A NOTE ON THE RESTRICTED k-MULTIPARTITION FUNCTION

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Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a sequence of positive integers and $k \geq 2$ an integer. We study $p_{k,\mathbf{a}}(n)$, the restricted k-multipartition function associated to $\mathbf{a}$ and k. We prove new formulas for $p_{k,\mathbf{a}}(n)$, its waves $W_j(n, k, \mathbf{a})$'s and its polynomial part $P_{k,\mathbf{a}}(n)$. Also, we give a lower bound for the density of the set $\{n \geq 0 \ : \ p_{k,\mathbf{a}}(n) \not\equiv 0 (\bmod m)\}$, where $m \geq 2$ is an integer.

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1. Introduction

Let n be a positive integer. We denote $[n] = \{1, 2, \ldots, n\}$. A *partition* of n is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of positive integers such that $|\lambda| = \lambda_1 + \cdots + \lambda_m = n$. We define $p(n)$ as the number of partitions of n and for convenience, we define $p(0) = 1$. This notion has the following generalization:

Let $k \geq 2$ be an integer. A *k-component multipartition* of n is a k-tuple $\lambda = (\lambda^1, \ldots, \lambda^k)$ of partitions of n such that $|\lambda| = |\lambda^1| + \cdots + |\lambda^k| = n$; see [1]. We denote $p_k(n)$, the number or k-component multipartitions of n and $p_k(0) = 1$.

Let $\mathbf{a} := (a_1, a_2, \ldots, a_r)$ be a sequence of positive integers, $r \geq 1$. The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \ldots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \geq 0$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})}, \ |z| < 1. \tag{1.1}$$

The *restricted k-multipartion function* associated to $\mathbf{a}$ is $p_{k,\mathbf{a}}(n) : \mathbb{N} \to \mathbb{N}$, $p_{k,\mathbf{a}}(n) :=$ the number of vector solutions $(x^1, \ldots, x^k)$ of

$$\sum_{j=1}^{k} \sum_{i=1}^{r} a_i x_i^j = n, \text{ where } x^j = (x_1^j, \ldots, x_r^j) \in \mathbb{N}^r \text{ for } 1 \leq j \leq k.$$

The aim of the paper is to study the properties of the function $p_{k,\mathbf{a}}(n)$, following the methods used in our previous paper [8]. We consider the sequence

$$\mathbf{a}[k] := (a_1^{[k]}, a_2^{[k]}, \ldots, a_r^{[k]}), \tag{1.2}$$

where $\ell^{[k]}$ denotes k copies of ℓ.

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It is easy to see that $p_{k,\mathbf{a}}(n) = p_{\mathbf{a}[k]}(n)$ and therefore, from (1.1) and (1.2) we have

$$\sum_{n=0}^{\infty} p_{k,\mathbf{a}}(n) z^n = \frac{1}{(1-z^{a_1})^k \cdots (1-z^{a_r})^k},\ |z|<1. \tag{1.3}$$

In Proposition 2.**1** we show that

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) := \prod_{i=1}^{k} \zeta_{\mathbf{a}}(s,w_i) = \sum_{n=0}^{\infty} \sum_{n_1+\cdots+n_k=n} \frac{p_{k,\mathbf{a}}(n)}{(n_1+w_1)^s \cdots (n_k+w_k)^s},$$

where $\zeta_{\mathbf{a}}(s,w)$ is the Barnes zeta function (see [2]). In Proposition 2.**4** we express $\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k)$ in terms of Hurwitz zeta functions.

Let D be the least common multiple of $a_1,\ldots,a_r$. In Proposition 2.**5** we note that

$$p_{k,\mathbf{a}}(n) = d_{k,\mathbf{a},rk-1}(n) n^{rk-1} + \cdots + d_{k,\mathbf{a},1}(n) n + d_{k,\mathbf{a},0}(n),$$

is a quasi-polynomial of period D. From this result, we deduce a new expression for $\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k)$ in Corollary 2.**6**.

In Theorem 3.**1** we prove formulas for the periodic functions $d_{k,\mathbf{a},m}(n)$.

Using the fact that $p_{k,\mathbf{a}}(n) = p_{\mathbf{a}[k]}(n)$, in Theorem 3.**2** we prove a formula for $p_{k,\mathbf{a}}(n)$. In Proposition 3.**3** we show that if a certain determinant is nonzero, then $p_{k,\mathbf{a}}(n)$ can be expressed in terms of values of Bernoulli polynomials and Bernoulli-Barnes numbers. Using a result from [9], in Corollary 3.**4** we show that

$$\lim_{N\to\infty} \frac{\#\{n \leq N \ :\ p_{k,\mathbf{a}}(n) \not\equiv 0 (\mathrm{mod}\ m)\}}{N} \geq \frac{1}{k\sum_{i=1}^{r} a_i},$$

where $m > 1$ is an integer.

Similarly to $p_{\mathbf{a}}(n)$ we consider the *Sylvester decomposition* (see [12], [13] and [14]) of $p_{k,\mathbf{a}}(n)$ as a sums of "waves", i.e.

$$p_{k,\mathbf{a}}(n) = \sum_{j\geq 1} W_j(k,\mathbf{a},n),$$

where $W_j(k,\mathbf{a},n) := W_j(\mathbf{a}[k],n)$. In Theorem 4.**1** we prove a formula for $W_j(k,\mathbf{a},n)$.

The polynomial part of $p_{k,\mathbf{a}}(n)$ is $P_{k,\mathbf{a}}(n) := W_1(k,\mathbf{a},n)$. In Theorem 4.**2** and Theorem 4.**3** we prove new formulas for $P_{k,\mathbf{a}}(n)$.

2. **Preliminary results**

Let $r \geq 1$ and $k \geq 2$ be two integers. Let $\mathbf{a} = (a_1,\ldots,a_r)$ be a sequence of positive integers. Let D be the least common multiple of $a_1,\ldots,a_r$.

For $0 \leq j_1 \leq \frac{D}{a_1}-1$, $0 \leq j_2 \leq \frac{D}{a_2}-1, \ldots, 0 \leq j_r \leq \frac{D}{a_r}-1$ let, by Euclidean division, $\mathfrak{q}(j_1,\ldots,j_r)$ and $\mathfrak{r}(j_1,\ldots,j_r)$ be the unique integers such that

$$a_1 j_1 + \cdots + a_r j_r = \mathfrak{q}(j_1,\ldots,j_r) D + \mathfrak{r}(j_1,\ldots,j_r),\ \ 0 \leq \mathfrak{r}(j_1,\ldots,j_r) \leq D-1. \tag{2.1}$$

We denote the rising factorial by $x^{(r)} := (x+1)(x+2)\cdots(x+r-1)$, $x^{(0)} = 1$. It holds that

$$\binom{n+r-1}{r-1} = \frac{1}{(r-1)!} n^{(r)} = \frac{1}{(r-1)!}\left(\begin{bmatrix} r \\ r \end{bmatrix} n^{r-1} + \cdots + \begin{bmatrix} r \\ 2 \end{bmatrix} n + \begin{bmatrix} r \\ 1 \end{bmatrix} \right), \tag{2.2}$$

where $\begin{bmatrix} r \\ k \end{bmatrix}$'s are the *unsigned Stirling numbers* of the first kind.

Let $w > 0$ be a real number. The *Barnes zeta function* associated to $\mathbf{a}$ and w is

$$\zeta_{\mathbf{a}}(s,w) := \sum_{u_1,\ldots,u_r \geq 0} \frac{1}{(a_1u_1 + \cdots + a_ru_r + w)^s},\ Re(s) > r. \tag{2.3}$$

For basic properties of the Barnes zeta function see [2], [10] and [11].

Let $w_1,\ldots,w_k > 0$ be some real numbers. We consider the function

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) := \zeta_{\mathbf{a}}(s,w_1)\cdots\zeta_{\mathbf{a}}(s,w_k). \tag{2.4}$$

Proposition 2.1. *We have that*

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) = \sum_{n=0}^{\infty}\sum_{n_1+\cdots+n_k=n}\frac{p_{k,\mathbf{a}}(n)}{(n_1+w_1)^s\cdots(n_k+w_k)^s}.$$

Proof. For $1\leq j\leq k$, we have that

$$\zeta_{\mathbf{a}}(s,w_j) = \sum_{n_j=0}^{\infty}\frac{p_{\mathbf{a}}(n_j)}{(n_j+w_j)^s}.$$

So, the conclusion follows from the definitions of $p_{k,\mathbf{a}}(n)$ and $\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k)$. □

Remark 2.2. Note that, if $r=1$ and $1\leq j\leq k$ then

$$\zeta_{\mathbf{a}}(s,w_j) = \sum_{u_j=0}^{\infty}\frac{1}{(a_1u_j+w_j)^s} = \frac{1}{a_1^s}\sum_{u_j=0}^{\infty}\frac{1}{(u_j+\frac{w_j}{a_1})^s} = \frac{1}{a_1^s}\zeta\left(s,\frac{a_1}{w_j}\right),$$

where

$$\zeta(s,w) := \sum_{n=0}^{\infty}\frac{1}{(n+w)^s}, Re(s)>1$$

is the Hurwitz zeta function. It follows that

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) = \frac{1}{a_1^{ks}}\zeta\left(s,\frac{a_1}{w_1}\right)\cdots\zeta\left(s,\frac{a_k}{w_j}\right).$$

We consider the set

$$\mathbf{B} := \{(j_1,\ldots,j_r)\ :\ 1\leq j_1\leq \frac{D}{a_1}-1,\ldots,1\leq j_r\leq\frac{D}{a_r}-1\}.$$

We recall the following result from [4]. Also, we mention that the definition of Stirling numbers is slightly different there; see [5] for more details.

Lemma 2.3. *(*[4, Lemma 2.2]*) We have*

$$\zeta_{\mathbf{a}}(s,w) = \frac{1}{D^s(r-1)!}\sum_{(j_1,\ldots,j_r)\in\mathbf{B}}\sum_{k=0}^{r-1}\begin{bmatrix} r \\ k+1\end{bmatrix}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\times$$
$$\times\left(\frac{a_1j_1+\cdots+a_rj_r+w}{D}\right)^j\zeta(s-k+j,\frac{\mathfrak{r}(j_1,\ldots,j_r)+w}{D}).$$

From (2.4) and Lemma 2.**3** it follows that:

Proposition 2.4. *We have*

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) = \frac{1}{D^s(r-1)!}\sum_{m=0}^{r-1}\begin{bmatrix} r \\ m+1\end{bmatrix}\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}\times$$
$$\times\prod_{i=1}^{k}\sum_{(j_1^i,\ldots,j_r^i)\in\mathbf{B}}\left(\frac{a_1j_1^i+\cdots+a_rj_r^i+w_i}{D}\right)^\ell\zeta(s-m+\ell,\frac{\mathfrak{r}(j_1^i,\ldots,j_r^i)+w_i}{D}).$$

Proposition 2.5. *$p_{k,\mathbf{a}}(n)$ is a quasi-polynomial of degree $rk-1$, with the period D, i.e.*

$$p_{k,\mathbf{a}}(n) = d_{k,\mathbf{a},rk-1}(n)n^{rk-1}+\cdots+d_{k,\mathbf{a},1}(n)n+d_{k,\mathbf{a},0}(n),$$

where $d_{k,\mathbf{a},m}(n+D) = d_{k,\mathbf{a},m}(n)$ for $0\leq m\leq rk-1$ and $n\geq 0$, and $d_{k,\mathbf{a},rk-1}(n)$ is not identically zero.

Proof. Since $p_{k,\mathbf{a}}(n) = p_{\mathbf{a}[k]}(n)$, where $\mathbf{a}[k] = (a_1^{[k]},\ldots,a_r^{[k]})$ (see (1.2)), the conclusion follows from the classical result of Bell [3]. □

Corollary 2.6. *We have that* $\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) =$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{rk-1} d_{k,\mathbf{a},m}(n) \sum_{\substack{n_1+\cdots+n_k=n \\ \ell_1+\cdots+\ell_k=m}} \binom{m}{\ell_1,\ldots,\ell_k} \prod_{j=1}^{k} \frac{1}{(n_j+w_j)^{s-\ell_j}(1+\frac{w_j}{n_j})^{\ell_j}}.$$

Proof. From Proposition 2.**1** we have that

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) = \sum_{n=0}^{\infty}\sum_{n_1+\cdots+n_k=n} \frac{p_{k,\mathbf{a}}(n)}{(n_1+w_1)^s\cdots(n_k+w_k)^s}.$$

Therefore, from Proposition 2.**5** it follows that

$$\zeta_{k,\mathbf{a}}(s,w_1,\ldots,w_k) = \sum_{n=0}^{\infty}\sum_{m=0}^{rk-1}\sum_{n_1+\cdots+n_k=n} \frac{d_{k,\mathbf{a},m}(n)(n_1+\cdots+n_k)^m}{(n_1+w_1)^s\cdots(n_k+w_k)^s} =$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{rk-1} d_{k,\mathbf{a},m}(n) \sum_{\substack{n_1+\cdots+n_k=n \\ \ell_1+\cdots+\ell_k=m}} \binom{m}{\ell_1,\ldots,\ell_k} \frac{n_1^{\ell_1}\cdots n_k^{\ell_k}}{(n_1+w_1)^s\cdots(n_k+w_k)^s}.$$

The conclusion follows immediately. □

We fix two integers $N \geq 1$ and we consider the numbers

$$f_{N,\ell} = \#\{(i_1,\ldots,i_k) \;:\; i_1+\cdots+i_k=\ell,\ 0\leq i_t\leq N-1\} \text{ where } 0\leq \ell\leq k(N-1). \quad (2.5)$$

It is clear that $f_{N,\ell}$ is the coefficient of t^ℓ of the polynomial

$$f_N(t) = (1+t+\cdots+t^{N-1})^k. \quad (2.6)$$

Using the binomial expansion, we have

$$f_N(t) = (1-t^N)^k(1-t)^{-k} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}t^{iN}\sum_{j=0}^{\infty}\binom{j+k-1}{j}t^j. \quad (2.7)$$

Proposition 2.7. *With the above notations, we have that*

$$f_{N,\ell} = \sum_{i,j\geq 0,\ iN+j=\ell} (-1)^i\binom{k}{i}\binom{j+k-1}{j}.$$

Proof. The conclusion follows from (2.5), (2.6) and (2.7). □

3. **Main results**

We use the notations from the previous section.

Theorem 3.1. *For* $n \geq 0$ *we have that*

$$d_{k,\mathbf{a},m}(n) = \frac{1}{(rk-1)!} \sum_{\substack{(\ell_1,\ldots,\ell_r)\in\mathbf{C} \\ a_1\ell_1+\cdots+a_r\ell_r\equiv n (\bmod D)}} \prod_{s=1}^{r} \sum_{i_s,j_s\geq 0,\ i_s\frac{D}{a_s}+j_s=\ell_s} (-1)^{i_s}\binom{k}{i_s}\times$$

$$\times\binom{j_s+k-1}{j_s}\sum_{t=m}^{rk-1}\begin{bmatrix} rk \\ t+1\end{bmatrix}(-1)^{t-m}\binom{k}{m}D^{-t}(a_1\ell_1+\cdots+a_r\ell_r)^{t-m},$$

where $\mathbf{C} = \{(\ell_1,\ldots,\ell_r) \;:\; 0\leq\ell_1\leq k(\frac{D}{a_1}-1),\ldots,0\leq\ell_r\leq k(\frac{D}{a_r}-1)\}$.

Proof. We consider the set

$$\mathbf{B}[\mathbf{k}] := \{(j_1,\ldots,j_{rk}) \ : \ 0 \leq j_1 \leq \frac{D}{a_1}-1,\ldots,0 \leq j_k \leq \frac{D}{a_1}-1,\ldots$$
$$\ldots,0 \leq j_{rk-k+1} \leq \frac{D}{a_r}-1,\ldots 0 \leq j_{rk} \leq \frac{D}{a_r}-1\}.$$

According to [4, Theorem 2.8] and Proposition 2.**5** we have that

$$d_{k,\mathbf{a},m} = \frac{1}{(rk-1)!} \sum_{\substack{(j_1,\ldots,j_{rk})\in\mathbf{B}[\mathbf{k}] \\ a_1(j_1+\cdots+j_k)+\cdots+a_r(j_{rk-k+1}+\cdots+j_{rk})\equiv n (\bmod\ D)}} \sum_{t=m}^{rk-1} \begin{bmatrix} rk \\ t+1 \end{bmatrix} \times$$
$$\times(-1)^{t-m}\binom{k}{m}D^{-t}(a_1(j_1+\cdots+j_k)+\cdots+a_r(j_{rk-k+1}+\cdots+j_{rk}))^{t-m}. \qquad (3.1)$$

We let $\ell_1 := j_1+\cdots+j_k$, $\ell_2 := j_{k+1}+\cdots+j_{2k},\ldots,\ell_r = j_{rk-k+1}+\cdots+j_{rk}$. It is clear that $(j_1,\ldots,j_{rk}) \in \mathbf{B}[\mathbf{k}]$ implies $(\ell_1,\ldots,\ell_r) \in \mathbf{C}$.

Since the cardinality of the set

$$\{(j_{ks-k+1},\ldots,j_{ks}) \ : \ j_{ks-k+1}+\cdots+j_{ks} = \ell_s, \ 0 \leq j_t \leq \frac{D}{a_s} \text{ for } ks-k+1 \leq t \leq ks\}$$

is $f_{\frac{D}{a_s},\ell_s}$, the conclusion follows from (3.1) and Proposition 2.**7**. □

Theorem 3.2. *For $n \geq 0$ we have that*

$$p_{k,\mathbf{a}}(n) = \frac{1}{(rk-1)!} \sum_{\substack{(\ell_1,\ldots,\ell_r)\in\mathbf{C} \\ a_1\ell_1+\cdots+a_r\ell_r\equiv n (\bmod\ D)}} \prod_{s=1}^{r} \sum_{i_s,j_s\geq 0,\ i_s\frac{D}{a_s}+j_s=\ell_s} (-1)^{i_s}\binom{k}{i_s}\times$$
$$\times\binom{j_s+k-1}{j_s}\prod_{t=1}^{rk-1}\left(\frac{n-a_1\ell_1-\cdots-a_r\ell_r}{D}+t\right),$$

where $\mathbf{C} = \{(\ell_1,\ldots,\ell_r) \ : \ 0 \leq \ell_1 \leq k(\frac{D}{a_1}-1),\ldots,0 \leq \ell_r \leq k(\frac{D}{a_r}-1)\}$.

Proof. The proof is similar to the proof of Theorem 3.**1**, using [4, Corollary 2.10] and Proposition 2.**7**. □

The *Bernoulli polynomials* are defined by

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k.$$

For $\mathbf{a} = (a_1,\ldots,a_r)$, the Bernoulli-Barnes numbers (see [2]) are

$$B_j(\mathbf{a}) = \sum_{i_1+\cdots+i_r=j}\binom{j}{i_1,\ldots,i_r}B_{i_1}\cdots B_{i_r}a_1^{i_1}\cdots a_r^{i_r}. \qquad (3.2)$$

From (1.2) and (3.2) it follows that

$$B_j(\mathbf{a}[k]) = \sum_{i_1+\cdots+i_{rk}=j}\binom{j}{i_1,\ldots,i_{rk}}B_{i_1}\cdots B_{i_{rk}}a_1^{i_1+\cdots+i_k}\cdots a_r^{i_{rk-k+1}+\cdots+i_{rk}} =$$
$$= \sum_{\ell_1+\cdots+\ell_r=j}\binom{j}{\ell_1,\ldots,\ell_r}a_1^{\ell_1}\cdots a_r^{\ell_r}\sum_{i_1+\cdots+i_k=\ell_1,\ldots,i_{rk-k+1}+\cdots+i_{rk}=\ell_r}\binom{\ell_1}{i_1,\ldots,i_k}\times$$
$$\times\cdots\binom{\ell_r}{i_{rk-k+1},\ldots,i_{rk}}B_{i_1}\cdots B_{i_{rk}}. \qquad (3.3)$$

We consider the $rkD \times rkD$ determinant:

$$\Delta(r,k,D) := \begin{vmatrix} \frac{B_1(\frac{1}{D})}{1} & \cdots & \frac{B_1(1)}{1} & \cdots & \frac{B_{rk}(\frac{1}{D})}{rk} & \cdots & \frac{B_{rk}(1)}{rk} \\ \frac{B_2(\frac{1}{D})}{2} & \cdots & \frac{B_1(1)}{1} & \cdots & \frac{B_{rk+1}(\frac{1}{D})}{rk+1} & \cdots & \frac{B_{rk+1}(1)}{rk+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{B_{rkD}(\frac{1}{D})}{rkD} & \cdots & \frac{B_{rkD}(1)}{rkD} & \cdots & \frac{B_{rkD+rk-1}(\frac{1}{D})}{rkD+rk-1} & \cdots & \frac{B_{rkD+rk-1}(1)}{rkD+rk-1} \end{vmatrix}. \quad (3.4)$$

Proposition 3.3. *If $\Delta(r,k,D) \neq 0$ then $p_{k,\mathbf{a}}(n)$ can be expressed in terms of $B_j\left(\frac{v}{D}\right)$ where $1 \leq v \leq D$ and $1 \leq j \leq rkD + rk - 1$, and $B_j(\mathbf{a}[k])$ with $rk \leq j \leq rkD + rk - 1$.*

Proof. According to [6, (1.8)], we have that

$$\sum_{m=0}^{rk-1}\sum_{v=1}^{D} d_{k,\mathbf{a},m}(n)D^{n+m}\frac{B_{n+m+1}\left(\frac{v}{D}\right)}{n+m+1} = \frac{(-1)^{rk}n!}{(n+rk)!}B_{rk+n}(\mathbf{a}[k]) - \delta_{0n}, \quad (3.5)$$

where $\delta_{0n} = \begin{cases} 1, & n=0 \\ 0, & n>0 \end{cases}$.

Taking $n = 0, 1, \ldots, rkD - 1$ in (3.5) and seing $d_{k,\mathbf{a},m}(n)$'s as unknowns, we obtain a linear system of type $rkD \times rkD$, whose determinant is $\Delta(r,k,D)$. Therefore, if $\Delta(r,k,D) \neq 0$, then $d_{k,\mathbf{a},m}(n)$'s are the solutions of the above system. Since, by Proposition 2.**5**, we have

$$p_{k,\mathbf{a}}(n) = d_{k,\mathbf{a},rk-1}(n)n^{rk-1} + \cdots + d_{k,\mathbf{a},1}(n)n + d_{k,\mathbf{a},0}(n),$$

we get the required result. □

We end this section with the following nice corollary of a result from [9].

Corollary 3.4. *If $m > 1$ is a positive integer, then*

$$\lim_{N\to\infty}\frac{\#\{n \leq N \;:\; p_{k,\mathbf{a}}(n) \not\equiv 0 (\text{mod } m)\}}{N} \geq \frac{1}{k\sum_{i=1}^{r} a_i}.$$

Proof. It follows from the fact that $p_{k,\mathbf{a}}(n) = p_{\mathbf{a}[k]}(n)$ (see (1.2)) and [9, Theorem 5.2]. □

4. The polynomial part and the waves of $p_{k,\mathbf{a}}(n)$

Let $\mathbf{a} = (a_1, \ldots, a_r)$. Sylvester [12, 13, 14] decomposed the restricted partition function $p_{\mathbf{a}}(n)$ in a sum of "waves",

$$p_{\mathbf{a}}(n) = \sum_{j\geq 1} W_j(n,\mathbf{a}), \quad (4.1)$$

where the sum is taken over all distinct divisors j of the components of $\mathbf{a}$ and showed that for each such j, $W_j(n,\mathbf{a})$ is the coefficient of t^{-1} in

$$\sum_{0\leq\nu<j,\ \gcd(\nu,j)=1}\frac{\left(\frac{2\pi\nu a_1 i}{j}\right)^{-\nu n}\cdot e^{nt}}{(1-e^{-a_1t+\frac{2\pi\nu a_1 i}{j}})\cdots(1-e^{-a_rt+\frac{2\pi\nu a_r i}{j}})},$$

where $\gcd(0,0) = 1$ by convention. Note that $W_j(n,\mathbf{a})$'s are quasi-polynomials of period j. Also, $W_1(n,\mathbf{a})$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$ and it is denoted by $P_{\mathbf{a}}(n)$. We define:

$$W_j(n,k,\mathbf{a}) := W_j(n,\mathbf{a}[k]) \text{ for } j \geq 1 \text{ and } P_{k,\mathbf{a}}(n) := W_1(n,k,\mathbf{a}), \quad (4.2)$$

the waves, respectively the polynomial part, of $p_{k,\mathbf{a}}(n)$.

Theorem 4.1. *For any positive integer j with $j|a_i$ for some $1 \leq i \leq r$, we have that:*

$$W_j(n,k,\mathbf{a}) = \frac{1}{D(rk-1)!} \sum_{m=1}^{rk-1} \sum_{\ell=1}^{j} e^{\frac{2\pi \ell i}{j}} \sum_{t=m-1}^{rk-1} \begin{bmatrix} rk \\ t+1 \end{bmatrix} \binom{t}{m-1} \times$$

$$\times \sum_{\substack{(\ell_1,\ldots,\ell_r)\in\mathbf{C} \\ a_1\ell_1+\cdots+a_r\ell_r \equiv n (\bmod D)}} \prod_{s=1}^{r} \sum_{i_s,j_s\geq 0,\ i_s\frac{D}{a_s}+j_s=\ell_s} (-1)^{i_s}\binom{k}{i_s}\binom{j_s+k-1}{j_s} \times$$

$$\times D^{-k}(a_1\ell_1+\cdots+a_r\ell_r)^{t-m+1}n^{m-1},$$

where $\mathbf{C} = \{(\ell_1,\ldots,\ell_r) \ :\ 0 \leq \ell_1 \leq k(\frac{D}{a_1}-1),\ldots,0\leq \ell_r \leq k(\frac{D}{a_r}-1)\}$.

Proof. The proof is similar to the proof of Theorem 3.**1**, using [7, Proposition 4.2] and Proposition 2.**7**. □

Theorem 4.2. *For $n \geq 0$ we have that*

$$P_{k,\mathbf{a}}(n) = \frac{1}{(rk-1)!} \sum_{(\ell_1,\ldots,\ell_r)\in\mathbf{C}} \prod_{s=1}^{r} \sum_{i_s,j_s\geq 0,\ i_s\frac{D}{a_s}+j_s=\ell_s} (-1)^{i_s}\binom{k}{i_s}\binom{j_s+k-1}{j_s} \times$$

$$\times \prod_{t=1}^{rk-1}\left(\frac{n-a_1\ell_1-\cdots-a_r\ell_r}{D}+t\right),$$

where $\mathbf{C} = \{(\ell_1,\ldots,\ell_r) \ :\ 0 \leq \ell_1 \leq k(\frac{D}{a_1}-1),\ldots,0\leq \ell_r \leq k(\frac{D}{a_r}-1)\}$.

Proof. The proof is similar to the proof of Theorem 3.**1**, using [4, Corollary 3.6] and Proposition 2.**7**. □

Theorem 4.3. *We have*

$$P_{k,\mathbf{a}}(n) = \frac{1}{(a_1\cdots a_r)^k} \sum_{u=0}^{rk-1} \frac{(-1)^u}{(rk-1-u)!} n^{rk-1-u} \sum_{\ell_1+\cdots+\ell_r=u} a_1^{\ell_1}\cdots a_r^{\ell_r} \times$$

$$\times \sum_{\substack{i_1+\cdots+i_k=\ell_1 \\ \vdots \\ i_{rk+k-1}+\cdots+i_{rk}=\ell_r}} \frac{B_{i_1}\cdots B_{i_{rk}}}{i_1!\cdots i_{rk}!}.$$

Proof. From [4, Corollary 3.11] it follows that

$$P_{k,\mathbf{a}}(n) = \frac{1}{(a_1\cdots a_r)^k} \sum_{u=0}^{rk-1} \frac{(-1)^u}{(rk-1-u)!} \sum_{i_1+\cdots+i_{rk}=u} \frac{B_{i_1}\cdots B_{i_{rk}}}{i_1!\cdots i_{rk}!} \times$$

$$\times a_1^{i_1+\cdots+i_k}\cdots a_r^{i_{rk-k+1}\cdots i_{rk}} n^{rk-1-u}$$

The conclusion follows immediately. □

5. Conclusions

We proved new formulas for $p_{k,\mathbf{a}}(n)$, the restricted k-multipartition function associated to a sequence of positive integers $\mathbf{a} = (a_1,\ldots,a_r)$ and to an integer $k \geq 2$, its Sylvester's waves and, in particular, its polynomial part. Also, we give a lower bound for the density of the set $\{n \geq 0 \ :\ p_{k,\mathbf{a}}(n) \not\equiv 0 (\bmod m)\}$, where $m \geq 2$.

Our methods are suitable for study other (restricted) integer partition functions.

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