A SELF-ADAPATIVE SUBGRADIENT EXTRAGRADIENT ALGORITHM WITH INERTIAL EFFECTS FOR VARIATIONAL INEQUALITIES

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In this paper, we introduce a new self-adaptive subgradient extragradient algorithm with inertial effects for variational inequality, for which the stepsize is chosen by a new way. The weak convergence of the algorithm is established. The numerical examples are given which illustrate the efficiency and advantage of the proposed algorithms.

Keywords: Variational inequality, Extragradient algorithm, Subgradient extragradient algorithm, Inertial type algorithm, Weak convergence.

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1. Introduction

Let \( C \) be a nonempty, closed and convex set in a real Hilbert space \( H \) and \( f : H \to H \) be a given mapping. In this article, we consider the classical variational inequality (\( VI(C, f) \))

\[
\text{Find } x^* \in C, \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.
\]

The theory of variational inequality has applications in many fields such as mathematical economy, physics, society, and engineering, and provides a simple, natural and unified framework for many issues, such as the minimization problems, fixed point problems, equilibrium problems and so on, see \([1, 3, 5, 6, 11, 21, 24–26, 29, 30, 32–37, 42–48, 50]\).

A great deal of projection methods for solving \( VI(C, f) \) have been studied (see, e.g., \([4, 8, 10, 13, 14, 17, 27, 38–41, 49]\)), where the simplest one is

\[
x_{k+1} = P_C(x_k - \tau f(x_k)), \quad k \geq 0
\]

(2)

where \( \tau \) is some positive real number and \( P_C \) is the metric projection onto \( C \) (see its definition in Definition 2.1).

If \( f \) is Lipschitz continuous and strongly monotone, then the sequence \( \{x_k\}_{k \in \mathbb{N}} \) generated by (2) converges to the solution of the problem (1). However, if the strong monotonicity hypothesis reduces to the plain monotonicity, then the sequence may be divergent. In order to deal with this situation, Korpelevich \([16]\) proposed the well-known extragradient algorithm

\[
\begin{align*}
  y_k &= P_C(x_k - \tau f(x_k)), \\
  x_{k+1} &= P_C(x_k - \tau f(y_k)),
\end{align*}
\]

(3)

which is convergent when \( f \) is Lipschitz continuous and monotone. If \( C \) is a general closed and convex set, then a minimal distance problem has to be solved (twice) in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method.

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To overcome this problem, Censor et al. [4] introduced a subgradient extragradient algorithm which replaces the second projection onto $C$ of the extragradient method (3) by a projection onto a specific constructible half-space $T_k$. The subgradient extragradient algorithm is defined by

$$
\begin{align*}
    y^k &= P_C(x^k - \tau f(x^k)), \\
    T^k &= \{ w \in H \mid \langle x^k - \tau f(x^k) - y^k, w - y^k \rangle \leq 0 \}, \\
    x^{k+1} &= P_{T_k}(x^k - \tau f(y^k)),
\end{align*}
$$

(4)

where $\tau$ is a positive real number.

To accelerate the speed of the extragradient-type algorithms, the inertial extrapolation technique was combined with the projection methods by some authors [7, 9, 31] for solving the variational inequality problems. These inertial projection algorithms have excellent numerical performance.

Recently, Thong and Hieu [27] proposed the following inertial subgradient extragradient algorithm:

$$
\begin{align*}
    w^k &= x^k + \alpha_n (x^k - x^{k-1}), \\
    y^k &= P_C(w^k - \tau f(w^k)), \\
    T_n &= \{ x \in H \mid \langle w^k - \tau f(w^k) - y^k, x - y^k \rangle \leq 0 \}, \\
    x^{k+1} &= P_{T_n}(w^k - \tau f(y^k)),
\end{align*}
$$

(5)

where $\tau$ is a positive real number.

Thong and Hieu [28] also applied the self-adaptive technique to give the stepsize $\tau_k$ which is the largest $\tau \in \{ \gamma, \gamma l, \gamma l^2, \ldots \}$ satisfying $\tau \| f(w^k) - f(y^k) \| \leq \mu \| w^k - y^k \|$, $l, \mu \in (0, 1)$.

Yang [31] introduced another self-adaptive technique as follows:

$$
\tau_{k+1} = \begin{cases} 
    \min \left\{ \frac{\mu (\| w^k - y^k \|^2 + \| x^{k+1} - y^k \|^2)}{2 (f(w^k) - f(y^k), x^{k+1} - y^k)}, \tau_n \right\}, & \text{if } (f(w^k) - f(y^k), x^{k+1} - y^k) > 0, \\
    \tau_n, & \text{otherwise},
\end{cases}
$$

(6)

where $\mu \in (0, 1)$.

The main purpose of this article is to introduce a new inertial subgradient extragradient algorithm, for which the stepsize is chosen through a different way from (6) and that in [28].

This paper is organized as follows. In Section 2, we recall some definitions and preliminary results used in the proof of the main results. Section 3 introduces an inertial subgradient extragradient algorithm and shows its weak convergence. In section 4, we provide two numerical experiments to illustrate the behaviors of the proposed algorithm by comparing with other methods.

2. Preliminaries

We use $x_k \rightharpoonup x$ ($x_k \to x$) to indicate that the sequence $(x_k)_{k\in\mathbb{N}}$ converges weakly (strongly) to $x$.

Let $C$ be a closed convex subset of real Hilbert space $H$. Denote by $N_C(v)$ the normal cone ( [19], p.76) of $C$ at $v \in C$, i.e.,

$$
N_C(v) := \{ d \in H \mid \langle d, y - v \rangle \leq 0, \quad \forall y \in C \}.
$$

Recall that in a Hilbert space $H$

$$
\| \lambda x + (1 - \lambda) y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda (1 - \lambda) \| x - y \|^2,
$$

(7)
for all \(x, y \in H\) and \(\lambda \in \mathbb{R}\) (see Corollary 2.14 in [2]). There also holds

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.
\]  

(8)

**Definition 2.1.** Let \(C\) be a closed convex subset of real Hilbert space \(H\). \(P_C\) is called the (metric or nearest point) projection from \(H\) onto \(C\) if for \(x \in H\), \(P_Cx\) is the unique point in \(C\) such that

\[
\|x - P_Cx\| = \inf\{\|x - z\| : z \in C\}.
\]

**Lemma 2.1.** Given \(x \in H\) and \(z \in C\). Then \(z = P_Cx\) if and only if there holds the relation:

\[
\langle x - z, y - z \rangle \leq 0, \quad \text{for all } y \in C.
\]

**Lemma 2.2** ([4]). For any \(x, y \in H\) and \(z \in C\), it holds

(i) \(\|P_C(x) - P_C(y)\| \leq \|x - y\|\);

(ii) \(\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2\).

(iii) \(\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle\).

**Lemma 2.3** ([2]). Assume that \(\{u_k\}, \{v_k\}\) and \(\{\alpha_k\}\) are three sequences in \([0, \infty)\) such that

(i) \(u_{k+1} - u_k \leq \alpha_k(u_k - u_{k+1}) + v_k\) for all \(k \geq 0\);

(ii) \(0 \leq \alpha_k \leq \alpha (\forall k \geq 0)\) for some \(\alpha \in (0, 1)\);

(iii) \(\sum_{k=0}^{\infty} v_k < \infty\).

Then \(\lim_{k \to \infty} u_k\) exists.

**Lemma 2.4** ([20]). Let \(f : H \to H\) be a monotone and \(L\)-Lipschitz continuous mapping. Assume that the sequence \(\{x^k\} \subset H\) satisfies \(x^k \rightharpoonup u^1\) and \(x^k - P_C(I - \tau f)x^k \to 0 (\tau > 0)\). Then \(u^1 \in SOL(f, C)\).

**Lemma 2.5.** (Opial’s lemma) Let \(C\) be a nonempty set of \(H\) and \(\{x^k\}\) be a sequence in \(H\) such that the following two conditions hold:

(i) \(\forall u \in C, \lim_{k \to \infty} \|x^k - u\|\) exists;

(ii) \(\omega_w(x^k) \subset C\).

Then \(\{x^k\}\) converges weakly to a point in \(C\).

3. The main results

In this section, we introduce a new self-adaptive inertial subgradient extragradient algorithm and establish its weak convergence.

We firstly impose the following assumptions on the variational inequality.

**Condition 3.1** The solution set of (1), denoted by \(SOL(C, f)\), is nonempty.

**Condition 3.2** The mapping \(f\) is monotone on \(H\), i.e.

\[
\langle f(x) - f(y), x - y \rangle \geq 0, \quad \forall x, y \in H.
\]

**Condition 3.3** The mapping \(f\) is Lipschitz continuous on \(H\) with constant \(L > 0\), that is,

\[
\|f(x) - f(y)\| \leq L\|x - y\|, \quad \forall x, y \in H.
\]

3.1. Algorithm

Now we present the self-adaptive inertial subgradient extragradient algorithm.

**Algorithm 3.1.**

**Initialization:** Let \(\{\alpha_k\}\) be a sequence in \([0, \infty)\). Let \(\mu, l \in (0, 1), \sigma > 0\) and \(x_0, x_{-1} \in H\) be arbitrary.

**Iterative Steps:** Assume that \(x^k\) has been given. Calculate \(x^{k+1}\) as follows:
Step 1. Compute
\[ w^k = x^k + \alpha_k(x^k - x^{k-1}) , \]
and
\[ y^k = P_C(w^k - \tau_k f(w^k)) , \]
where \( \tau_k \) is the largest \( \tau \in \{ \sigma, \sigma^2, \ldots \} \) satisfying
\[ \tau \|f(w^k) - f(y^k)\| \leq \mu \|w^k - y^k\| . \]
If \( y^k = w^k \) then stop. Otherwise, go to Step 2.

Step 2. Construct the half-space
\[ T_k := \{ z \in H : (w^k - \gamma \tau_k \beta_k f(w^k) - y^k, z - y^k) \leq 0 \} \]
and compute
\[ x^{k+1} = P_{T_k}(w^k - \gamma \tau_k \beta_k f(y^k)) , \]
where \( \gamma \) is a positive real number and
\[ \beta_k := \begin{cases} \frac{\varphi(w^k, y^k)/\|d(w^k, y^k)\|^2}{1 - \mu}, & \text{if } d(w^k, y^k) \neq 0 , \\ 1 , & \text{if } d(w^k, y^k) = 0 , \end{cases} \]
and
\[ \varphi(w^k, y^k) := \langle w^k - y^k, d(w^k, y^k) \rangle , \]
and
\[ d(w^k, y^k) := \langle w^k - y^k, \tau_k(f(w^k) - f(y^k)) \rangle . \]
Let \( k := k + 1 \) and return to Step 1.

Remark 3.1. We have the following remarks for the stepsize and the inertial parameters \( \alpha_k \).

(i) The stepsizes in \( y^k \) and \( x^{k+1} \) in Algorithm 3.1 are different and the choice of the stepsizes of our algorithm is different from (6) and that in (5).

(ii) By (10), it is easy to show that \( \frac{\mu \|d\|}{T} \leq \tau_k \leq \sigma \).

(iii) In Algorithm 3.1, \( \{\alpha_k\}_{k \in \mathbb{N}} \) is assumed to be nondecreasing with \( \alpha_1 = 0 \) and satisfy \( 0 \leq \alpha_k \leq \alpha < \sqrt{5} - 2 \). The parameters \( \mu \) and \( \alpha \) satisfy the following inequality
\[ 0 < \gamma \leq \frac{(1 - \mu)^2(1 - 4\alpha - \alpha^2 - 2\delta)}{\mu(1 - \alpha)^2} , \]
where \( \delta \in \left( 0, \frac{1 - 4\alpha - \alpha^2}{2} \right) \).

From (13), it is easy to verify that
\[ 0 < \gamma \mu \frac{(1 - \mu)^2}{(1 - \alpha)^2} \leq \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1 - \alpha)^2} < 1 . \]

Remark 3.2. By Lemma 2.1, we know that \( w^\dagger \in SOL(C, f) \iff w^\dagger = P_C(w^\dagger - \tau f(w^\dagger)) \) for some \( \tau > 0 \). Thus, if at some iterative step \( w^k = y^k = P_C(w^k - \tau_k f(w^k)) \), then \( w^k \in SOL(C, f) \).

Remark 3.3. We assume that \( f \) is \( L \)-Lipschitz continuous. However, the information of \( L \) is not necessary priority to be known. That is, we need not to estimate the value of \( L \).

Lemma 3.1. Let \( \{\beta_k\}_{k \in \mathbb{N}} \) be generated by (12). Under the Condition 3.2, it holds
\[ \beta_k \leq \frac{1}{(1 - \mu)^2} . \]
Proof. By the monotonicity of \( f \), we have
\[
\langle f(w^k) - f(y^k), w^k - y^k \rangle \geq 0.
\]
Using Condition 3.2 and the definition of \( d(w^k, y^k) \), we obtain
\[
\varphi(w^k, y^k) = \langle d(w^k, y^k), w^k - y^k \rangle
\]
\[
= \langle w^k - y^k, w^k - y^k \rangle - \tau_k \langle f(w^k) - f(y^k), w^k - y^k \rangle
\]
\[
\leq \| w^k - y^k \|^2,
\]
and
\[
\| d(w^k, y^k) \|^2 = \| (w^k - y^k) - \tau_k (f(w^k) - f(y^k)) \|^2
\]
\[
= \| w^k - y^k \|^2 + \tau_k^2 \| f(w^k) - f(y^k) \|^2 - 2 \tau_k \| f(w^k) - f(y^k), w^k - y^k \|
\]
\[
\geq (\| w^k - y^k \|^2 - \tau_k \| f(w^k) - f(y^k) \| \| w^k - y^k \|)
\]
\[
\geq (1 - \mu) \| w^k - y^k \|^2.
\]
From (16) and (17), (15) follows. \qed

3.2. The convergence analysis

We present some convergence results for Algorithm 3.1 as follows.

Lemma 3.2. Let \( u \in SOL(C, f) \) and \( \{x^k\} \in \mathbb{N} \) be the sequence generated by Algorithm 3.1. Then, under Conditions 3.1, 3.2 and 3.3, we have
\[
\| x^{k+1} - u \|^2 \leq \| x^k - u \|^2 - \frac{1}{2} \left( 1 - \frac{\gamma \mu}{(1 - \mu)^2} \right) \| x^{k+1} - w^k \|^2.
\]

Proof. Applying Lemma 2.2(ii) to (11), we obtain
\[
\| x^{k+1} - u \|^2 \leq \| x^k - \gamma \tau_k \beta_k f(y^k) - u \|^2 - \| x^k - \gamma \tau_k \beta_k f(y^k) - x^{k+1} \|^2
\]
\[
= \| x^k - u \|^2 - 2 \gamma \tau_k \beta_k \langle w^k - u, f(y^k) \rangle + \gamma^2 \tau_k^2 \beta_k^2 \| f(y^k) \|^2
\]
\[
- \| x^k - x^{k+1} \|^2 + 2 \gamma \tau_k \beta_k \langle w^k - x^{k+1}, f(y^k) \rangle - \| x^k - x^{k+1} \|^2
\]
\[
= \| x^k - u \|^2 + 2 \gamma \tau_k \beta_k \langle u - x^{k+1}, f(y^k) \rangle - \| x^k - x^{k+1} \|^2
\]
\[
\geq \| x^k - u \|^2 + 2 \gamma \tau_k \beta_k \langle u - y^k, f(y^k) - f(u) \rangle - \| x^k - x^{k+1} \|^2
\]
\[
+ 2 \gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) \rangle.
\]

Since \( f \) is monotone, \( \langle u - y^k, f(y^k) - f(u) \rangle \leq 0 \). Noting that \( u \in SOL(C, f) \) and \( y^k \in C \), we deduce that \( \langle u - y^k, f(u) \rangle \leq 0 \). It follows from (18) that
\[
\| x^{k+1} - u \|^2 \leq \| x^k - u \|^2 - \| x^k - x^{k+1} \|^2 + 2 \gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) \rangle
\]
\[
= \| w^k - u \|^2 - \| w^k - y^k + y^k - x^{k+1} \|^2 + 2 \gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) \rangle
\]
\[
= \| w^k - u \|^2 - \| w^k - y^k \|^2 - 2 \| w^k - y^k, y^k - x^{k+1} \| - \| y^k - x^{k+1} \|^2
\]
\[
+ 2 \gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) \rangle.
\]
By the definition of \( T_k \) and \( x^{k+1} \in T_k \), we have
\[
\langle w^k - \gamma \tau_k \beta_k f(w^k) - y^k, x^{k+1} - y^k \rangle \leq 0,
\]
which implies that
\[
\langle w^k - y^k, y^k - x^{k+1} \rangle \geq \gamma \tau_k \beta_k \langle f(w^k), y^k - x^{k+1} \rangle.
\]
By virtue of (19) and (20), we obtain
\[
\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 - \|w^k - y^k\|^2 - 2\gamma \tau_k \beta_k \langle f(w^k), y^k - x^{k+1} \rangle \\
- \|y^k - x^{k+1}\|^2 + 2\gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) \rangle \\
\leq \|w^k - u\|^2 + 2\gamma \tau_k \beta_k (y^k - x^{k+1}, f(y^k) - f(u^k)) \\
- \|w^k - y^k\|^2 - \|y^k - x^{k+1}\|^2.
\]

In the light of (10), we deduce
\[
2\gamma \tau_k \beta_k \langle y^k - x^{k+1}, f(y^k) - f(u^k) \rangle \leq 2\gamma \beta_k \tau_k \|f(y^k) - f(u^k)\| \|y^k - x^{k+1}\| \\
\leq 2\gamma \beta_k \mu \|y^k - w^k\| \|y^k - x^{k+1}\| \\
\leq \gamma \beta_k \mu \|y^k - w^k\|^2 + \gamma \beta_k \mu \|y^k - x^{k+1}\|^2.
\]

This together with (21) implies that
\[
\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 + \gamma \beta_k \mu \|y^k - w^k\|^2 + \gamma \beta_k \mu \|y^k - x^{k+1}\|^2 \\
- \|w^k - y^k\|^2 - \|y^k - x^{k+1}\|^2
\]
\[
= \|w^k - u\|^2 - (1 - \gamma \beta_k \mu) \|w^k - y^k\|^2 - (1 - \gamma \beta_k \mu) \|x^{k+1} - y^k\|^2.
\]

By the mean value inequality and the trigonometric inequality,
\[
\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 - \frac{1}{2}(1 - \gamma \beta_k \mu) (\|w^k - y^k\|^2 + \|x^{k+1} - y^k\|^2)
\]
\[
\leq \|w^k - u\|^2 - \frac{1}{2}(1 - \gamma \beta_k \mu) \|x^{k+1} - w^k\|^2.
\]

Combining (15) with (23), we have
\[
\|x^{k+1} - u\|^2 \leq \|w^k - u\|^2 - \frac{1}{2} \left(1 - \frac{\gamma \mu}{(1 - \mu)^2}\right) \|x^{k+1} - w^k\|^2.
\]

The proof is complete. \qed

**Theorem 3.1.** Assume that Conditions 3.1, 3.2 and 3.3 hold. Then the sequence \( \{x^k\} \in \mathbb{N} \) generated by Algorithm 3.1 converges weakly to SOL\((C, f)\).

**Proof.** From the definition of \( w^k \), we have
\[
\|x^{k+1} - w^k\|^2 = \|x^{k+1} - (x^k + \alpha_k (x^k - x^{k-1}))\|^2 \\
= \|x^k - x^{k+1}\|^2 + \alpha_k^2 \|x^k - x^{k+1}\|^2 + 2\alpha_k \langle x^k - x^{k+1}, x^k - x^{k-1} \rangle \\
\geq (1 - \alpha_k) \|x^{k+1} - x^k\|^2 + (\alpha_k^2 - \alpha_k) \|x^k - x^{k-1}\|^2.
\]

Using (7), we obtain
\[
\|w^k - u\|^2 = (1 + \alpha_k) \|w^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \alpha_k (1 + \alpha_k) \|x^k - x^{k-1}\|^2.
\]

Then, from (14) and Lemma 3.2, we get
\[
\|x^{k+1} - u\|^2 \leq (1 + \alpha_k) \|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \alpha_k (1 + \alpha_k) \|x^k - x^{k-1}\|^2 \\
- \frac{1}{2} \left(1 - \frac{\gamma \mu}{(1 - \mu)^2}\right) \left[(1 - \alpha_k) \|x^{k+1} - x^k\|^2 + (\alpha_k^2 - \alpha_k) \|x^k - x^{k-1}\|^2\right]
\]
\[
\leq (1 + \alpha_k) \|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 - \zeta_k \|x^{k+1} - x^k\|^2 + \xi_k \|x^k - x^{k-1}\|^2,
\]

where
\[
\zeta_k := \frac{1}{2} \left(1 - \frac{\gamma \mu}{(1 - \mu)^2}\right) (1 - \alpha_k) \geq 0.
\]
Combining (31) and (32), we deduce

\[ \zeta_k := \alpha_k (1 + \alpha_k) - \frac{1}{2} \left( 1 - \frac{\gamma \mu}{(1 - \mu)^2} \right) (\alpha_k^2 - \alpha_k) \geq 0. \]  

(27)

Let \( h = \frac{1}{2} \left( 1 - \frac{\gamma \mu}{(1 - \mu)^2} \right) \). By (13), it is easy to show

\[ \zeta_k - \zeta_{k+1} = h(1 - \alpha_k) - \alpha_{k+1} (1 + \alpha_{k+1}) + h(\alpha_{k+1}^2 - \alpha_k) \]

\[ \geq h(1 - \alpha_{k+1}) - \alpha_{k+1} (1 + \alpha_{k+1}) + h(\alpha_{k+1}^2 - \alpha_k) \]

\[ = h(1 - \alpha_{k+1})^2 - \alpha_{k+1} - \alpha_k \]

\[ \geq h(1 - \alpha)^2 - \alpha - \alpha^2 \]

\[ \geq -\delta. \]  

(28)

We define the sequence, for all \( k \geq 1 \),

\[ \Gamma^k := \|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \zeta_k \|x^k - x^{k-1}\|^2. \]  

(29)

Using the monotonicity of \( \{\alpha_k\}_{k \geq 1} \), we get

\[ \Gamma^{k+1} - \Gamma^k \leq \|x^{k+1} - u\|^2 - \alpha_k \|x^k - u\|^2 + \zeta_{k+1} \|x^{k+1} - x^k\|^2 \]

\[ - \|x^{k+1} - u\|^2 + \alpha_{k+1} \|x^{k-1} - u\|^2 - \zeta_k \|x^k - x^{k-1}\|^2 \]

\[ = \|x^{k+1} - u\|^2 - (1 + \alpha_k) \|x^k - u\|^2 + \alpha_k \|x^{k-1} - u\|^2 \]

\[ + \zeta_{k+1} \|x^{k+1} - x^k\|^2 - \zeta_k \|x^k - x^{k-1}\|^2 \]

\[ \leq - \zeta_k \|x^{k+1} - x^k\|^2 + \zeta_{k+1} \|x^{k+1} - x^k\|^2 \]

\[ = - (\zeta_k - \zeta_{k+1}) \|x^{k+1} - x^k\|^2 \]

\[ \leq - \delta \|x^{k+1} - x^k\|^2 \leq 0. \]  

Hence, \( \{\Gamma^k\} \) is a monotone decreasing sequence.

According to (29), we get

\[ \|x^k - u\|^2 \leq \alpha_k \|x^{k-1} - u\|^2 + \Gamma^k \]

\[ \leq \alpha \|x^{k-1} - u\|^2 + \Gamma^1 \]

\[ \leq \ldots \]

\[ \leq \alpha^k \|x^0 - u\|^2 + (1 + \alpha + \cdots + \alpha^{k-1}) \Gamma^1 \]

\[ \leq \alpha^k \|x^0 - u\|^2 + \frac{1}{1 - \alpha} \Gamma^1. \]  

(31)

Note that

\[ \Gamma^{k+1} = \|x^{k+1} - u\|^2 - \alpha_{k+1} \|x^k - u\|^2 + \zeta_{k+1} \|x^{k+1} - x^k\|^2 \]

\[ \geq -\alpha_{k+1} \|x^k - u\|^2. \]  

(32)

Combining (31) and (32), we deduce

\[ -\Gamma^{k+1} \leq \alpha_{k+1} \|x^k - u\|^2 \leq \alpha \|x^k - u\|^2 \leq \alpha^k \|x^0 - u\|^2 + \frac{\alpha}{1 - \alpha} \Gamma^1. \]
This together with (30) implies that
\[ \delta \sum_{k=1}^{i} \|x^{k+1} - x^k\|^2 \leq \sum_{k=1}^{i} (\Gamma^k - \Gamma^{k+1}) = \Gamma^1 - \Gamma^{k+1} \]
\[ \leq \alpha^{i+1} \|x^0 - u\|^2 + \frac{1}{1 - \alpha} \Gamma^1 \]
\[ \leq \|x^0 - u\|^2 + \frac{1}{1 - \alpha} \Gamma^1. \]
which implies that
\[ \sum_{k=1}^{\infty} \|x^{k+1} - x^k\|^2 < \infty. \tag{33} \]
Therefore,
\[ \lim_{k \to \infty} \|x^{k+1} - x^k\| = 0. \tag{34} \]
Note that
\[ \|w^k - x^k\| = \|\alpha_k(x^k - x^{k-1})\| \leq \alpha_k \|x^k - x^{k-1}\| \leq \alpha \|x^k - x^{k-1}\|. \tag{35} \]
By (34) and (35), we derive
\[ \lim_{k \to \infty} \|x^k - w^k\| = 0. \tag{36} \]
By the definition (27) of \(\xi_k\), there exists a positive constant \(\xi\) such that \(\xi_k \leq \xi\) for all \(k \geq 0\).
From (25), we have
\[ \|x^{k+1} - u\|^2 \leq (1 + \alpha_k) \|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \xi_k \|x^k - x^{k-1}\|^2 \]
\[ \leq (1 + \alpha_k) \|x^k - u\|^2 - \alpha_k \|x^{k-1} - u\|^2 + \xi \|x^k - x^{k-1}\|^2. \]
It follows that
\[ \|x^{k+1} - u\|^2 - \|x^k - u\|^2 \leq \alpha_k (\|x^k - u\|^2 - \|x^{k-1} - u\|^2) + \xi \|x^k - x^{k-1}\|^2. \tag{37} \]
Applying Lemma 2.3 to (37), we deduce that the limit \(\lim_{k \to \infty} \|x^k - u\|\) exists. Thus, the sequence \(\{x^k\}\) is bounded.
From (22), we obtain
\[ (1 - \gamma \mu \beta_k) \|w^k - y^k\|^2 \leq \|w^k - u\|^2 - \|x^{k+1} - u\|^2 \leq \|w^k - x^{k+1}\|(\|w^k - u\| + \|x^{k+1} - u\|). \tag{38} \]
By (14) and (15), \(\gamma \mu \beta_k \leq \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1 - \alpha^2)} < 1.\) Thus, \(1 - \gamma \mu \beta_k > 1 - \frac{(1 - 4\alpha - \alpha^2 - 2\delta)}{(1 - \alpha^2)} > 0).\)
Thanks to (34) and (36), \(\lim_{k \to \infty} \|w^k - x^{k+1}\| = 0.\) At the same time, \(\{\|w^k - u\|\}\) and \(\{\|x^k - u\|\}\) are all bounded. Therefore, from (38), we get
\[ \lim_{k \to \infty} \|w^k - y^k\| = 0. \tag{39} \]
Let \(u^i \in \omega_w(x_n)\). There exists a subsequence \(\{x^{k_i}\} \subset \{x^k\}\) such that \(x^{k_i} \to u^i\). Thus, \(w^{k_i} \to u^i\) due to (36). Applying Lemma 2.4 to (39), we conclude that \(u^i \in SOL(f, C)\).
Hence, \(\lim_{k \to \infty} \|x^k - u^i\|\) exists. Therefore, by Lemma 2.5, we deduce that \(x^k\) weakly converges to an element in \(SOL(f, C)\). The proof is completed. \(\square\)
4. Numerical illustrations

In this section, we provide two numerical examples to show the practicability and the advantage of our proposed algorithm by comparing it with Algorithm 1 in [22] and Algorithm 3.1 in [23].

In numerical results listed in the following tables, ‘Iter.’ and ‘Sec.’ denote the number of iterations and the cpu time in seconds, respectively.

We take $\alpha_k = 0.1$ in these Algorithms and $\sigma = 0.2$ in Algorithm 3.1 and Algorithm 1 in [22]. Take $\gamma = 0.9$ and $\mu = 0.5$ in Algorithm 3.1 and choose $\mu = 0.5$ in Algorithm 3.1 in [23]. Set $l = 0.01$ in Example 4.1 and $l = 0.01$ in Example 4.2, respectively.

**Example 4.1.** Let the operator $f(x) := Mx + q$, $x \in \mathbb{R}^m$. This example is taken from [12] and has been considered by many authors for numerical experiments (see, for example, [15, 18, 23]), where

$$M = BB^T + F + D,$$

and $B$ is an $m \times m$ matrix, $F$ is an $m \times m$ skew-symmetric matrix, $D$ is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so $M$ is positive semidefinite), $q$ is a vector in $\mathbb{R}^m$. The feasible set $C \subset \mathbb{R}^m$ is a closed and convex subset defined by $C := \{x \in \mathbb{R}^m : Qx \leq b\}$, where $Q$ is an $l \times m$ matrix and $b$ is a nonnegative vector. It is clear that $f$ is monotone and $L$-Lipschitz-continuous with $L = \|M\|$. Let $q = 0$. Then, the solution set $\Gamma := \{0\}$.

Take $\|x^k\| < 10^{-5}$ as the stopping criterion in Table 1. Figure 1 and Table 1 show that Algorithm 3.1 is better than Algorithm 1 in [22] and Algorithm 3.1 in [23] in the number of iterations and the cpu time.

Next we give an example in an infinite dimensional Hilbert space.
Example 4.2. Suppose that $H = L^2([0, 1])$ with norm $\|x\| := \left( \int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}}$ and inner product $\langle x, y \rangle := \int_0^1 x(t)y(t)dt$, $x, y \in H$. Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball. Define an operator $A : C \to H$ by

$$A(x)(t) = \int_0^1 (x(t) - F(t,s)f(x(s)))ds + g(t), \quad x \in C, t \in [0, 1],$$

(40)

where

$$F(t, s) = \frac{2ts e^{t+s}}{e^{\sqrt{e^2 - 1}}}, \quad f(x) = \cos x, \quad g(t) = \frac{2te^t}{e^{\sqrt{e^2 - 1}}}.$$ 

$A$ is monotone and $L$-Lipschitz-continuous with $L = 2$ (hence uniformly continuous) and $\Gamma = \{0\}$.

Take $\|x^k\| < 10^{-15}$ as the stopping criterion in Table 2. Figure 2 and Table 2 illustrate that Algorithm 3.1 is better than Algorithm 1 in [22] and Algorithm 3.1 in [23].
Algorithm 3.1 Algorithm 1 in [22] Algorithm 3.1 in [23]

<table>
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<th>x(t)</th>
<th>Iter.</th>
<th>Sec.</th>
<th>Iter.</th>
<th>Sec.</th>
<th>Iter.</th>
<th>Sec.</th>
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<td>0.159966</td>
<td>124</td>
<td>0.163477</td>
<td>186</td>
<td>0.246614</td>
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<tr>
<td>(\frac{1}{2})</td>
<td>83</td>
<td>0.163891</td>
<td>124</td>
<td>0.171908</td>
<td>193</td>
<td>0.260201</td>
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<tr>
<td>(t^2)</td>
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<td>125</td>
<td>0.187206</td>
<td>189</td>
<td>0.271120</td>
</tr>
<tr>
<td>(e^t)</td>
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<td>0.186619</td>
<td>127</td>
<td>0.189535</td>
<td>183</td>
<td>0.281754</td>
</tr>
</tbody>
</table>

**Table 2.** Comparison of Algorithm 3.1 and Algorithm 1 in [22] and Algorithm 3.1 in [23] in different initial point.

**References**


