SET-VALUED OPTIMIZATION PROBLEMS UNDER CONE CONVEXITY

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In this paper, we study set-valued optimization problems, set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semi-infinite programming problems, and set-valued minimax programming problems via generalized cone convexity assumptions. We establish the existing results of optimality conditions of scalar and vector optimization problems in the setting of set-valued maps.

Keywords: convex cone, set-valued map, contingent epiderivative, weak subdifferential, set-valued optimization problem, approximate quasi efficient solutions, optimality conditions.


1. Introduction

Set-valued optimization problem which is being a new branch of optimization problem attracts the attention of researchers to an increasing extent in the last few years. Set-valued optimization problem makes a bridge between different areas in optimization theory. It plays an important role in multiobjective programming problems, functional analysis, statistics, the theory of decision making, game theory, and approximation theory.


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Optimization problems involving difference of convex mappings are called D. C. optimization problems. The sufficient optimality conditions of generalized D. C. multiobjective optimization problems have been studied by Guo et al. [19] via the notion of subdifferential.

Semi-infinite and minimax programming problems are some classes of optimization problems. Mishra and Jaiswal [22] have established the sufficient optimality conditions and Mond-Weir type duality theorems of nondifferentiable multiobjective semi-infinite programming problems via generalized convexity assumptions.

Das and Nahak [7–18] introduced the notion of ρ-cone convexity of set-valued maps. They establish the sufficient KKT conditions and develop the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions.

In this paper, we study set-valued optimization problems, set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semiinfinite programming problems, and set-valued minimax programming problems via ρ-cone convexity assumptions.

2. Definition and Preliminaries

Let $Y$ be a real normed space and $K$ be a nonempty subset of $Y$. Then $K$ is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Furthermore, $K$ is called non-trivial if $K \neq \{0\}$, proper if $K \neq Y$, pointed if $K \cap (-K) = \{0\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$, and convex if $\lambda K + (1 - \lambda)K \subseteq K$, for all $\lambda \in [0, 1]$, where $\text{int}(K)$ and $\overline{K}$ denote the interior and closure of $K$, respectively and $\theta_Y$ is the zero element of $Y$.

Let $K$ be a solid pointed convex cone in $Y$. There are two types of cone-orderings in $Y$ with respect to $K$. For any two elements $y_1, y_2 \in Y$, we have

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in K$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(K).$$

The following notions of minimality are mainly used with respect to a solid pointed convex cone $K$ in a real normed space $Y$.

**Definition 2.1.** Let $B$ be a nonempty subset of a real normed space $Y$. Then minimal and weakly minimal points of $B$ are defined as

(i) $y' \in B$ is a minimal point of $B$ if there is no $y \in B \setminus \{y'\}$, such that $y \leq y'$.

(ii) $y' \in B$ is a weakly minimal point of $B$ if there is no $y \in B$, such that $y < y'$.

We recall the notions of contingent cone and second-order contingent set in a real normed space.

**Definition 2.2.** [2,3] Let $Y$ be a real normed space, $\emptyset \neq B \subseteq Y$, and $y' \in \overline{B}$. The contingent cone to $B$ at $y'$ is denoted by $T(B, y')$ and is defined as follows:

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in $Y$, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$ in $\mathbb{R}$, with $t_n > 0$ and $\{y'_n\}$ in $B$, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

**Definition 2.3.** [2,3,5] Let $Y$ be a real normed space, $\emptyset \neq B \subseteq Y$, $y' \in \overline{B}$, and $u \in Y$. The second-order contingent set to $B$ at $y'$ in the direction $u$ is denoted by $T^2(B, y', u)$ and defined as
An element \( y \in T^2(B, y', u) \) if there exist sequences \( \{\lambda_n\} \) in \( \mathbb{R} \), with \( \lambda_n \to 0^+ \) and \( \{y_n\} \) in \( Y \), with \( y_n \to y \), such that
\[
y' + \lambda_n u + \frac{1}{2} \lambda_n^2 y_n \in B, \forall n \in \mathbb{N},
\]
or, there exist sequences \( \{t_n\}, \{t'_n\} \) in \( \mathbb{R} \), with \( t_n, t'_n > 0, t_n \to \infty, t'_n \to \infty, \frac{t_n}{t'_n} \to 2 \), and \( \{y'_n\} \) in \( B \), with \( y'_n \to y' \), such that
\[
t_n(y'_n - y') \to u \text{ and } t'_n(t_n(y'_n - y') - u) \to y.
\]

Let \( X, Y \) be real normed spaces, \( 2^Y \) be the set of all subsets of \( Y \), and \( K \) be a solid pointed convex cone in \( Y \). Let \( F : X \to 2^Y \) be a set-valued map from \( X \) to \( Y \), i.e., \( F(x) \subseteq Y \), for all \( x \in X \). The effective domain, image, graph, and epigraph of \( F \) are defined by
\[
\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\},
\]
\[
F(A) = \bigcup_{x \in A} F(x), \text{ for any } \emptyset \neq A \subseteq X,
\]
\[
\text{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\},
\]
and
\[
\text{epi}(F) = \{(x, y) \in X \times Y : y \in F(x) + K\}.
\]
Let \( A \) be a nonempty subset of \( X \), \( x' \in A \), \( F : X \to 2^Y \) be a set-valued map, with \( A \subseteq \text{dom}(F) \), and \( y' \in F(x') \). Jahn and Rauh [21] introduced the notion of contingent epiderivative of set-valued maps which plays a vital role in various aspects of set-valued optimization problems.

**Definition 2.4.** [21] A single-valued map \( D_1F(x', y') : X \to Y \) whose epigraph coincides with the contingent cone to the epigraph of \( F \) at \( (x', y') \), i.e.,
\[
\text{epi}(D_1F(x', y')) = T(\text{epi}(F), (x', y')),
\]
is said to be the contingent epiderivative of \( F \) at \( (x', y') \).

Jahn et al. [20] introduced the notion of second-order contingent epiderivative of set-valued maps which also has a fundamental role in set-valued optimization problems.

**Definition 2.5.** [20] A single-valued map \( D_2^2F(x', y', u, v) : X \to Y \) whose epigraph coincides with the second-order contingent set to the epigraph of \( F \) at \( (x', y') \) in \( \text{gr}(F) \) in a direction \( (u, v) \in X \times Y \), i.e.,
\[
\text{epi}(D_2^2F(x', y', u, v)) = T^2(\text{epi}(F), (x', y'), (u, v)),
\]
is said to be the second-order contingent epiderivative of \( F \) at \( (x', y') \) in the direction \( (u, v) \).

We now turn our attention to the notion of cone convexity of set-valued maps, introduced by Borwein [4].

**Definition 2.6.** [4] Let \( A \) be a nonempty convex subset of a real normed space \( X \). A set-valued map \( F : X \to 2^Y \), with \( A \subseteq \text{dom}(F) \), is called \( K \)-convex on \( A \) if \( \forall x_1, x_2 \in A \) and \( \lambda \in [0, 1] \),
\[
\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.
\]
3. Set-Valued Fractional Programming Problems

Let $X$ be a real normed space and $A$ be a nonempty subset of $X$. Let $F : X \to 2^{\mathbb{R}^m}$, $G : X \to 2^{\mathbb{R}^m}$, and $H : X \to 2^{\mathbb{R}^k}$ be set-valued maps, with

$$A \subseteq \text{dom}(F) \cap \text{dom}(G) \cap \text{dom}(H).$$

Let $F = (F_1, F_2, \ldots, F_m)$, $G = (G_1, G_2, \ldots, G_m)$, and $H = (H_1, H_2, \ldots, H_k)$, where the set-valued maps $F_i : X \to 2^{\mathbb{R}}$, $G_i : X \to 2^{\mathbb{R}}$; $i = 1, 2, \ldots, m$, and $H_j : X \to 2^{\mathbb{R}}$; $j = 1, 2, \ldots, k$, are defined by

$$\text{dom}(F_i) = \text{dom}(F), \text{dom}(G_i) = \text{dom}(G), \text{and dom}(H_j) = \text{dom}(H),$$

$$x \in A, y = (y_1, y_2, \ldots, y_m) \in F(x) \implies y_i \in F_i(x), \forall i = 1, 2, \ldots, m,$$

$$z = (z_1, z_2, \ldots, z_m) \in G(x) \implies z_i \in G_i(x), \forall i = 1, 2, \ldots, m,$$

and

$$w = (w_1, w_2, \ldots, w_k) \in H(x) \implies w_j \in H_j(x), \forall j = 1, 2, \ldots, k.$$

Assume that $F_i(x) \subseteq \mathbb{R}^+$ and $G_i(x) \subseteq \text{int}(\mathbb{R}^+), \forall i = 1, 2, \ldots, m$ and $x \in A$. Let $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m) \in \mathbb{R}^m$. Define elements $\frac{y}{z}, \lambda'z \in \mathbb{R}^k$ and a subset $\lambda'G(x)$ of $\mathbb{R}^m$ by

$$\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, \ldots, \frac{y_m}{z_m}\right),$$

$$\lambda'z = (\lambda'_1z_1, \lambda'_2z_2, \ldots, \lambda'_mz_m),$$

and

$$\lambda'G(x) = \{\lambda'z : z \in G(x)\}.$$

For $x \in A$, define a subset $F(x)G(x)$ of $\mathbb{R}^m$ by

$$F(x)G(x) = \left\{\frac{y}{z} = \left(\frac{y_1}{z_1}, \frac{y_2}{z_2}, \ldots, \frac{y_m}{z_m}\right) : y = (y_1, y_2, \ldots, y_m) \in F(x), z = (z_1, z_2, \ldots, z_m) \in G(x)\right\}.$$

Consider a set-valued fractional programming problem (FP):

$$\text{minimize } \frac{F(x)}{G(x)} \quad \text{subject to } \quad H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \quad \text{(FP)}$$

The feasible set of the problem (FP) is given by

$$S_{FP} = \{x \in A : H(x) \cap (-\mathbb{R}^k_+) \neq \emptyset\}.$$ 

**Definition 3.1.** A point $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$, with $x' \in S_{FP}$, $y' \in F(x')$, and $z' \in G(x')$, is called a minimizer of the problem (FP) if for all $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$, with $x \in S_{FP}$, $y \in F(x)$, and $z \in G(x)$,

$$\frac{y}{z} - \frac{y'}{z'} \notin (-\mathbb{R}^m_+) \setminus \{0_{\mathbb{R}^m}\}.$$

**Definition 3.2.** A point $(x', \frac{y'}{z'}) \in X \times \mathbb{R}^m$, with $x' \in S_{FP}$, $y' \in F(x')$, and $z' \in G(x')$, is called a weak minimizer of the problem (FP) if for all $(x, \frac{y}{z}) \in X \times \mathbb{R}^m$, with $x \in S_{FP}$, $y \in F(x)$, and $z \in G(x)$,

$$\frac{y}{z} - \frac{y'}{z'} \notin (-\text{int}(\mathbb{R}^m_+)).$$
Definition 3.3. Let $A$ be a nonempty convex subset of a real normed space $X$, $e \in \text{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then $F$ is said to be $\rho$-$K$-convex with respect to $e$ on $A$ if there exists $\rho \in \mathbb{R}$, such that

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho \lambda(1 - \lambda)||x_1 - x_2||^2 e + K,$$

$$\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

We construct an example of $\rho$-cone convex set-valued map, which is not cone convex.

Example 3.1. Consider a set-valued map $F : [-1, 1] \subseteq \mathbb{R} \to 2^{\mathbb{R}^2}$ defined by

$$F(t) = \begin{cases} \{(x - 2t^2, x^2 - 2t^2) : x \geq 0\}, & \text{if } 0 \leq t \leq 1, \\
\{(x - 2t^2, x^2 - 2t^2) : x \leq 0\}, & \text{if } -1 \leq t < 0.\end{cases}$$

We prove that $F$ is not $\mathbb{R}^2_+\text{-convex on } [-1, 1]$ but is $(-2)\mathbb{R}^2_+\text{-convex with respect to } e = I_{\mathbb{R}^2} = (1, 1) \text{ on } [-1, 1].$ \hfill $\square$

Let $\lambda^\prime \in \mathbb{R}^m_+$ and $G : X \to 2^{\mathbb{R}^m}$ be a set-valued map. Define a set-valued map $(-\lambda^\prime G) : X \to 2^{\mathbb{R}^m}$ by $(-\lambda^\prime G)(x) = -\lambda^\prime G(x), \forall x \in \text{dom}(G).$

We establish the sufficient optimality conditions of the problem (FP) by using the notion of contingent epiderivable and $\rho$-cone convexity assumptions.

Theorem 3.1. (Sufficient optimality conditions) Let $A$ be a nonempty convex subset of a real normed space $X$, $x^\prime$ be an element of the feasible set $S_{FP}$ of the problem (FP), $y^\prime \in F(x^\prime)$, $z^\prime \in G(x^\prime)$, $\lambda^\prime \in F(x^\prime)$, $w^\prime \in H(x^\prime) \cap (-L)$, and $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$. Assume that $F$, $-\lambda^\prime G$ are $\rho_1\mathbb{R}^m_+\text{-convex, } \rho_2\mathbb{R}^m_+\text{-convex, respectively, with respect to } I_{\mathbb{R}^m} \text{ and } H$ is $\rho_3\mathbb{R}^k_+\text{-convex with respect to } I_{\mathbb{R}^k}$, on $A$. Let $F$, $-\lambda^\prime G$ be contingent epiderivable at $(x^\prime, y^\prime)$, $(x^\prime, -\lambda^\prime z^\prime)$, respectively and $H$ be contingent epiderivable at $(x^\prime, w^\prime)$. Suppose that there exists $(y^\prime, z^\prime) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^\prime \neq 0_{\mathbb{R}^m}$, and

$$\begin{align*}
\langle p_1 + p_2, y^\prime, I_{\mathbb{R}^m} \rangle + p_2(z^\prime, I_{\mathbb{R}^k}) & \geq 0, \quad (3.1) \\
\langle y^\prime, D_1F(x^\prime, y^\prime)(x - x^\prime) + D_1(-\lambda^\prime G)(x^\prime, -\lambda^\prime z^\prime)(x - x^\prime) \rangle \\
+ \langle z^\prime, D_1H(x^\prime, w^\prime)(x - x^\prime) \rangle & \geq 0, \forall x \in A, \quad (3.2) \\
y^\prime - \lambda^\prime z^\prime & = 0_{\mathbb{R}^m}, \quad (3.3) \\
\langle z^\prime, w^\prime \rangle & = 0. \quad (3.4)
\end{align*}$$

Then $(x^\prime, \frac{w^\prime}{\rho_3})$ is a weak minimizer of the problem (FP).

4. Optimization Problems with Difference of Set-Valued Maps

Let $X, Y, Z$ be real normed spaces and $A$ be a nonempty convex subset of $X$. Let $K$ and $L$ be solid pointed convex cones in $Y$ and $Z$, respectively. Suppose that $F_1 : X \to 2^Y$, $F_2 : X \to 2^Z$, $G_1 : X \to 2^Z$, and $G_2 : X \to 2^Z$ are set-valued maps with $A \subseteq \text{dom}(F_1) \cap \text{dom}(F_2) \cap \text{dom}(G_1) \cap \text{dom}(G_2)$. Consider an optimization problem (DP) with the difference of set-valued maps:

$$\begin{align*}
\text{minimize} & \quad F_1(x) - F_2(x) \\
\text{subject to} & \quad (G_1(x) - G_2(x)) \cap (-L) \neq \emptyset. \quad (DP)
\end{align*}$$
Here, the feasible set $S_{DP}$ of the problem (DP) is defined by
\[ S_{DP} = \{ x \in A : (G_1(x) - G_2(x)) \cap (-L) \neq \emptyset \}. \]

**Definition 4.1.** A point $(x', y_1'-y_2') \in X \times Y$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, is called a minimizer of the problem (DP) if for all $(x, y_1-y_2) \in X \times Y$, with $x \in S_{DP}$, $y_1 \in F_1(x)$, and $y_2 \in F_2(x)$,
\[ (y_1-y_2) - (y_1'-y_2') \notin (-K) \setminus \{ \emptyset \}. \]

**Definition 4.2.** A point $(x', y_1'-y_2') \in X \times Y$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, is called a weak minimizer of the problem (DP) if for all $(x, y_1-y_2) \in X \times Y$, with $x \in S_{DP}$, $y_1 \in F_1(x)$, and $y_2 \in F_2(x)$,
\[ (y_1-y_2) - (y_1'-y_2') \notin (-\text{int}(K)). \]

Let $y^* \in Y^*$, $z^* \in Z^*$, $x' \in A$, $y_1' - y_2'$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, be a weak minimizer of the problem (DP) and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with $z_1' - z_2' \in (-L)$.

Suppose that $F_1 : X \to 2^Y$, $G_1 : X \to 2^Z$ are $K$-convex, $L$-convex, respectively, on $A$. Also, suppose that $F_2 : X \to 2^Y$, $G_2 : X \to 2^Z$ are strongly $\rho_2$-$K$-convex, strongly $\rho_2$-$L$-convex, with respect to $e_2, e_2', \text{respectively, on } A$. Assume that $\partial_s F_2(x'; y_2') \neq \emptyset$ and $\partial_s G_2(x'; z_2') \neq \emptyset$. Then there exists $(\theta_Y, \theta_Z, \cdot) \neq (y^*, z^*) \in K^+ \times L^+$, such that
\[ y^* T_1 + z^* T_2 \in \partial(y^* F_1 + z^* G_1)(x'; (y^*, y_1') + (z^*, z_1')), \]
\[ \forall T_1 \in \partial_s F_2(x'; y_2') \text{ and } T_2 \in \partial_s G_2(x'; z_2') \]
and
\[ \langle z^*, z_1' - z_2' \rangle = 0. \]

We establish the sufficient KKT conditions of the set-valued D. C. optimization problem (DP) by using $\rho$-cone convexity assumptions.

**Theorem 4.1.** Let $A$ be a nonempty convex subset of a real normed space $X$, $e_2 \in \text{int}(K)$, $e_2' \in \text{int}(L)$, and $\rho_2, \rho_2' \in \mathbb{R}$. Let $(x', y_1' - y_2')$, with $x' \in S_{DP}$, $y_1' \in F_1(x')$, and $y_2' \in F_2(x')$, be a weak minimizer of the problem (DP) and there exist $z_1' \in G_1(x')$ and $z_2' \in G_2(x')$, with $z_1' - z_2' \in (-L)$.

Suppose that $F_1 : X \to 2^Y$, $G_1 : X \to 2^Z$ are $K$-convex, $L$-convex, respectively, on $A$. Also, suppose that $F_2 : X \to 2^Y$, $G_2 : X \to 2^Z$ are strongly $\rho_2$-$K$-convex, strongly $\rho_2$-$L$-convex, with respect to $e_2, e_2'$, respectively, on $A$. Assume that $\partial_s F_2(x'; y_2') \neq \emptyset$ and $\partial_s G_2(x'; z_2') \neq \emptyset$. Then there exists $(\theta_Y, \cdot) \neq (y^*, z^*) \in K^+ \setminus \{ \emptyset \}$ and $z^* \in L^+$, satisfying $\rho_2(y^*, e_2) + \rho_2'(z^*, e_2') \geq 0$, (4.5)

such that
\[ y^* T_1 + z^* T_2 \in \partial(y^* F_1 + z^* G_1)(x'; (y^*, y_1') + (z^*, z_1')) \]
\[ \forall T_1 \in \partial_s F_2(x'; y_2'), T_2 \in \partial_s G_2(x; z_2), x \in A, y_2 \in F_2(x), \text{ and } z_2 \in G_2(x) \]
(4.6)
and
\[
\langle z^*, z_1^* - z_2^* \rangle = 0,
\]
then \((x', y_1' - y_2')\) is a weak minimizer of the problem (DP).

5. Set-Valued Semi-Infinite Programming Problems

We consider semi-infinite programming problems in the setting of set-valued maps. Let \(U\) be a countably infinite subset of \(\mathbb{R}^p\), \(\emptyset \neq A \subseteq \mathbb{R}^p\), and \(F = (F_1, F_2, ..., F_m) : \mathbb{R}^n \to 2^{\mathbb{R}^m}\), \(G : \mathbb{R}^n \times U \to 2^\mathbb{R}\) be two set-valued maps with
\[
A \subseteq \text{dom}(F) \text{ and } A \times U \subseteq \text{dom}(G).
\]
Let \(B_1, B_2, ..., B_m\) be \(n \times n\) positive semi-definite (symmetric) real matrices. Consider a set-valued semi-infinite programming problem (SP):

minimize \(x \in A\)

\[
(F_1(x) + (x^T B_1 x)\frac{1}{2}, F_2(x) + (x^T B_2 x)\frac{1}{2}, ..., F_m(x) + (x^T B_m x)\frac{1}{2})
\]

subject to \(G(x, u) \cap (-\mathbb{R}_+^m) \neq \emptyset, \forall u \in U\).

The feasible set of the problem (SP) is defined by
\[
S_{SP} = \{x \in A : G(x, u) \cap (-\mathbb{R}_+^m) \neq \emptyset, \forall u \in U\}.
\]

Definition 5.1. A point \((x', y') \in \mathbb{R}^n \times \mathbb{R}^m\), with \(x' \in S_{SP}\) and \(y' = (y_1', y_2', ..., y_m') \in F(x')\), is called a minimizer of the problem (SP) if for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\), with \(x \in S_{SP}\) and \(y = (y_1, y_2, ..., y_m) \in F(x)\),
\[
(y_1 + (x^T B_1 x)\frac{1}{2}, y_2 + (x^T B_2 x)\frac{1}{2}, ..., y_m + (x^T B_m x)\frac{1}{2})
\]
\[- (y_1' + (x'^T B_1 x')\frac{1}{2}, y_2' + (x'^T B_2 x')\frac{1}{2}, ..., y_m' + (x'^T B_m x')\frac{1}{2}) \notin (-\mathbb{R}_+^m) \setminus \{0^m\}.
\]

Definition 5.2. A point \((x', y') \in \mathbb{R}^n \times \mathbb{R}^m\), with \(x' \in S_{SP}\) and \(y' = (y_1', y_2', ..., y_m') \in F(x')\), is called a weak minimizer of the problem (SP) if for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\), with \(x \in S_{SP}\) and \(y = (y_1, y_2, ..., y_m) \in F(x)\),
\[
(y_1 + (x^T B_1 x)\frac{1}{2}, y_2 + (x^T B_2 x)\frac{1}{2}, ..., y_m + (x^T B_m x)\frac{1}{2})
\]
\[- (y_1' + (x'^T B_1 x')\frac{1}{2}, y_2' + (x'^T B_2 x')\frac{1}{2}, ..., y_m' + (x'^T B_m x')\frac{1}{2}) \notin (-\text{int}(\mathbb{R}_+^m)).
\]

Let \(J\) be the index set, such that \(U = \{u_j : j \in J\}\). Let \(x' \in A\). Denote a set \(J(x')\) by
\[
J(x') = \{j \in J : 0 \in G(x', u_j)\}.
\]

Throughout this chapter, we assume that \(J(x') \neq \emptyset\). Let \(\pi_i \in \mathbb{R}^n\), \(i = 1, 2, ..., m\). Define maps \(\tilde{T}_i B_i \pi_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., m\), by
\[
(\tilde{T}_i B_i \pi_i)(x) = x^T B_i \pi_i, \forall x \in \mathbb{R}^n.
\]

The gradient vector of \(\tilde{T}_i B_i \pi_i\), denoted by \(\nabla(\tilde{T}_i B_i \pi_i)\), is given by
\[
\nabla(\tilde{T}_i B_i \pi_i) = B_i \pi_i.
\]

Let \(x' \in A\) and \(j \in J(x')\). Define a set-valued map \(G(., u_j) : \mathbb{R}^n \to 2^\mathbb{R}\) by
\[
G(., u_j)(x) = G(x, u_j), \forall x \in \text{dom}(G).
\]

We establish the sufficient KKT conditions of the set-valued semi-infinite programming problem (SP) by using \(\rho\)-cone convexity assumptions.
Theorem 5.1. (Sufficient optimality conditions) Let $A$ be a nonempty convex subset of $\mathbb{R}^n$, $x' \in SSP$, and $y' = (y'_1, y'_2, \ldots, y'_m) \in F(x')$. Let $\pi_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$ and $z_j' = (z_j')_{j \in J}$, with $z_j' \in G(x'_j, u_j) \cap (-\mathbb{R}_+^n)$. Let $\rho_i, \rho_j' \in \mathbb{R}$, for $i = 1, 2, \ldots, m$ and $j \in J(x')$. Suppose that $F_i, \pi_i \tau, i = 1, 2, \ldots, m$, and $G(., u_j), j \in J(x')$, are $\rho_i$-convex, $\rho_i'$-convex, and $\rho_i'$-$\mathbb{R}_+$-convex valued set-valued maps, respectively, with respect to 1, on $A$. Assume that the contingent epiderivatives $D_{\tau}F_i(x'_j, y'_j')$ and $D_{\tau}G(., u_j)(x'_j, z_j')$ exist. If there exist $y_i' > 0$, $i = 1, 2, \ldots, m$, and $z_j' \geq 0$, $j \in J(x')$, with $z_j' \neq 0$, for finitely many $j$, and
\[
\sum_{i=1}^{m} y_i' (\rho_i + \rho_i') + \sum_{j \in J(x')} z_j' \rho_j'' \geq 0, \tag{5.8}
\]
satisfying the following conditions
\[
\left( \sum_{i=1}^{m} y_i' (D_{\tau}F_i(x'_j, y'_j') + (B_{i, \pi_i})'\tau) + \sum_{j \in J(x')} z_j' D_{\tau}G(., u_j)(x'_j, z_j') \right)(x-x') \geq 0, \forall x \in A, \tag{5.9}
\]
\[
z_j' z_j' = 0, \forall j \in J(x'), \tag{5.10}
\]
\[
\pi_i' B_{i, \pi_i} \leq 1, i = 1, 2, \ldots, m, \tag{5.11}
\]
and
\[
(x'^T B_{i, \pi_i})^{\frac{1}{2}} = x'^T B_{i, \pi_i}, i = 1, 2, \ldots, m. \tag{5.12}
\]
Then $(x', y')$ is a weak minimizer of the problem (SP).

6. Set-Valued Minimax Programming Problems

Definition 6.1. A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called upper semicontinuous if $F^+(V) = \{ x \in \mathbb{R}^n : F(x) \subseteq V \}$ is open in $\mathbb{R}^n$ for any open set $V$ in $\mathbb{R}^m$.

Definition 6.2. Let $B$ be a nonempty subset of $\mathbb{R}^m$. Then $B$ is called $\mathbb{R}^m_+$-semicompact if every open cover of complements of the form $\{(y_i + \mathbb{R}^m_+)^c : y_i \in B, i \in I\}$ has a finite subcover.

Definition 6.3. A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called $\mathbb{R}^m_+$-semicompact-valued if $F(x)$ is $\mathbb{R}^m_+$-semicompact, for all $x \in \text{dom}(F)$.


Theorem 6.1. [6] Let $X, Y$ be real topological vector spaces, $A$ be a nonempty compact subset of $X$, and $K$ be an acute, (i.e., $K$ is pointed) convex cone in $Y$. Let $F : X \to 2^Y$ be a $K$-semicompact-valued and upper semicontinuous set-valued map. Then there exists a maximal point of the problem $\max_{x \in A} F(x)$.

For simplicity, let us assume $X = \mathbb{R}^m$, $Y = \mathbb{R}$, and $K = \mathbb{R}_+$.

Let $A$ be a nonempty subset of $\mathbb{R}^n$ and $B$ be a nonempty compact subset of $\mathbb{R}^m$. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to 2^\mathbb{R}$ and $G : \mathbb{R}^n \to 2^{\mathbb{R}^p}$ be two set-valued maps with $A \times B \subseteq \text{dom}(\Phi)$ and $A \subseteq \text{dom}(G)$.

Consider a set-valued minimax programming problem (MP):
\[
\begin{align*}
\text{minimize} & \quad \max_{y \in B} \bigcup_{x \in A} \Phi(x, y) \\
\text{subject to} & \quad G(x) \cap (-\mathbb{R}_+^p) \neq \emptyset.
\end{align*}
\]
where the set-valued map $\Phi(x, \cdot) : \mathbb{R}^m \to 2^\mathbb{R}$ is $\mathbb{R}_+$-semicompact-valued and upper semi-continuous on $B$, for all $x \in A$. Therefore, by Theorem 6.1, $\max_{y \in B} \Phi(x, y)$ always exists, for all $x \in A$. As $\Phi(x, y) \subseteq \mathbb{R}$, for each $x \in A$ there exists only one maximal point of the problem $\max_{y \in B} \Phi(x, y)$.

The feasible set of the problem (MP) is given by

$$S_{MP} = \{ x \in A : G(x) \cap (-\mathbb{R}_+^p) \neq \emptyset \}.$$ 

For $x \in A$, define following sets by

$$I(x) = \{ j : 0 \in G_j(x), 1 \leq j \leq p \},$$

$$J(x) = \{ 1, 2, \ldots, p \} \setminus I(x),$$

and

$$B(x) = \{ b \in B : \max_{y \in B} \Phi(x, y) \in \Phi(x, b) \}.$$ 

Under the assumptions, $B(x) \neq \emptyset$, for all $x \in A$.

**Definition 6.4.** Let $x' \in S_{MP}$ and $z' = \max_{y \in B} \Phi(x', y)$. Then $(x', z')$ is called a minimizer of the problem (MP) if for all $x \in S_{MP}$ and $z = \max_{y \in B} \Phi(x, y)$,

$$z' \leq z.$$ 

The sufficient KKT conditions of the set-valued minimax programming problem (MP) are established by using $\rho$-cone convexity assumptions.

**Theorem 6.2.** (Sufficient optimality conditions) Let $A$ be a nonempty convex subset of $\mathbb{R}^n$, $x' \in S_{MP}$, and $z' = \max_{y \in B} \Phi(x', y)$. Assume that there exist a positive integer $k$, $z^*_i \geq 0$, $y_i \in B(x')$, $(1 \leq i \leq k)$ with $\sum_{i=1}^k z^*_i \neq 0$, and $w^*_j \geq 0$, $w'_j \in G_j(x') \cap (-\mathbb{R}_+)$, $(1 \leq j \leq p)$, such that

$$\sum_{i=1}^k z^*_i D_T \Phi(\cdot, y_i)(x' - x') + \sum_{j=1}^p w^*_j D_T G_j(x', w'_j)(x' - x') \geq 0, \quad \forall x \in A \quad (6.13)$$

and

$$w^*_j w'_j = 0, \forall j = 1, 2, \ldots, p. \quad (6.14)$$

Let $\rho_i, \rho'_j \in \mathbb{R}$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$. If $\Phi(\cdot, y_i)$, $(1 \leq i \leq k)$ and $G_j$, $(1 \leq j \leq p)$ are $\rho_i-\mathbb{R}_+$-convex and $\rho'_j-\mathbb{R}_+$-convex, respectively, with respect to 1, on $A$, with

$$\sum_{i=1}^k z^*_i \rho_i + \sum_{j=1}^p w^*_j \rho'_j \geq 0, \quad (6.15)$$

then $(x', z')$ is a minimizer of the problem (MP).
7. Set-Valued Optimization Problems via Contingent Epiderivative

Let $X$, $Y$, $Z$ be real normed spaces and $A$ be a nonempty subset of $X$. Let $K$ and $L$ be solid pointed convex cones of $Y$ and $Z$, respectively. Suppose that $F : X \to 2^Y$ and $G : X \to 2^Z$ are two set-valued maps, with $A \subseteq \text{dom}(F) \cap \text{dom}(G)$.

Consider the set-valued optimization problem (P):

\[
\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} & \quad G(x) \cap (-L) \neq \emptyset.
\end{align*}
\]

The feasible set of the problem (P) is given by

\[S_P = \{x \in A : G(x) \cap (-L) \neq \emptyset\}.
\]

We introduce the notions of $\rho(\eta, \theta)$-cone preinvexity and $\rho(\eta, \theta)$-cone invexity of set-valued maps.

**Definition 7.1.** Let $A$ be an invex subset of a real normed space $X$ with respect to $\eta : A \times A \to X$, $e \in \text{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then $F$ is said to be $\rho(\eta, \theta)$-$K$-preinvex with respect to $e$ on $A$ if there exist a map $\theta : A \times A \to X$ and $\rho \in \mathbb{R}$, such that

\[
\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(x_2 + \lambda \eta(x_1, x_2)) + \rho\lambda(1 - \lambda)\|\theta(x_1, x_2)\|^2 e + K,
\]

\[
\forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].
\]

**Definition 7.2.** Let $A$ be a nonempty subset of a real normed space $X$, $e \in \text{int}(K)$, and $F : X \to 2^Y$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Let $x' \in A$ and $y' \in F(x')$. Assume that $F$ is contingent epiderivable at $(x', y')$. Then $F$ is said to be $\rho(\eta, \theta)$-$K$-invex with respect to $e$ on $A$ if there exist maps $\eta, \theta : A \times A \to X$ and $\rho \in \mathbb{R}$, with

\[
\eta(A, x') \subseteq \text{dom}(D_x F(x', y')),
\]

such that

\[
F(x) - y' \subseteq D_x F(x', y')(\eta(x, x')) + \rho\|\theta(x, x')\|^2 e + K, \forall x \in A.
\]

We give the following example of $\rho(\eta, \theta)$-cone invex set-valued map for some suitable $\rho$, $\eta$ and $\theta$, which is not an $\eta$-invex set-valued map for any $\eta$.

**Example 7.1.** Consider a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ defined by

\[
F(\lambda) = \begin{cases} 
\{(x, \sqrt{x}) : x \geq 0\}, & \text{if } \lambda \geq 0, \\
\{(x, \sqrt{-x}) : x \in [-4, 0]\}, & \text{if } \lambda < 0.
\end{cases}
\]

Take $K = \mathbb{R}_+^2$. Here $F$ is not $\eta$-invex at $(0, (0, 0))$ on $\mathbb{R}$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\rho = -1$, and $e = 1_{\mathbb{R}^2}$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 
0, & \text{if } \lambda \geq 0, \\
2, & \text{if } \lambda < 0.
\end{cases}$

We prove that $F$ is $\rho(\eta, \theta)$-$\mathbb{R}_+^2$-invex with respect to $1_{\mathbb{R}^2}$ at $(0, (0, 0))$ on $\mathbb{R}$.

We establish the KKT sufficient optimality conditions of the problem (P) by using contingent epiderivative and $\rho(\eta, \theta)$-cone preinvexity assumptions.

**Theorem 7.1.** (Sufficient optimality conditions) Let $A$ be an invex subset of a real normed space $X$ with respect to a map $\eta : A \times A \to X$, $x' \in S_P$, $y' \in F(x')$, and $z' \in G(x') \cap (-L)$. Let $e \in \text{int}(K)$ and $e' \in \text{int}(L)$. Let $\rho_1, \rho_2 \in \mathbb{R}$ and $\eta, \theta : A \times A \to X$ be two maps. Assume
that \( F, G \) are \( \rho_1(\eta, \theta)\)-\( K \)-preinvex, \( \rho_2(\eta, \theta)\)-\( L \)-preinvex with respect to \( e, e' \), respectively, on \( A \). Let \( F \) and \( G \) be contingent epiderivable at \((x', y')\) and \((x', z')\), respectively, with 
\[
\eta(A, x') \subseteq \text{dom}(D_1 F(x', y')) \cap \text{dom}(D_1 G(x', z')).
\]
If there exists \((y^*, z^*) \in K^+ \times L^+\), with \( y^* \neq \theta y^* \) and 
\[
\rho_1(y^*, e) + \rho_2(z^*, e') \geq 0,
\] 
(7.16) such that
\[
\langle y^*, D_1 F(x', y')(\eta(x, x')) \rangle + \langle z^*, D_1 G(x', z')(\eta(x, x')) \rangle \geq 0, \forall x \in A
\] 
(7.17) and
\[
\langle z^*, z' \rangle = 0.
\] 
(7.18) Then \((x', y')\) is a weak minimizer of the problem \((P)\).

We illustrate the Theorem 7.1 by the following example.

**Example 7.2.** We consider a primal problem \((P)\), where \( A = \mathbb{R}, F : \mathbb{R} \rightarrow 2^{\mathbb{R}^2}\) is defined like Example 7.1, and \( G : \mathbb{R} \rightarrow 2^{\mathbb{R}^2}\) is defined as
\[
G(\lambda) = \begin{cases} 
\{ (x, x^2) : x \geq 0 \}, & \text{if } \lambda \geq 0, \\
\{ (x, x) : x > 8 \}, & \text{if } \lambda < 0.
\end{cases}
\]
Take \( K = \mathbb{R}^2_+ \), and \( L = \mathbb{R}^2_+ \). Choose any map \( \eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \rho_1 = -1, \) and \( \rho_2 = 2 \). We also choose a map \( \theta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) in such a way that,
\[
\theta(\lambda, 0) = \begin{cases} 
0, & \text{if } \lambda \geq 0, \\
2, & \text{if } \lambda < 0.
\end{cases}
\]
We prove that \( G \) is \( \rho_2(\eta, \theta)\)-\( \mathbb{R}^2_+ \)-invex with respect to \( \mathbb{I}_{\mathbb{R}^2} \) at \((1, 0, 0)\) on \( \mathbb{R} \). From Example 7.1, \( F \) is \( \rho_1(\eta, \theta)\)-\( \mathbb{R}^2_+ \)-invex with respect to \( \mathbb{I}_{\mathbb{R}^2} \) at \((1, 0, 0)\) on \( \mathbb{R} \). It is clear that for \( y^* = z^* = (1, 1) \), Eqs. (7.17) and (7.18) are satisfied. Therefore, \((x', (x', y')) = (1, (0, 0))\) is a weak minimizer of the problem \((P)\).

8. Set-Valued Optimization Problems via Second-Order Contingent Epi-derivative

We introduce second-order \( \rho-(\eta, \theta) \)-cone invexity of set-valued maps via second-order contingent epiderivative.

**Definition 8.1.** Let \( A \) be a nonempty subset of a real normed space \( X \), \( e \in \text{int}(K) \), and \( F : X \rightarrow 2^Y \) be a set-valued map, with \( A \subseteq \text{dom}(F) \). Let \( x', u \in A, \ y' \in F(x'), \) and \( v \in F(u) + K \). Assume that \( F \) is second-order contingent epiderivable at \((x', y')\) in the direction \((u - x', v - y')\). Then \( F \) is said to be second-order \( \rho-(\eta, \theta)\)-\( K \)-invex with respect to \( e \) at \((x', y')\) in the direction \((u - x', v - y')\) on \( A \) if there exist maps \( \eta, \theta : A \times A \rightarrow X \), and \( \rho \in \mathbb{R}, \) with 
\[
\eta(A, x') \subseteq \text{dom}(D_2^2 F(x', y', u - x', v - y')),
\]
such that
\[
F(x) - y' \subseteq D_2 F(x', y', u - x', v - y')(\eta(x, x')) + \rho\|\theta(x, x')\|^2 e + K, \forall x \in A.
\]

We construct the following set-valued map \( F : \mathbb{R} \rightarrow 2^{\mathbb{R}^2}, \) which is second-order \( \rho-(\eta, \theta)\)-\( \mathbb{R}^2_+ \)-invex for some \( \rho, \eta \) and \( \theta, \) but is not second-order \( \eta \)-invex, for any \( \eta. \)
Example 8.1. Let a set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be defined by

$$F(\lambda) = \begin{cases} \{(x, \sqrt{x}) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, \sqrt{-x}) : x \in [-4, 0]\}, & \text{if } \lambda < 0. \end{cases}$$

Let $K = \mathbb{R}^2_+$. Here $F$ is not second-order $\eta$-invex at $(0, (0,0))$ in the direction $(-1, (0,0))$ on $\mathbb{R}$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\rho = -1$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$ We prove that $F$ is second-order $\rho(\eta, \theta)$-$\mathbb{R}^2_+$-invex with respect to $1_{\mathbb{R}^2}$ at $(0, (0,0))$ in the direction $(-1, (0,0))$ on $\mathbb{R}$. \hfill \Box

We establish the second-order KKT sufficient optimality conditions of the problem (P) via second-order contingent epiderivative and second-order $\rho(\eta, \theta)$-cone invexity assumptions.

Theorem 8.1. (Sufficient optimality conditions) Let $x'$ be a feasible point of the problem (P), $y' \in F(x')$, and $z' \in G(x') \cap (-L)$. Let $u \in X$, $v \in F(u) + K$, $w \in G(u) + L$, $e \in \text{int}(K)$, and $e' \in \text{int}(L)$. Let $\rho_1, \rho_2 \in \mathbb{R}$ and $\eta, \theta : A \times X \to \mathbb{R}$ be two maps. Assume that $F$ is second-order $\rho_1(\eta, \theta)$-$K$-invex with respect to $e$ at $(x', y')$ in the direction $(u - x', v - y')$ and $G$ is second-order $\rho_2(\eta, \theta)$-$L$-invex with respect to $e'$ at $(z', z')$ in the direction $(u - x', w - z')$, on $A$. Suppose that there exists $(y^*, z^*) \in K^+ \times L^+$, with $y^* \neq \theta y^*$, satisfying

$$\rho_1(y^*, e) + \rho_2(z^*, e') \geq 0, \tag{8.19}$$

such that

$$\langle y^*, D^2 F(x', y', u - x', v - y') \eta(x, x') \rangle + \langle z^*, D^2 G(x', z', u - x', w - z') \eta(x, x') \rangle \geq 0, \forall x \in A \tag{8.20}$$

and

$$\langle z^*, z' \rangle = 0. \tag{8.21}$$

Then $(x', y')$ is a weak minimizer of the problem (P).

We illustrate the Theorem 8.1 by the following example.

Example 8.2. We consider a primal problem (P), where $X = \mathbb{R}$, the set-valued map $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ is given in Example 8.1, and $G : \mathbb{R} \to 2^{\mathbb{R}^2}$ is defined as

$$G(\lambda) = \begin{cases} \{(x, x^2) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, x) : x \geq 8\}, & \text{if } \lambda < 0. \end{cases}$$

Let $K = \mathbb{R}^2_+$ and $L = \mathbb{R}^2_+$. Choose any map $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\rho_1 = -1$, and $\rho_2 = 2$. We also choose a map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in such a way that, $\theta(\lambda, 0) = \begin{cases} 0, & \text{if } \lambda \geq 0, \\ 2, & \text{if } \lambda < 0. \end{cases}$ It is clear that for $y^* = z^* = (1, 1)$, Eqs. (8.20) and (8.21) are satisfied. Therefore, $(x', (x', y')) = (0, (0, 0))$ is a weak minimizer of the problem (P). \hfill \Box

9. Conclusions

In this paper, we introduce set-valued fractional programming problems, set-valued D. C. optimization problems, set-valued semi-infinite programming problems, and set-valued minimax programming problems. We establish the optimality conditions of various types of set-valued optimization problems via $\rho$-cone convexity assumptions.

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