ON UNCERTAINTY PRINCIPLE OF ORTHONORMAL SEQUENCES FOR THE $q$-DUNKL TRANSFORM

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In this work, using some elements of the $q$-harmonic analysis and the $q$-Dunkl transform, for fixed $q \in [0,1]$, we establish a $q$-analogue of uncertainty inequalities for orthonormal sequences and prove a quantitative version of Shapiro’s uncertainty principle for the $q$-Dunkl transform. As a side result, we prove a variation of Donoho-Stark’s uncertainty inequality, in particular, if $f$ is $\varepsilon_S$-concentrated on $S$ and $F^q_D (f)$ is $\varepsilon_\Sigma$-concentrated on $\Sigma$ with $\|f\|_{L^2_{q}} = 1$ and $\varepsilon_S + \varepsilon_\Sigma < 1$, then $|S| |\Sigma| \geq (1 - (\varepsilon_S + \varepsilon_\Sigma))^2$.

1. Notations and preliminaries

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function $f$ and its Fourier transform $\mathcal{F}(f)$. Every discussion of the uncertainty principle must necessarily begin with the classical uncertainty principle, called the Heisenberg-Pauli-Weil inequality in which concentration is measured in terms of dispersions. It states that for $f \in L^2(\mathbb{R}^d)$

$$\|x|f\|_{L^2(\mathbb{R}^d)}^2 \|\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{d}{2} \|f\|_{L^2(\mathbb{R}^d)}^2,$$  \hspace{1cm} (1.1)

with equality if and only if $f$ is a multiple of a suitable Gaussian function, where $\mathcal{F}$ is the classical Fourier transform defined by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i \langle x, \xi \rangle} f(x) \, dx,$$

and $\langle \cdot , \cdot \rangle$, $|\cdot |$ are the usual inner product and norm on $\mathbb{R}^d$.

Generalizations of this result in both classical and quantum analysis have been treated and many versions of Heisenberg-Pauli-Weil type uncertainty inequalities have been obtained for several generalized Fourier transforms.

Recently, considerable attention has been devoted to proving new mathematical formulations and new contexts for the uncertainty principle. In [18], H. Shapiro proved a number of uncertainty inequalities for orthonormal sequences that are stronger than corresponding inequalities for a single function. More precisely, if $(\varphi_n)_{n=1}^\infty$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sup_n \left( \|x|\varphi_n\|_{L^2(\mathbb{R}^d)}^2 + \|\xi|\mathcal{F}(\varphi_n)\|_{L^2(\mathbb{R}^d)}^2 \right) = \infty.$$

A quantitative version of Shapiro’s result (1.2) has been proved by Jaming and Powell [11] and then by Malinnikova [14],

$$\forall s > 0, \forall N \geq 1, \sum_{n=1}^{N} \left( \|x|^{s} \varphi_n\|_{L^2(\mathbb{R}^d)}^2 + \|\xi|^{s} \mathcal{F}(\varphi_n)\|_{L^2(\mathbb{R}^d)}^2 \right) \geq C N^{1+s/d}.$$  \hspace{1cm} (1.3)
In [5, 8], the authors established a Heisenberg uncertainty inequality for the $q$-Dunkl transform. In particular we have
\[ \|x|^a f\|_{L_q^2} + \|\alpha^p F_{D_q} f\|_{L_q^2} \geq c_1(s, q)\|f\|_{L_q^2}, \]  
(1.4)
or equivalently
\[ \|x|^a f\|_{L_q^2} + \|\alpha^p F_{D_q} f\|_{L_q^2} \geq c_2(s, q)\|f\|_{L_q^2}. \]  
(1.5)

Motivated by these results, our purpose in this paper is to prove a quantitative version of Shapiro’s mean dispersion theorem for the $q$-Dunkl Rubin transform. Also, the Donoho-Stark’s uncertainty principle for the $q$-Dunkl transform is proved.

This paper is organized as follows: in this section, we present some notations and results used in $q$-theory and useful in the sequel. Also, we recall some basic properties of the $q$-Dunkl operator and the $q$-Dunkl transform introduced in [3]. In Section 2, we prove some Donoho-Stark’s inequalities and finally in Section 3, we prove a $q$-analogue of uncertainty inequalities for orthonormal sequences and show a quantitative version of Shapiro’s uncertainty principle for the $q$-Dunkl transform.

For the convenience of the reader, we collect here some usual notions and notations used in the $q$-theory and useful in the sequel. For more information on the $q$-theory we refer the reader to [10, 12, 17]. Throughout this paper, we will fix $q \in [0, 1]$ such that $\frac{L_q(1-q)}{L_q(n)} \in 2\mathbb{Z}$.

We note
\[ \mathbb{R}_q = \{ \pm q^n; n \in \mathbb{Z} \}; \mathbb{R}_{q,+} = \{ q^n; n \in \mathbb{Z} \} \) and $\mathbb{R}_q = \mathbb{R}_q \cup \{0\}$. 

For $a \in \mathbb{C}$; the $q$-shifted factorials are defined by
\[ (a, q)_0 = 1; \quad (a, q)_n = \prod_{k=0}^{n-1} (1 - aq^n); \quad (a, q) = \prod_{n=0}^{\infty} (1 - aq^n). \]

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$
\[ [x]_q \frac{1 - q^n}{1 - q}; \quad [n]_q! = \frac{(q, q)_n}{(1 - q)^n}. \]

In [16], Rubin defined a $q$-analogue differential operator by
\[ \partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz)}{2(1 - q)z}. \]  
(1.6)

The $q$-Jackson integrals are defined by [10]
\[ \int_a^b f(x) d_q x = (1 - q)a \sum_{k \in \mathbb{Z}} f(aq^n) q^n, \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \]
and
\[ \int_{-\infty}^{\infty} f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} q^n (f(q^n) + f(-q^n)) \]
provided the sums converge absolutely.

Using the $q$-Jackson integrals, we denote by:

- $L_{p,q}^\infty(\mathbb{R}_q) = \{ f : \|f\|_{L_{p,q}^\infty} = \sup |f(x)| : x \in \mathbb{R}_q < \infty \}$,
- $L_{\alpha,q}^\infty(\mathbb{R}_q) = \{ f : \|f\|_{L_{\alpha,q}^\infty} = \sup \|f(x)\| : x \in \mathbb{R}_q < \infty \}$.

For the particular case $p = 2$, we denote by $\langle f, g \rangle$ the inner product of the Hilbert space $L_{\alpha,q}^2(\mathbb{R}_q)$ as
\[ \langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) g(x) d_q x. \]
In the following, we recall some basic properties of the $q$-Dunkl operator and the $q$-Dunkl transform introduced in [1].

For $\alpha \geq -\frac{1}{2}$, the $q$-Dunkl operator is defined by

$$
\Lambda_{\alpha,q}(f)(x) = \partial_q[H_{\alpha,q}(f)][x] + [2\alpha + 1]_q f(x) - f(-x),
$$

where

$$
H_{\alpha,q} : f = f_e + f_o + q^{2\alpha+1} f_o.
$$
f_e and $f_o$ are the even and odd parts respectively of the function $f$.

It was shown in [1] that for each $\lambda \in \mathbb{C}$, the function

$$
\psi_{\lambda}^{\alpha,q} : x \rightarrow j_\alpha(\lambda x, q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha + 1}(\lambda x, q^2)
$$
is the unique solution of the $q$-differential-difference equation

$$
\Lambda_{\alpha,q}(f) = i\lambda f, \quad f(0) = 1
$$
where $j_\alpha(x, q^2)$ is the normalized third Jackson’s $q$-Bessel function given by

$$
j_\alpha(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2, q^2)_n(q^{2\alpha+1}, q^2)_n} (1 - q)x^{2n}.
$$

The $q$-Dunkl transform $F^{\alpha,q}_D$ is defined on $L^1_{\alpha,q}(\mathbb{R}_q)$ by (see [1])

$$
F^{\alpha,q}_D(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}d_qx
$$
for all $\lambda \in \mathbb{R}_q$ where $c_{\alpha,q} = \frac{(1+q)^{-\alpha}}{\Gamma_q(\alpha+1)}$.

This transform satisfies the following properties(see [17]):

- For $f \in L^1_{\alpha,q}(\mathbb{R}_q)$:

$$
\|F^{\alpha,q}_D(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{\langle q, q\rangle_{\infty}} \|f\|_{1,q}. \tag{1.9}
$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$ such that $xf \in L^1_{\alpha,q}(\mathbb{R}_q)$

$$
F^{\alpha,q}_D(\Lambda_{\alpha,q}(f)) = i\lambda F^{\alpha,q}_D(f)(\lambda), \quad \Lambda_{\alpha,q}(F^{\alpha,q}_D(f)) = -iF^{\alpha,q}_D(f(x)). \tag{1.10}
$$

- The $q$-Dunkl transform $F^{\alpha,q}_D$ is an isomorphism from $L^2_{\alpha,q}(\mathbb{R}_q)$ onto itself and satisfies

the following Plancherel formula:

$$
\|F^{\alpha,q}_D(f)\|_{2,q} = \|f\|_{2,q}. \tag{1.11}
$$

- $\forall x \in \mathbb{R}_q$, $\quad (F^{\alpha,q}_D)^{-1}(F^{\alpha,q}_D(f))(x) = f(x) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{\infty} F^{\alpha,q}_D(f)(\lambda)(\psi_{\lambda}^{\alpha,q}(\lambda)|x|^{2\alpha+1}d_q\lambda.$

The $q$-analogue of the Heisenberg uncertainly principle is given by (see [5])

**Theorem 1.1.** Let $s > 0, \quad \alpha \geq -\frac{1}{2}$ and $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ then

$$
\|x^s f\|_{2,q}\|\lambda^s F^{\alpha,q}_D(f)\|_{2,q} \geq c(s,q)\|f\|_{2,q} \tag{1.12}
$$
where $c(s,q)$ is a constant which depends on $s$ and $q$.  


2. An \( L_{o,q}^2(\mathbb{R}_q) \) Donoho-Stark’s uncertainty principle

In this section we will prove a \( q \)-analogue of the variation on Donoho-Stark’s uncertainty principle for \( L_{o,q}^2(\mathbb{R}_q) \).

A subset \( T \subset \mathbb{R}_q \) is said to be measurable subset of \( \mathbb{R}_q \)
\[
|T| = \int_{-\infty}^{\infty} \chi_T(x)|x|^{2\alpha+1}d_qx < \infty,
\]
where \( \chi_T \) is the characteristic function of \( T \).

The time-limiting and the frequency-limiting operators on \( L_{o,q}^2(\mathbb{R}_q) \) are defined by
\[
E_S f = \chi_S f; \quad F_S f = (F_D^\alpha)^{-1}[\chi_S F_D^\alpha(f)]
\]
where \( S \) and \( \Sigma \) are measurable subsets of \( \mathbb{R}_q \).

**Definition 2.1.** Let \( 0 < \epsilon < 1 \) and \( f \in L_{o,q}^2(\mathbb{R}_q) \) then

1. A function \( f \) is said to be \( \epsilon \)-concentrated on \( S \) if: \( \|E_S f\|_{2,q} \leq \epsilon \|f\|_{2,q} \).
2. \( F_D^\alpha(f) \) is said to be \( \epsilon \)-concentrated on \( \Sigma \) if: \( \|F_D^\alpha f\|_{2,q} \leq \epsilon \|f\|_{2,q} \).

**Lemma 2.1.** \( F_S E_S \) is a Hilbert-Schmidt operator with the kernel:
\[
k(t, \lambda) = \frac{c_{\alpha,q}}{2} \chi_S(t) \int_{\Sigma} \psi^\alpha_{\lambda}(x)\psi^\alpha_{-\lambda}(t)|x|^{2\alpha+1}d_qx
\]
\[
\|F_S E_S\|_{H^2}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k(t, \lambda)|^2|t|^{2\alpha+1}|\lambda|^{2\alpha+1}d_qtd_q\lambda \leq \frac{4c_{\alpha,q}^2}{(q,q)_\infty^2} |S||\Sigma|.
\]

**Proof.** We have
\[
(F_S E_S f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{\Sigma} \psi^\alpha_{\lambda}(x)(F_D^\alpha(E_S f))(x)|x|^{2\alpha+1}d_qx
\]
\[
= \frac{c_{\alpha,q}^2}{4} \int_S \psi^\alpha_{\lambda}(x)(\int_S \psi^\alpha_{-\lambda}(t)f(t)|t|^{2\alpha+1}d_qt)|x|^{2\alpha+1}d_qx.
\]
By Fubini’s theorem we obtain
\[
(F_S E_S f)(\lambda) = \frac{c_{\alpha,q}^2}{4} \int_S f(t)(\int_S \psi^\alpha_{\lambda}(x)\psi^\alpha_{-\lambda}(t)|x|^{2\alpha+1}d_qx)|t|^{2\alpha+1}d_qt
\]
so that
\[
(F_S E_S f)(\lambda) = \int_{-\infty}^{\infty} f(t)k(t, \lambda)|t|^{2\alpha+1}d_qt
\]
where
\[
k(t, \lambda) = \frac{c_{\alpha,q}^2}{4} \chi_S(t) \int_{\Sigma} \psi^\alpha_{\lambda}(x)\psi^\alpha_{-\lambda}(t)|x|^{2\alpha+1}d_qx.
\]
Let \( g_t(\lambda) = k(t, \lambda) \) then the inversion formula shows that
\[
F_D^\alpha(g_t)(\lambda) = \frac{c_{\alpha,q}}{2} \chi_S(t)\chi_{\Sigma}(\lambda)\psi^\alpha_{-\lambda}(t).
\]
By the Plancherel formula it follows
\[
\int_{-\infty}^{\infty} |g_t(s)|^2s|^{2\alpha+1}d_qs = \int_{-\infty}^{\infty} |F_D^\alpha(g_t)(\lambda)|^2|\lambda|^{2\alpha+1}d_q\lambda \leq \frac{4c_{\alpha,q}^2}{(q,q)_\infty^2} |S||\Sigma|.
\]
Hence, integrating over \( t \in S \), we obtain the desired result.

**Theorem 2.1.** Let \( S \) and \( \Sigma \) be a \( q \)-mesurable sets such that \( f \) is \( \epsilon_S \)-concentrated on \( S \) and \( F_D^\alpha(f) \) is \( \epsilon_\Sigma \)-concentrated on \( \Sigma \) and suppose that \( \|f\|_{2,q} = 1 \) and \( \epsilon_S + \epsilon_\Sigma < 1 \). Then
\[
|S||\Sigma| \geq (1 - (\epsilon_S + \epsilon_\Sigma))^2.
\]
Proof. The norm of an operator $Q$ is defined by:

$$
\|\|Q\|\| = \sup_{g \in L^2_{\alpha,q}(\mathbb{R}_q)} \frac{\|Qg\|_{2,q}}{\|g\|_{2,q}}.
$$

By the triangle inequality we have

$$
\|f - F_{\Sigma}Esf\|_{2,q} \leq \|f - F_{\Sigma}f\|_{2,q} + \|F_{\Sigma}f - F_{\Sigma}Esf\|_{2,q}
$$

then

$$
\|F_{\Sigma}Esf\|_{2,q} \geq 1 - (\epsilon\Sigma + \epsilon_S).
$$

Using Holder’s inequality, we obtain

$$
\|F_{\Sigma}Esf\|_{2,q} \geq \|f\|_{2,q} \|Esf\|_{2,q} \leq \|f\|_{2,q} \|f\|_{2,q} \|Esf\|_{2,q}
$$

By Fubini-Tonnelli’s theorem and Lemma 2.1

$$
\|F_{\Sigma}Esf\|_{2,q} \leq \|f\|_{2,q} \|F_{\Sigma}Esf\|_{H} \leq \frac{2\epsilon\Sigma}{(q,q)_{\infty}} \sqrt{|\Sigma|}.
$$

Thus, the proof is complete. \qed

Definition 2.2. For $f \in L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)$. We say that

1. $f$ is $\epsilon$-timelimited on $S$ if

$$
\|f\chi_S\|_{1,q} \leq \epsilon\|f\|_{1,q}.
$$

2. $f$ is $\epsilon$-bandlimited on $\Sigma$ if

$$
\|F^\alpha(f)\chi_{\Sigma}\|_{2,q} \leq \epsilon\|f\|_{2,q}.
$$

(2.20)

Theorem 2.2. Let $0 < \epsilon_1, \epsilon_2 < 1$, $S, \Sigma$ be a pair of $q$-measurable subsets of $\mathbb{R}_q$ and $f \in L^1_{\alpha,q}(\mathbb{R}_q) \cap L^2_{\alpha,q}(\mathbb{R}_q)$. If $f$ is $\epsilon_1$-time limited on $S$ and $\epsilon_2$ -bandlimited on $\Sigma$ then:

$$
|\Sigma| = (1 - \epsilon_1)^2(1 - \epsilon_2^2)(q,q)_{\infty}^2.
$$

Proof. We have

$$
\|f\chi_S\|_{1,q} = \|f\|_{1,q} - \|f\chi_S^c\|_{1,q}
$$

hence

$$
\|f\chi_S\|_{1,q} \geq (1 - \epsilon_1)\|f\|_{1,q}.
$$

Using the Cauchy-Schwarz inequality, we obtain

$$
\|f\chi_S\|_{1,q} \leq \|f\|_{2,q} \|\chi_S\|_{2,q}.
$$

So

$$
|\Sigma| \|f\|_{2,q} \geq (1 - \epsilon_1)^2\|f\|_{1,q}^2.
$$

(2.21)

On the other hand, we have

$$
\|F^\alpha(f)\chi_{\Sigma}\|_{2,q}^2 = \|F^\alpha(f)\|_{2,q}^2 - \|F^\alpha(f)\chi_S^c\|_{2,q}^2
$$

From (1.11) and (2.20) we get

$$
\|F^\alpha(f)\chi_{\Sigma}\|_{2,q}^2 \geq (1 - \epsilon_2^2)\|f\|_{2,q}^2.
$$
Then
\[ \|F_D^{\alpha,q}(f)\chi_{\Sigma}\|_{2,q}^2 \leq |\Sigma|\|F_D^{\alpha,q}(f)\|_{\infty,q}^2 \leq \frac{4c_{\alpha,q}^2|\Sigma|}{(q,q)_{\infty}^2}\|f\|_{1,q}^2. \]

Consequently
\[ \frac{4c_{\alpha,q}^2|\Sigma|}{(q,q)_{\infty}^2}\|f\|_{1,q}^2 \geq (1 - \epsilon_2^2)\|f\|_{2,q}^2. \]

By (2.21) we obtain:
\[ |\Sigma||S| \geq \frac{(q,q)_{\infty}^2}{4c_{\alpha,q}^2}(1 - \epsilon_1^2)(1 - \epsilon_2^2). \]

3. Uncertainty principle for orthonormal bases

In this section we will prove an uncertainty principle for orthonormal bases for \( L_{\alpha,q}^2(\mathbb{R}_q) \).

**Theorem 3.1.** Let \( (\varphi_n)_{n=1}^N \) be an orthonormal system in \( L_{\alpha,q}^2(\mathbb{R}_q) \) and let \( S \) and \( \Sigma \) be two measurable subsets of \( \mathbb{R}_q \). Assume that
\[ \|E_S^c\varphi_n\|_{2,q} \leq a_n; \quad \|F_D^{\alpha,q}\varphi_n\|_{2,q} \leq b_n \]
then
\[ \sum_{n=1}^N (1 - \frac{3}{2}a_n - \frac{3}{2}b_n) \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2}|S||\Sigma|. \]

**Proof.** We consider the corresponding self-adjoint operator
\[ Q = (F_D^cE_S)^*(F_D^cE_S) = E_SE_CE_S. \]
Since
\[ \text{tr}(Q) = \|F_D^cE_S\|_{HS}^2 \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2}|S||\Sigma| \]
we have
\[ \sum_{n=1}^N \langle Q\varphi_n, \varphi_n \rangle \leq \text{tr}(Q) = \|F_D^cE_S\|_{HS}^2 \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2}|S||\Sigma|. \]

On the other hand,
\[ \langle Q\varphi_n, \varphi_n \rangle = \langle F_D^cE_SE_S\varphi_n, E_SE_S\varphi_n \rangle = \langle \varphi_n, \varphi_n \rangle - \langle \varphi_n - E_SE_S\varphi_n, \varphi_n \rangle - \langle E_S\varphi_n, \varphi_n - F_D^cE_S\varphi_n \rangle - \langle F_D^cE_S\varphi_n, \varphi_n - E_S\varphi_n \rangle. \]

It follows that
\[ \langle Q\varphi_n, \varphi_n \rangle \geq 1 - 2a_n - b_n \]
and
\[ \sum_{n=1}^N (1 - 2a_n - b_n) \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2}|S||\Sigma|. \]  \hspace{1cm} (3.22)

Now, if we consider the operator
\[ \tilde{Q} = (E_SE_D)^*(E_SE_D) \]
we get similarly
\[ \sum_{n=1}^N (1 - a_n - 2b_n) \leq \frac{4c_{\alpha,q}^2}{(q,q)_{\infty}^2}|S||\Sigma|. \]  \hspace{1cm} (3.23)
The desired result follows by combining (3.22) and (3.23).

The following corollary is an immediate consequence of Theorem 3.1

**Corollary 3.1.** Let $r_0, r_1 > 0$ and let $0 < \epsilon_1, \epsilon_2 < 1$ such that $\epsilon_1 + \epsilon_2 < \frac{2}{3}$, let $(\varphi_n)_{n=1}^{N}$ be an orthonormal system in $L_{\alpha,q}^{2}(\mathbb{R}_q)$ such that $\varphi_n$ is $\epsilon_1$-concentrated on a set $B_{r_0} = \mathbb{R}_q \setminus [-r_0, r_0]$ and $F_{D}^{\alpha,q}(\varphi_n)$ is $\epsilon_2$-concentrated on a set $B_{r_1} = \mathbb{R}_q \setminus [-r_1, r_1]$ for each $n = 1, \ldots, N$, i.e.

$$
\int_{B_{r_0}} |\varphi_n(t)|^2 |t|^{2\alpha+1} d_qt \geq 1 - \epsilon_1^2 : \int_{B_{r_1}} |F_{D}^{\alpha,q}(\varphi_n)(w)|^2 |w|^{2\alpha+1} d_qw \geq 1 - \epsilon_2^2. \quad (3.24)
$$

Then

$$
N \leq \frac{16c_{\alpha,q}^2(r_0r_1)^{2\alpha+2}}{(1 - 3^{2\alpha+2})(q,q)_\infty^2 [2\alpha + 2]^2}. \quad (3.25)
$$

Another immediate application of the localization inequality is the $q$-analogue quantitative version of Shapiro Umbrella Theorem.

**Corollary 3.2.** Let $A, B > 0$ and $(\varphi_n)_{n=1}^{N}$ be an orthonormal system in $L_{\alpha,q}^{2}(\mathbb{R}_q)$ such that

$$
||x||^s \varphi_n ||_{2,q} \leq A^s \quad \text{and} \quad ||\lambda||^s F_{D}^{\alpha,q}(\varphi_n) ||_{2,q} \leq B^s.
$$

Then

$$
N \leq c(s,q)(AB)^{2\alpha+2},
$$

where $c(s,q) = 64 A^s B^s (q,q)_\infty 4^{2\alpha+1}.\quad (\text{Corollary 3.1})$

**Proof.** Let $r_0 = 4\frac{1}{2} A$ and $r_1 = 4\frac{1}{2} B$, we have

$$
\int_{B_{r_0}} |\varphi_n(x)|^2 |x|^{2\alpha+1} d_qx = \int_{B_{r_0}} |x|^{-2s} |x|^s |\varphi_n(x)|^2 |x|^{2\alpha+1} d_qx \leq \frac{1}{16 A^s} ||x|^s \varphi_n ||_{2,q}^2 \leq \frac{1}{16}
$$

so $\varphi_n$ is $\frac{1}{4}$-concentrated on $B_{r_0}$ and in the same way we prove that $F_{D}^{\alpha,q}$ is $\frac{1}{4}$-concentrated on $B_{r_1}$. The desired result follows from Corollary 3.1.

**Theorem 3.2.** Let $s$ be a positive real and $(\varphi_n)_{n}$ be an orthonormal sequence in $L_{\alpha,q}^{2}(\mathbb{R}_q)$. Then there exists a constant $C(\alpha, s, q)$ such that

$$
\sum_{n=1}^{N} (||x|^s \varphi_n ||_{2,q}^2 + ||\lambda|^s F_{D}^{\alpha,q}(\varphi_n) ||_{2,q}^2) \geq C(\alpha, s, q) N^{1+ \frac{s}{\alpha+q}}. \quad (3.26)
$$

This theorem implies in particular that, if the elements of an orthonormal sequence and their q-Dunkl-Fourier transforms have uniformly bounded dispersions then the sequence is finite.

**Proof.** Let $(\varphi_n)_{n}$ be an orthonormal sequence in $L_{\alpha,q}(\mathbb{R}_q)$. For each $k \in \mathbb{Z}$ we define

$$
P_k = \left\{ n : \max \left\{ ||x|^s \varphi_n ||_{2,q}, ||\lambda|^s F_{D}^{\alpha,q}(\varphi_n) ||_{2,q} \right\} \in (2^{(k-1)}, 2^{sk}) \right\}.
$$

then

$$
||x|^s \varphi_n ||_{2,q} \leq 2^{sk} \quad \text{and} \quad ||\lambda|^s F_{D}^{\alpha,q}(\varphi_n) ||_{2,q} \leq 2^{sk}
$$

whenever $n \in P_k$. From Corollary 3.2, deduce that the number of elements in $\bigcup_{j=-\infty}^{k} P_j$ is less than $c_1(\alpha, s, q) 2^{(k+1)}$ where $c_1(\alpha, s, q)$ is a constant that does not depend on $k$. This shows that when $c_1(\alpha, s, q) 2^{(k+1)} < 1$, the number of elements in $\bigcup_{j=-\infty}^{k} P_j$ is null.
Consequently, there exists $k_0$ such that $P_k$ is empty for all $k < k_0$. For given $N > 2c_1(\alpha, s, q)4^{2k(\alpha+1)}$ choose $k$ such that $2c_1(\alpha, s, q)4^{2k(\alpha+1)} \geq N > 2c_1(\alpha, s, q)4^{2(k-1)(\alpha+1)}$. Then at least half of $\{1, \ldots, N\}$ does not belong to $\bigcup_{j=1}^{k-1} P_j$ and we obtain

$$\sum_{n=0}^{N} \left( |||\varphi_n|^s \varphi_n||_{2,q}^2 + \left| |\lambda|^{2\alpha+1} F_{D}^{\alpha,q}(\varphi_n) \right|_{2,q}^2 \right) \geq \frac{N}{2} 4^{s(k-1)} \geq a(\alpha, s, q) N^{1+\frac{s}{2(\alpha+1)}}$$

For $N \leq 2c_1(\alpha, s, q)$ we have

$$\sum_{n=1}^{N} \left( |||\varphi_n|^s \varphi_n||_{2,q}^2 + \left| |\lambda|^{2\alpha+1} F_{D}^{\alpha,q}(\varphi_n) \right|_{2,q}^2 \right) \geq N 4^{s(k_0-1)} \geq \frac{4^{s(k_0-1)}}{(2c_1(\alpha, s, q)) \frac{1}{2(\alpha+1)}} N^{1+\frac{s}{2(\alpha+1)}}$$

this achieve the proof. 

We deduce the following result

**Corollary 3.3.** Let $(\varphi_n)_{n \geq 1}$ be an orthonormal sequence in $L^2_{\alpha,q}(\mathbb{R}_q)$, then

$$\sup_n \left( |||\varphi_n|^s \varphi_n||_{2,q}^2 + |||\xi|^s F_{D}^{\alpha,q}(\varphi_n) ||_{2,q}^2 \right) = \infty.$$ (3.27)

**REFERENCES**


