COMMON FIXED POINTS FOR MAPPINGS OF CYCLIC FORM SATISFYING LINEAR CONTRACTIVE CONDITIONS WITH OMEGA-DISTANCE

by Wasfi Shatanawi[1], Georgeta Maniu[2], Anwar Bataihah[3] and Feras Bani Ahmad[4]

In this paper we utilize the concept of cyclic form and Ω-distance to derive and prove some common fixed point theorems for self mappings of cyclic form by using the concept of Ω-distance. Our results are extensions on some results on Ω-distance.

Keywords: Common fixed point, cyclic mapping, Omega-distance.
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1. Introduction

In 2006 Mustafa and Sims introduced a new generalization of the usual metric spaces named G-metric spaces and studied some fixed point results: please, see [1]. After that, many authors studied fixed and common fixed point results in complete G-metric spaces: Mustafa and Sims [2]; Aydi *et al.* [3, 4]; Abbas *et al.* [5, 6, 7]; Karapinar and Agarwal [8]; Bilgili *et al.* [9, 10]; Chandok *et al.* [11]; Pourhadi [12]; Popa and Patriciu [13]; Tu *et al.* [14]; Thangthong and Charoensawan [15]; Shatanawi [16, 17], Shatanawi and Postolache [18]. But Jleli and Samet [19] and Samet *et al.* [20] in their clever papers showed that there are some fixed point theorems in the setting of G-metric spaces which can be obtained from well-known fixed point theorems in metric spaces or quasi metric spaces. Thereafter, Karapinar and Agarwal in their interesting paper [8] showed that the smart technique of Samet *et al.* [19, 20] cannot be used to all contractive conditions. For this instance, they introduced some contractive conditions where the technique of Samet *et al.* [19, 20] does not work.

In 2010 Saadati *et al.* [21] introduced the concept of Ω-distance and proved some fixed point results in a complete G-metric space. After that, many authors utilized the concept of Ω-distance in a complete G-metric space to prove some fixed and coupled fixed point results: Gholizadeh *et al.* [22]; Shatanawi *et al.* [23, 24, 25]; Gholizadeh [26]. These results cannot be evolved by the technique used in [19, 20]. Recently, many authors proved fixed and common fixed point theorems for mappings of cyclic form in different metric spaces, for example see [27]-[42]. In this paper we utilize the concept of cyclic form and Ω-distance to derive and prove some common fixed point theorems for self mappings of cyclic form by using the concept of Ω-distance.

[1]Department of Mathematics, Faculty of Science, Hashemite University, Zarqa, Jordan, Email: swasfi@hu.edu.jo, wshatanawi@yahoo.com; Department of Mathematics and General Courses, Prince Sultan University, Riyadh, Saudi Arabia, E-mail: wshatanawi@psu.edu.sa
[2]Department of Mathematics and Informatics, University Politehnica of Bucharest, Bucharest, 060042, Romania; Department of Computer Science, Information Technology, Mathematics and Physics, Petroleum-Gas University of Ploieşti, Email: maniugeorgeta@gmail.com
[3]Department of Mathematics, Faculty of Science, Irbid National University, Irbid, Jordan, Email: anwerbataihah@gmail.com
[4]Department of Mathematics, Faculty of Science, Hashemite University, Zarqa, Jordan, Email: fbaniahmad@hu.edu.jo

2. **Preliminaries**

Now, we recall the concept of cyclic mappings.

Definition 2.1. Let A and B be two nonempty subsets of a space X. A mapping $T\colon A\cup B \to A\cup B$ is called cyclic if $T(A)\subseteq B$ and $T(B)\subseteq A$.

The notion of G-metric spaces was given in 2006 by Z. Mustafa and B. Sims [1] as follows:

Definition 2.2 ([1])**.** Let X be a nonempty set, and let $G\colon X\times X\times X\to \mathbb{R}^+$ be a function satisfying:

(G1) $G(x,y,z)=0$ if $x=y=z$;

(G2) $G(x,x,y)>0$ for all x, $y\in X$ with $x\neq y$;

(G3) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$;

(G4) $G(x,y,z)=G(p\{x,y,z\})$, for each permutation of x,y,z (the symmetry);

(G5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z),\quad \forall\, x,y,z,a\in X$ (the rectangle inequality).

Then the function G is called a *generalized metric space*, or more specifically G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.3 ([1])**.** Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if for any $\epsilon>0$, there exists $k\in\mathbb{N}$ such that $G(x,x_n,x_m)<\epsilon$, for all $n,m\geq k$.

Definition 2.4 ([1])**.** Let (X,G) be a G-metric space. A sequence $(x_n)\subseteq X$ is said to be G-Cauchy if for every $\epsilon>0$, there exists $k\in N$ such that $G(x_n,x_m,x_l)<\epsilon$ for all $n,m,l\geq k$.

Definition 2.5 ([2])**.** A G-metric space (X,G) is said to be G-complete or complete G-metric space if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

In 2010, R. Saadati *et al.* [21] introduced the concept of Ω-distance and prove some fixed point results. The definition of the Ω-distance is given as follows:

Definition 2.6 ([21])**.** Let (X,G) be a G-metric space. Then a function $\Omega\colon X\times X\times X\to [0,\infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x,y,z)\leq\Omega(x,a,a)+\Omega(a,y,z),\quad \forall\, x,y,z,a\in X$,

(b) for any $x,y\in X$, the functions $\Omega(x,y,\cdot)$, $\Omega(x,\cdot,y)\colon X\to X$ are lower semi continuous,

(c) for each $\epsilon>0$, there exists $\delta>0$ such that $\Omega(x,a,a)\leq\delta$ and $\Omega(a,y,z)\leq\delta$ imply $G(x,y,z)\leq\epsilon$.

Definition 2.7 ([21])**.** Let (X,G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $M\geq 0$ such that $\Omega(x,y,z)\leq M$ for all $x,y,x\in X$.

Lemma 2.1 ([21])**.** *Let X be a metric space endowed with the metric G and Ω be an Ω-distance on X. Let (x_n), (y_n) be sequences in X, (α_n), (β_n) be sequences in $[0,\infty)$ converging to zero and let $x,y,z,a\in X$. Then we have the following:*

(1) *If $\Omega(y,x_n,x_n)\leq\alpha_n$ and $\Omega(x_n,y,z)\leq\beta_n$ for $n\in\mathbb{N}$ then $G(y,y,z)<\epsilon$ and hence $y=z$;*

(2) *If $\Omega(y_n,x_n,x_n)\leq\alpha_n$ and $\Omega(x_n,y_m,z)\leq\beta_n$ for any $m>n\in\mathbb{N}$, then $G(y_n,y_m,z)\to 0$ and hence $y_n\to z$;*

(3) *If $\Omega(x_n,x_m,x_l)\leq\alpha_n$ for any $m,n,l\in\mathbb{N}$ with $n\leq m\leq l$, then (x_n) is a G-Cauchy sequence;*

(4) *If $\Omega(x_n,a,a)\leq\alpha_n$ for any $n\in\mathbb{N}$, then (x_n) is a G-Cauchy sequence.*

3. Main Results

We start with the following result.

Theorem 3.1. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$ and $A \cap B \neq \phi$. Suppose that $f, g\colon A \cup B \to A \cup B$ are two mappings such that $f(A) \subseteq B$ and $g(B) \subseteq A$, and suppose that there exists $r \in [0, \frac{1}{2})$ such that the following conditions hold true*

$$\Omega(fx, gfx, gy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, gy, gy)\,]\ \forall x \in A \text{ and } \forall y \in B, \tag{3.1}$$

$$\Omega(gx, fgx, fy) \leq r\,[\,\Omega(x, gx, gx) + \Omega(y, fy, fy)\,]\ \forall y \in A \text{ and } \forall x \in B, \tag{3.2}$$

$$\Omega(fx, gfx, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x, y \in A, \tag{3.3}$$

and

$$\Omega(gx, fgx, gy) \leq r\,[\,\Omega(x, gx, gx) + \Omega(y, gy, gy)\,]\ \forall x, y \in B. \tag{3.4}$$

If f and g are continuous, then f and g have a unique common fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. Since $f(A) \subseteq B$, then $fx_0 = x_1 \in B$. Also, since $g(B) \subseteq A$, then $gx_1 = x_2 \in A$. Continuing this process we obtain a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$, $x_{2n} \in A$, $gx_{2n+1} = x_{2n+2}$ and $x_{2n+1} \in B$, $n \in \mathbb{N} \cup \{0\}$.

First, since X is Ω-bounded, then there exists $M \geq 0$ such that

$$\Omega(x, y, z) \leq M\ \ \forall x, y, z \in X.$$

Now, our claim is to show that $\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}\ M\ \forall n, s \in \mathbb{N}$, where $q = \dfrac{r}{1-r}$.

Let $n, s \in \mathbb{N}$. Then we have four cases:

Case 1: n is even and s is even. Therefore $n = 2t$ for some $t \in \mathbb{N}$. By (3.4), we have

$$\begin{aligned}
\Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t}, x_{2t+1}, x_{2t+s}) \\
&= \Omega(gx_{2t-1}, fgx_{2t-1}, gx_{2t+s-1}) \\
&\leq r\ [\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,].
\end{aligned} \tag{3.5}$$

Also, by (3.1), we get

$$\begin{aligned}
&\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\
&= \Omega(fx_{2t-2}, gfx_{2t-2}, gx_{2t-1}) + \Omega(fx_{2t+s-2}, gfx_{2t+s-2}, gx_{2t+s-1}) \\
&\leq r\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t-1}, x_{2t}, x_{2t})] \\
&+r\ [\Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\
&\leq \frac{r}{1-r}\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,] \\
&\leq q\ [\,\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,].
\end{aligned}$$

By applying the previous steps repeatedly we get

$$\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})].$$

Since X is Ω-bounded, then $\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq 2q^{n-1}M$. Having in mind that $r < \frac{1}{2}$, then the inequality (3.5) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M. \tag{3.6}$$

Case 2: n is odd, s is even. Therefore $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. By (3.3), we get

$$\begin{aligned} \Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t+1}, x_{2t+2}, x_{2t+s+1}) \\ &= \Omega(fx_{2t}, gfx_{2t}, fx_{2t+s}) \\ &\leq r\,[\,\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})\,]. \end{aligned} \tag{3.7}$$

By (3.2), we obtain

$$\begin{aligned} &\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \\ &= \Omega(gx_{2t-1}, fgx_{2t-1}, fx_{2t}) + \Omega(gx_{2t+s-1}, fgx_{2t+s-1}, fx_{2t+s}) \\ &\leq r\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t}, x_{2t+1}, x_{2t+1})] \\ &+ r\,[\Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})]. \end{aligned}$$

Therefore

$$\begin{aligned} &\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \\ &\leq \frac{r}{1-r}\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,] \\ &\leq q\,[\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,]. \end{aligned}$$

Hence, by applying the previous steps repeatedly we get

$$\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})].$$

Since X is Ω-bounded then $\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq 2q^{n-1}M$. But $r < \frac{1}{2}$, then the inequality (3.7) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M.$$

Case 3: n is even, and s is odd. Therefore $n = 2t$ for some $t \in \mathbb{N}$. By (3.2), we have

$$\begin{aligned} \Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t}, x_{2t+1}, x_{2t+s}) \\ &= \Omega(gx_{2t-1}, fgx_{2t-1}, fx_{2t+s-1}) \\ &\leq r\,[\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,]. \end{aligned} \tag{3.8}$$

By (3.1) and (3.2), we obtain

$$\begin{aligned} &\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\ &= \Omega(fx_{2t-2}, gfx_{2t-2}, gx_{2t-1}) + \Omega(gx_{2t+s-2}, fgx_{2t+s-2}, fx_{2t+s-1}) \\ &\leq r\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t-1}, x_{2t}, x_{2t})] \\ &+ r\,[\Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})]. \end{aligned}$$

Therefore

$$\begin{aligned} &\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\ &\leq \frac{r}{1-r}\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,] \\ &\leq q\,[\,\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,]. \end{aligned}$$

By applying the previous steps repeatedly we get

$$\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})].$$

Since X is Ω-bounded, then $\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq 2q^{n-1}M$. But $r < \frac{1}{2}$, so the inequality (3.8) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M.$$

Case 4: n is odd, s is odd. Therefore $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. By (3.1), we get

$$\begin{aligned} \Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t+1}, x_{2t+2}, x_{2t+s+1}) \\ &= \Omega(fx_{2t}, gfx_{2t}, gx_{2t+s}) \\ &\leq r\,[\,\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})\,]. \end{aligned} \tag{3.9}$$

By (3.1) and (3.2), we have

$$\begin{aligned}
&\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \\
&= \Omega(gx_{2t-1}, fgx_{2t-1}, fx_{2t}) + \Omega(fx_{2t+s-1}, gfx_{2t+s-1}, gx_{2t+s}) \\
&\leq r\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t}, x_{2t+1}, x_{2t+1})] \\
&+r\,[\Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})].
\end{aligned}$$

So

$$\begin{aligned}
&\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \\
&\leq \frac{r}{1-r}\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,] \\
&\leq q\,[\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,].
\end{aligned}$$

By applying the previous steps repeatedly we get

$$\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})].$$

But X is Ω-bounded. Then $\Omega(x_{2t}, x_{2t+1}, x_{2t+1})+\Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq 2q^{n-1}M$. Since $r < \frac{1}{2}$, then inequality (3.9) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M.$$

Thus in all cases we have

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M, \quad \forall n, s\ \in \mathbb{N}. \tag{3.10}$$

Now, for all $l \geq m \geq n$, we have

$$\begin{aligned}
\Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l) \\
&\leq q^{n-1}M + q^nM + \cdots + q^{m-2}M \\
&\leq \frac{q^{n-1}}{1-q}M.
\end{aligned}$$

Thus by Lemma 2.1 (x_n) is a G-Cauchy sequence. Therefore, there exists $u \in X$ such that (x_n) is G-convergent to u. Since (x_n) G-converges to u, then each subsequence of (x_n) also G-converges to u. So the subsequences $(x_{2n+1}) = (fx_{2n})$ and $(x_{2n+2}) = (gx_{2n+1})$ are G-convergent to u.

First, suppose that f is continuous. Then $\lim_{n\to\infty} fx_{2n} = fu$ and $\lim_{n\to\infty} x_{2n+1} = u$, by uniqueness of the limit we have $fu = u$.

Second, suppose that g is continuous. Then $\lim_{n\to\infty} gx_{2n+1} = gu$ and $\lim_{n\to\infty} x_{2n+2} = u$, by uniqueness of the limit we have $gu = u$.

Since $(x_{2n}) \subseteq A$ and A is closed, then $u \in A$. Also, since $(x_{2n+1}) \subseteq B$ and B is closed, then $u \in B$. Hence u is a common fixed point for f and g in $A \cap B$.

Now, we prove the uniqueness.

First, we show that if $w = fw = gw$, then $\Omega(w, w, w) = 0$. By (3.1), we have

$$\begin{aligned}
\Omega(w, w, w) = \Omega(fw, gfw, gw) &\leq r\,[\Omega(w, w, w) + \Omega(w, w, w)] \\
&\leq 2r\Omega(w, w, w).
\end{aligned}$$

Since $r < \frac{1}{2}$, then $\Omega(w, w, w) = 0$.

Now, let $v \in X$ be another common fixed point for f and g. Then by (3.1), we get

$$\Omega(v, v, u) = \Omega(fv, gfv, gu) \leq r\,[\,\Omega(v, v, v) + \Omega(u, u, u)\,].$$

Since $v = fv = gv$ and $u = fu = gu$, then $\Omega(v, v, v) = \Omega(u, u, u) = 0$. Therefore $\Omega(v, v, u) = 0$. Thus by the definition of Ω-distance we have $G(v, v, u) = 0$. Hence $u = v$. □

If we choose $X = A = B$ in Theorem 3.1, then we have the following result

Corollary 3.1. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Suppose that $f,g\colon X\to X$ be two mappings. Suppose that there exists $r\in[0,\frac{1}{2})$ such that the following conditions hold true*

$$\Omega(fx,gfx,gy)\le r\,[\,\Omega(x,fx,fx)+\Omega(y,gy,gy)\,]\ \forall x,y\in X,$$
$$\Omega(gx,fgx,fy)\le r\,[\,\Omega(x,gx,gx)+\Omega(y,fy,fy)\,]\ \forall x,y\in X,$$
$$\Omega(fx,gfx,fy)\le r\,[\,\Omega(x,fx,fx)+\Omega(y,fy,fy)\,]\ \forall x,y\in X,$$

and

$$\Omega(gx,fgx,gy)\le r\,[\,\Omega(x,gx,gx)+\Omega(y,gy,gy)\,]\ \forall x,y\in X.$$

If f or g is continuous, then f and g have a unique common fixed point in X.

If we replace g by f in Theorem 3.1, we get the following result.

Corollary 3.2. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X=A\cup B$ and $A\cap B\neq\phi$. Suppose that $f:A\cup B\to A\cup B$ is a cyclic mapping. Also, assume that there exists $r\in[0,\frac{1}{2})$ such that the following condition hold true*

$$\Omega(fx,f^2x,fy)\le r\,[\,\Omega(x,fx,fx)+\Omega(y,fy,fy)\,]\ \forall x,y\in A\cup B.$$

If f is continuous, then f has a unique fixed point in $A\cap B$.

By modifying the contractive condition in Theorem 3.1, we get the following result

Theorem 3.2. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with $X=A\cup B$. Suppose that $f,g\colon A\cup B\to A\cup B$ be two mappings such that $f(A)\subseteq B$ and $g(B)\subseteq A$, and suppose that the following conditions hold true*

$$\Omega(fx,gfx,gy)\le r\,[\,\Omega(x,fx,y)+\Omega(y,gy,x)\,],\ \forall x\in A,\ and\,\forall y\in B,\tag{3.11}$$
$$\Omega(fx,gfx,fy)\le r\,[\,\Omega(x,fx,y)+\Omega(y,fy,x)\,],\ \forall x,y\in A,\tag{3.12}$$
$$\Omega(gx,fgx,fy)\le r\,[\,\Omega(x,gx,y)+\Omega(y,fy,x)\,],\ \forall y\in A,\ and\,\forall x\in B,\tag{3.13}$$

and

$$\Omega(gx,fgx,gy)\le r\,[\,\Omega(x,gx,y)+\Omega(y,gy,x)\,],\ \forall x,y\in B.\tag{3.14}$$

If f and g are continuous, then f and g have a unique common fixed point in $A\cap B$.

Proof. Let $x_0\in A$. Since $f(A)\subseteq B$, then $fx_0=x_1\in B$. Also, since $g(B)\subseteq A$, then $gx_1=x_2\in A$. Continuing this way we obtain a sequence (x_n) in X such that $fx_{2n}=x_{2n+1}$, $x_{2n}\in A$, $gx_{2n+1}=x_{2n+2}$ and $x_{2n+1}\in B$, $n\in\mathbb{N}\cup\{0\}$.

Since X is Ω-bounded, then there exists $M\ge 0$ such that $\Omega(x,y,z)\le M$, for all $x,y,z\in X$.

Now, our aim is to show that $\Omega(x_n,x_{n+1},x_{n+s})\le(2r)^nM$.

Let $n,s\in\mathbb{N}$. Then we have four cases:

Case 1: n is even and s is even, therefore $n=2t$ for some $t\in\mathbb{N}$. By (3.14), we have

$$\begin{aligned}\Omega(x_n,x_{n+1},x_{n+s}) &= \Omega(x_{2t},x_{2t+1},x_{2t+s})\\ &= \Omega(gx_{2t-1},fgx_{2t-1},gx_{2t+s-1})\\ &\le r\,[\,\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\,].\end{aligned}\tag{3.15}$$

Also, by (3.12), we get

$$\begin{aligned}&\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\\ &=\Omega(fx_{2t-2},gfx_{2t-2},fx_{2t+s-2})+\Omega(fx_{2t+s-2},gfx_{2t+s-2},fx_{2t-2})\\ &\le r\,[\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})]\\ &+r\,[\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})+\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})]\\ &\le 2r\,[\,\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})\,].\end{aligned}$$

Hence by applying the previous steps repeatedly, we get

$$\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\leq(2r)^{n-1}\,[\,\Omega(x_0,x_1,x_s)+\Omega(x_s,x_{s+1},x_0)\,].$$

Since X is Ω-bounded, then $\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\leq 2M(2r)^{n-1}$. Thus inequality (3.15) becomes $\Omega(x_n,x_{n+1},x_{n+s})\leq(2r)^nM$.

Case 2: n is odd, s is even, therefore $n=2t+1$ for some $t\in\mathbb{N}\cup\{0\}$. From (3.12), we get

$$\begin{aligned}\Omega(x_n,x_{n+1},x_{n+s}) &= \Omega(x_{2t+1},x_{2t+2},x_{2t+s+1})\\ &= \Omega(fx_{2t},gfx_{2t},fx_{2t+s})\\ &\leq r\,[\,\Omega(x_{2t},x_{2t+1},x_{2t+s})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t})\,].\end{aligned}\qquad(3.16)$$

Also, by (3.14), we get

$$\begin{aligned}&\Omega(x_{2t},x_{2t+1},x_{2t+s})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t})\\ &=\Omega(gx_{2t-1},fgx_{2t-1},gx_{2t+s-1})+\Omega(gx_{2t+s-1},fgx_{2t+s-1},gx_{2t-1})\\ &\leq r\,[\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})]\\ &+r\,[\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})+\Omega(x_{2t-1},x_{2t},x_{2t+s-1})]\\ &\leq 2r[\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\,].\end{aligned}$$

Hence by applying the previous steps repeatedly, we get

$\Omega(x_{2t},x_{2t+1},x_{2t+s})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t})\leq(2r)^{n-1}\,[\Omega(x_0,x_1,x_s)+\Omega(x_s,x_{s+1},x_0)]$. Since X in Ω-bounded, then $\Omega(x_{2t},x_{2t+1},x_{2t+1})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})\leq 2M(2r)^{n-1}$. Thus inequality (3.16) becomes

$$\Omega(x_n,x_{n+1},x_{n+s})\leq(2r)^nM.$$

Case 3: n is even, s is odd, therefore $n=2t$ for some $t\in\mathbb{N}$. By (3.13), we get

$$\begin{aligned}\Omega(x_n,x_{n+1},x_{n+s}) &= \Omega(x_{2t},x_{2t+1},x_{2t+s})\\ &= \Omega(gx_{2t-1},fgx_{2t-1},fx_{2t+s-1})\\ &\leq r\,[\,\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\,].\end{aligned}\qquad(3.17)$$

Also, by (3.11) and (3.13), we get

$$\begin{aligned}&\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\\ &=\Omega(fx_{2t-2},gfx_{2t-2},gx_{2t+s-2})+\Omega(gx_{2t+s-2},fgx_{2t+s-2},fx_{2t-2})\\ &\leq r\,[\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})]\\ &+r\,[\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})+\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})]\\ &\leq 2r\,[\Omega(x_{2t-2},x_{2t-1},x_{2t+s-2})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t-2})\,].\end{aligned}$$

Hence by applying the previous steps repeatedly, we obtain

$$\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\leq(2r)^{n-1}\,[\Omega(x_0,x_1,x_s)+\Omega(x_s,x_{s+1},x_0)].$$

Since X is Ω-bounded, then $\Omega(x_{2t-1},x_{2t},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t-1})\leq 2M(2r)^{n-1}$. Thus inequality (3.17) becomes $\Omega(x_n,x_{n+1},x_{n+s})\leq(2r)^nM$.

Case 4: n is odd, s is odd, therefore $n=2t+1$ for some $t\in\mathbb{N}\cup\{0\}$. From (3.11), we get

$$\begin{aligned}\Omega(x_n,x_{n+1},x_{n+s}) &= \Omega(x_{2t+1},x_{2t+2},x_{2t+s+1})\\ &= \Omega(fx_{2t},gfx_{2t},gx_{2t+s})\\ &\leq r\,[\,\Omega(x_{2t},x_{2t+1},x_{2t+s})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t})\,].\end{aligned}\qquad(3.18)$$

Also, by (3.11) and (3.13), we have

$$\begin{aligned}
&\Omega(x_{2t}, x_{2t+1}, x_{2t+s}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t}) \\
&= \Omega(gx_{2t-1}, fgx_{2t-1}, fx_{2t+s-1}) + \Omega(fx_{2t+s-1}, gfx_{2t+s-1}, gx_{2t-1}) \\
&\leq r\,[\Omega(x_{2t-1}, x_{2t}, x_{2t+s-1}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t-1})] \\
&+r\,[\Omega(x_{2t+s-1}, x_{2t+s}, x_{2t-1}) + \Omega(x_{2t-1}, x_{2t}, x_{2t+s-1})] \\
&\leq 2r\,[\Omega(x_{2t-1}, x_{2t}, x_{2t+s-1}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t-1})\,].
\end{aligned}$$

Hence by applying the previous steps repeatedly, we get
$\Omega(x_{2t}, x_{2t+1}, x_{2t+s})+\Omega(x_{2t+s}, x_{2t+s+1}, x_{2t}) \leq (2r)^{n-1}\,[\Omega(x_0, x_1, x_s)+\Omega(x_s, x_{s+1}, x_0)]$.

Since X is Ω-bounded, then $\Omega(x_{2t}, x_{2t+1}, x_{2t+1})+\Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq 2M(2r)^{n-1}$. Thus inequality (3.18) becomes $\Omega(x_n, x_{n+1}, x_{n+s}) \leq (2r)^n M$.

Hence in all cases we have

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq (2r)^n M,\ \forall\ n, s\ \in \mathbb{N}.$$

Now $\forall\ l \geq m \geq n$, we get

$$\begin{aligned}
\Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l) \\
&\leq (2r)^n M + (2r)^{n+1} M + \cdots + (2r)^{m-1} M \\
&\leq \frac{(2r)^n}{1-(2r)} M.
\end{aligned}$$

Hence by Lemma 2.1, (x_n) is a G-Cauchy sequence. Therefore there exists $u \in X$ such that (x_n) is G-convergent to u. Since (x_n) G-converges to u, then each subsequence of (x_n) also G-converges to u. Therefore the subsequences $(x_{2n+1}) = (fx_{2n})$ and $(x_{2n+2}) = (gx_{2n+1})$ are G-converge to u.

First, suppose that f is continuous. Then $\lim\limits_{n\to\infty} x_{2n+1} = u$ and $\lim\limits_{n\to\infty} fx_{2n} = fu$, by uniqueness of the limit we have $fu = u$.

Second, suppose that g is continuous. Then $\lim\limits_{n\to\infty} x_{2n+2} = u$ and $\lim\limits_{n\to\infty} gx_{2n+1} = fu$, by uniqueness of the limit we have $gu = u$.

Since $(x_{2n}) \subseteq A$ and $(x_{2n+1}) \subseteq B$, and both A and B closed, then $u \in A \cap B$. Hence u is a common fixed point for f and g in $A \cap B$.

To prove the uniqueness, let $v \in X$ be an other common fixed point of f and g; that is $v = fv = gv$. Then by (3.11), we get

$$\begin{aligned}
\Omega(v, v, v) = \Omega(fv, gfv, gv) &\leq r\,[\,\Omega(v, v, v) + \Omega(v, v, v)\,] \\
&\leq \frac{r}{1-r}\Omega(v, v, v).
\end{aligned}$$

Since $r < \frac{1}{2}$, then $\Omega(v, v, v) = 0$.

Again, by (3.11), we get

$$\begin{aligned}
\Omega(v, v, u) = \Omega(fv, gfv, gu) &\leq r\,[\,\Omega(v, v, u) + \Omega(u, u, v)\,] \\
&\leq \frac{r}{1-r}\Omega(u, u, v).
\end{aligned}$$

Also, by (3.11) we get

$$\begin{aligned}
\Omega(u, u, v) &\leq r\,[\,\Omega(u, u, v) + \Omega(v, v, u)\,] \\
&\leq \frac{r}{1-r}\Omega(v, v, u).
\end{aligned}$$

Since $r < \frac{1}{2}$, then $\Omega(u, u, v) = \Omega(v, v, u) = 0$. Hence by the definition of Ω-distance we have $G(v, v, u) = 0$. Thus $u = v$. □

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