$ to X is a derivation.*

Proof. Let $\delta : \mathfrak{U} \to X$ be a bounded central derivation and $\{\mathbf{t}_\lambda\}_{\lambda\in\Lambda}$ be a symmetric approximate diagonal for $\mathfrak{U}^\sharp$. We extend δ to $\mathfrak{U}^\sharp$ by putting $\delta(1) = 0$. From the fact that δ is central, for each $a, b, c \in \mathfrak{U}^\sharp$ we have

$$\begin{aligned}\Phi_\delta(ab \otimes c) - \Phi_\delta(b \otimes ca) &= ab\delta(c) - b\delta(ca)\\ &= ab\delta(c) - \delta(ca)b\\ &= ab\delta(c) - c\delta(a)b - \delta(c)ab\\ &= cb\delta(a).\end{aligned}$$

So

$$\Phi_\delta(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) = \pi(\mathbf{t}_\lambda^\circ)\delta(a)$$

for all $a \in \mathfrak{U}^\sharp$. Since $\mathbf{t}_\lambda^\circ = \mathbf{t}_\lambda$ $(\lambda \in \Lambda)$, $a\mathbf{t}_\lambda - \mathbf{t}_\lambda a \longrightarrow 0$ and $\pi(\mathbf{t}_\lambda)\delta(a) \longrightarrow \delta(a)$, it follows that $\delta = 0$. □

The restriction of a central derivation to a subalgebra is central so we can extend Lemma 6.1 to the following.

Corollary 6.2. *Let $\mathfrak{U}$ be the smallest closed subalgebra which contains all the closed subalgebras $\mathfrak{V}$ of $\mathfrak{U}$ such that $\mathfrak{V}^\sharp$ is symmetrically pseudo-amenable. Then every bounded central derivation with domain $\mathfrak{U}$ is 0.*

Lemma 6.3. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable. Suppose that Y is a Banach $\mathfrak{U}$-bimodule and X is a closed $\mathfrak{U}$-subbimodule of Y. If $\delta : \mathfrak{U} \to Y$ is a bounded derivation and $\tau : \mathfrak{U} \to \mathcal{Z}_{\mathfrak{U}}(Y)$ is a linear map such that $(\delta+\tau)(\mathfrak{U}) \subseteq X$, then $\delta(\mathfrak{U}) \subseteq X$ and $\tau(\mathfrak{U}) \subseteq \mathcal{Z}_{\mathfrak{U}}(X)$.*

Proof. Let $\pi_X : Y \to Y/X$ be the quotient map where $W = Y/X$ is the quotient Banach $\mathfrak{U}$-bimodule. We have

$$0 = \pi_X \circ (\delta + \tau) = \pi_X \circ \delta + \pi_X \circ \tau.$$

Hence

$$\pi_X \circ \delta = -\pi_X \circ \tau.$$

Since π_X maps $\mathcal{Z}_{\mathfrak{U}}(Y)$ into $\mathcal{Z}_{\mathfrak{U}}(W)$ and $\tau(\mathfrak{U}) \subseteq \mathcal{Z}_{\mathfrak{U}}(Y)$, it follows that

$$\pi_X \circ \delta(\mathfrak{U}) = -\pi_X \circ \tau(\mathfrak{U}) \subseteq \mathcal{Z}_{\mathfrak{U}}(W).$$

So by the fact that π_X is a bounded module homomorphism, $\pi_X \circ \delta$ is a bounded central derivation from $\mathfrak{U}$ into W. According to Lemma 6.1, $\pi_X \circ \delta = 0$, and hence $\delta(\mathfrak{U}) \subseteq X$. Now from assumption and the obtained result, we have $\tau(\mathfrak{U}) \subseteq X \cap \mathcal{Z}_{\mathfrak{U}}(Y) = \mathcal{Z}_{\mathfrak{U}}(X)$. □

In the following theorem we state the main result of this section.

Theorem 6.4. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable with the a symmetric approximate diagonal $\{\boldsymbol{t}_\lambda\}_{\lambda\in\Lambda}$. Suppose X is a Banach $\mathfrak{U}$-bimodule such that for each $x \in X$ the net $\{\psi_x(\boldsymbol{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded, and $D : \mathfrak{U} \to X$ is a bounded Lie derivation such that the net $\{\Phi_D(\boldsymbol{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded. Then there exist a bounded derivation $d : \mathfrak{U} \to X$ and a bounded central trace $\tau : \mathfrak{U} \to \mathcal{Z}_{\mathfrak{U}}(X)$ such that $D = d + \tau$.*

Proof. Assume that $D : \mathfrak{U} \to X$ is a bounded Lie derivation. We extend D to $\mathfrak{U}^\sharp$ by putting $D(1) = 0$. Then

$$\Phi_D(a\mathbf{t}_\lambda - \mathbf{t}_\lambda a) = a\Phi_D(\mathbf{t}_\lambda) - \Phi_D(a \circ \mathbf{t}_\lambda) + \psi_{D(a)}(\mathbf{t}_\lambda) - \pi(\mathbf{t}_\lambda)D(a) + \Phi_D(\mathbf{t}_\lambda \circ a) - \Phi_D(\mathbf{t}_\lambda)a$$

for each $a \in \mathfrak{U}^\sharp$. Let $x_\lambda := \Phi_D(\mathbf{t}_\lambda)$. Since $a\mathbf{t}_\lambda - \mathbf{t}_\lambda a \longrightarrow 0$, $a \circ \mathbf{t}_\lambda - \mathbf{t}_\lambda \circ a \longrightarrow 0$ and $\pi(\mathbf{t}_\lambda)D(a) \longrightarrow D(a)$, it follows that

$$D(a) = \lim_\lambda((ax_\lambda - x_\lambda a) + \psi_{D(a)}(\mathbf{t}_\lambda)) \tag{6.1}$$

for $a \in \mathfrak{U}^\sharp$. Now, viewing X as a closed $\mathfrak{U}^\sharp$-subbimodule of X^{**}, and hence D is a bounded Lie derivation from $\mathfrak{U}^\sharp$ to X^{**}. In view of our assumptions, define $\Omega \in X^{**}$ and the bounded linear map $\tau : \mathfrak{U}^\sharp \to X^{**}$ by

$$\langle \Omega, f\rangle = Lim_\lambda \langle x_\lambda, f\rangle$$

and

$$\langle \tau(a), f\rangle = Lim_\lambda \langle \psi_{D(a)}(\mathbf{t}_\lambda), f\rangle,$$

where $a \in \mathfrak{U}^\sharp$, $f \in X^*$ and Lim_λ is a generalized limit on Λ. It follows from (6.1) that

$$\langle D(a), f\rangle = \langle a\Omega - \Omega a, f\rangle + \langle \tau(a), f\rangle$$

for any $a \in \mathfrak{U}^\sharp$ and $f \in X^*$. Consequentially,

$$D(a) = a\Omega - \Omega a + \tau(a)$$

for each $a \in \mathfrak{U}^\sharp$. The linear map $d : \mathfrak{U}^\sharp \to X^{**}$ defined by $d(a) = a\Omega - \Omega a$ is a continuous derivation, and therefore $D = d + \tau$. Also, with a proof similar to the proof of centrality of Δ in the proof of Theorem 5.1, we have $\tau(a) \in \mathcal{Z}_{\mathfrak{U}^\sharp}(X^{**}) \subseteq \mathcal{Z}_{\mathfrak{U}}(X^{**})$. So $\tau = D - d$ is a bounded Lie derivation, and from the fact that $\tau(\mathfrak{U}) \subseteq \mathcal{Z}_{\mathfrak{U}^\sharp}(X^{**})$, it follows that $\tau([a, b]) = 0$ for every $a, b \in \mathfrak{U}^\sharp$. The conditions of Lemma 6.3 hold for d and τ on $\mathfrak{U}$, hence d maps $\mathfrak{U}$ to X and τ maps $\mathfrak{U}$ to $\mathcal{Z}_{\mathfrak{U}}(X)$. □

The following result is immediate.

Corollary 6.5. *Let $\mathfrak{U}$ be a Banach algebra such that $\mathfrak{U}^\sharp$ is symmetrically pseudo-amenable with the a symmetric approximate diagonal $\{\boldsymbol{t}_\lambda\}_{\lambda\in\Lambda}$. Suppose X is a Banach $\mathfrak{U}$-bimodule such that*

(i) *for each $x \in X$ the net $\{\psi_x(\boldsymbol{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded, and*
(ii) *for each bounded Lie derivation $D : \mathfrak{U} \to X$ the net $\{\Phi_D(\boldsymbol{t}_\lambda)\}_{\lambda\in\Lambda}$ is bounded.*

Then for every bounded Lie derivation $D : \mathfrak{U} \to X$ there exist a bounded derivation $d : \mathfrak{U} \to X$ and a bounded central trace $\tau : \mathfrak{U} \to \mathcal{Z}_{\mathfrak{U}}(X)$ such that $D = d + \tau$.

Similar to the proof of Corollary 5.2, the following result is obtained, which is proved in [15, Theorem 9.2]. Therefore, it can be said that Theorem 6.4 (and Corollary 6.5) is a generalization of [15, Theorem 9.2].

Corollary 6.6. *Let $\mathfrak{U}$ be a symmetrically amenable Banach algebra and X be a Banach $\mathfrak{U}$-bimodule. Then for every bounded Lie derivation $D : \mathfrak{U} \to X$ there exist a bounded derivation $d : \mathfrak{U} \to X$ and a bounded central trace $\tau : \mathfrak{U} \to \mathcal{Z}_{\mathfrak{U}}(X)$ such that $D = d + \tau$.*

7. **Conclusions**

In this paper, we introduced the concept of *symmetric pseudo-amenability* for Banach algebras and developed its fundamental theory. We established several structural properties, provided examples that distinguish it from other notions of amenability, and identified classes of Banach algebras where symmetric pseudo-amenability naturally arises. Our results demonstrate that this notion forms a genuine intermediate concept between symmetric amenability and pseudo-amenability, with its own distinctive features and applications.

A central contribution of this paper is the application of symmetric pseudo-amenability to the study of Jordan and Lie derivations. By extending Johnson's classical theorems, we showed that symmetric pseudo-amenability provides a powerful framework for characterizing derivations and their decompositions in Banach algebras and C^*-algebras. This highlights the role of symmetric pseudo-amenability not only as a structural property but also as a tool for analysing cohomological problems.

Looking forward, several promising directions emerge. One natural question is to further clarify the relationship between symmetric pseudo-amenability and symmetric amenability, particularly in non-commutative settings. Another avenue is to explore connections with approximate amenability and other cohomological properties, which may reveal deeper structural insights. It would also be interesting to investigate symmetric pseudo-amenability in broader classes of operator algebras, group algebras, and Munn-type constructions, as well as its potential impact on the fixed point property and related functional analytic problems.

We believe that the notion of symmetric pseudo-amenability opens a new line of inquiry in the theory of Banach algebras. By bridging classical amenability concepts with modern cohomological techniques, it provides fertile ground for further exploration and may lead to new applications in both pure and applied functional analysis.

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