U.P.B. Sci. Bull., Series A, Vol. 79, Iss. 4, 2017 ISSN 1223-7027

CERTAIN GENERATING RELATIONS OF KONHAUSER MATRIX POLYNOMIALS FROM THE VIEW POINT OF LIE ALGEBRA METHOD

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This paper is devoted to construct Lie operators associated with Konhauser matrix polynomials of the first kind using Lie group theory. Furthermore, certain generating matrix functions, integral representations and matrix differential recurrence relations, new and known consequences for Konhauser matrix polynomials are derived and their applications are presented.

Keywords: Konhauser matrix polynomials, Matrix recurrence relations, Matrix differential equations, Generating matrix functions, Lie algebra method, Lie group theory.

MSC2010: 33C25, 33C35, 33C45, 33C80, 15A60.

1. **Introduction**

Special matrix functions are the solutions of a wide class of mathematically and physically relevant functional equations. Generating matrix functions play an important role in the study of special matrix functions. In the investigation of generating matrix functions, group theoretic method seems to be a potent one in comparison with analytic methods because of the fact that the unknown generating matrix functions can only be obtained by group theoretic method as well as the known generating matrix functions can be verified and the corresponding extension can be made by analytic method (see [2, 3, 4, 5, 12, 13, 15, 19, 20, 21, 24, 25]). In [16, 17], Konhauser also introduced two sets of polynomials $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$, which are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$, $\alpha > -1$ for k is a positive integer.

Motivated and inspired by the work of Erkuş-Duman and Çekim [7], Shehata [26], Varma et al. [27], and Varma and Taşdelen [28] and a recent work on representation of Lie algebra [1, 11, 14, 22, 23], in this paper, we derive some integral representations, matrix differential recurrence relations and certain generating matrix functions involving Konhauser matrix polynomials of the first kind by using Weisner's method [29]. In section 2, we discuss some integral representations and matrix differential recurrence relations with Konhauser matrix polynomials. In section 3, we derive certain generating matrix functions for Konhauser matrix polynomials by using the representation of the Lie group theory. The main interest in our results lies in the fact that a number of their special cases can be used to derive many new and known consequences for the Konhauser matrix polynomials of two variables, which we will obtain in section 4. The main results of our investigation are derived in sections 5.

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1.1. Preliminaries

In this subsection, we give the brief introduction related to Konhauser matrix polynomials and recall the following definitions, theorems, lemmas and some previously known concepts. During this work, the spectrum $\sigma(A)$ of a matrix A in $\mathbb{C}^{N\times N}$ symbolize the set of all eigenvalues of A. Furthermore, the identity matrix and the null matrix or zero matrix in $\mathbb{C}^{N\times N}$ will be symbolized by I and $\mathbf{0}$, respectively.

Definition 1.1. *If $A_0,\ A_1,\dots,A_n$ are elements in $\mathbb{C}^{N\times N}$ and $A_n\neq \mathbf{0}$, then the matrix polynomials of degree n in x (x is a real variable or complex variable) is an expression in the form*

$$P(x)=A_nx^n+A_{n-1}x^{n-1}+\ldots+A_1x+A_0.$$

Theorem 1.1. *(Dunford and Schwartz* [6], *Theorem 5, p.558) If $u(z)$ and $v(z)$ are holomorphic functions in an open set Ω of the complex plane $\mathbb{C}$, and P, Q are commutative matrices in $\mathbb{C}^{N\times N}$ with $\sigma(P)\subset\Omega$ and $\sigma(Q)\subset\Omega$, then*

$$u(P)v(Q)=v(Q)u(P).$$

Definition 1.2. *(Jódar and Cortés* [9], *p.89) If P is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix function $\Gamma(P)$ is defined as*

$$\Gamma(P)=\int_0^{\infty}e^{-t}t^{P-I}dt;\quad t^{P-I}=\exp\Big((P-I)\ln t\Big). \tag{1.1}$$

Definition 1.3. *For $A\in\mathbb{C}^{N\times N}$ such that $\sigma(A)$ does not contain 0 or a negative integer ($\sigma(A)\cap\mathbb{Z}^-=\emptyset$ where $\emptyset$ is an empty set), the matrix analogues of Pochhammer symbol or shifted factorial is defined as (see Jódar and Cortés* [10], *p.206)*

$$\begin{aligned}(A)_n&=A(A+I)(A+2I)\ldots(A+(n-1)I)\\&=\Gamma(A+nI)\Gamma^{-1}(A);\ n\geq 1;\ (A)_0=I,\end{aligned} \tag{1.2}$$

where $\Gamma(A)$ is an invertible matrix.

Definition 1.4. *(Jódar and Cortés* [9], *p.92) If P and Q are positive stable matrices in $\mathbb{C}^{N\times N}$, then the Beta matrix function $\boldsymbol{B}(P,Q)$ is defined as*

$$\boldsymbol{B}(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q-I}dt. \tag{1.3}$$

Lemma 1.1. *(Jódar and Cortés* [10], *Lemma 2, p.209) Let P, Q and $Q+P$ be positive stable matrices in $\mathbb{C}^{N\times N}$ satisfying $PQ=QP$ and $P+nI$, $Q+nI$ and $P+Q+nI$ are invertible matrices for all nonnegative integers n. Then*

$$\boldsymbol{B}(P,Q)=\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{1.4}$$

Definition 1.5. *(Jódar et al.* [8], *p.58) Let A be a matrix in $\mathbb{C}^{N\times N}$ such that*

$$-k\notin\sigma(A)\text{ for every integers }k>0, \tag{1.5}$$

and λ is a complex number with $Re(\lambda)>0$. Then the Laguerre matrix polynomials is defined as

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k(A+I)_n[(A+I)_k]^{-1}(\lambda x)^k}{k!(n-k)!}. \tag{1.6}$$

For the purpose of the present study, we recall the following explicit expression for the Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$ of the first kind in (Varma et al. [27], p. 197):

$$Z_n^{(A,\lambda)}(x;k) = \Gamma(A+(kn+1)I)\sum_{r=0}^{n}\frac{(-1)^r(\lambda\, x)^{rk}}{r!(n-r)!}\Gamma^{-1}(A+(kr+1)I), \tag{1.7}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$Re(\mu) > -1, \quad \text{for all eigenvalues } \mu \in \sigma(A) \tag{1.8}$$

and λ is a complex number with $Re(\lambda) > 0$, $k \in \mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

2. Some properties of Konhauser matrix polynomials

The integral representations for the Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$ of the first kind are derived as in the following theorems.

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8) and for $\left|\frac{t}{\lambda x}\right| < 1$. Then the Konhauser matrix polynomials has the following integral representation:*

$$Z_n^{(A,\lambda)}(x;k) = \frac{\Gamma(A+(kn+1)I)}{n!2\pi i}\int_C (t^k-(x\lambda)^k)^n e^t t^{-A-(kn+1)I}dt. \tag{2.1}$$

Proof. The contour integral representation for the reciprocal Gamma function is given as: (see [18], p. 115, No. (5.10.5))

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\, i}\int_C e^t t^{-z}dt \tag{2.2}$$

where C is the path around the origin in the positive direction, beginning at and returning to positive infinity with respect for the branch cut along the positive real axis.

Thus, from (2.2), we have the following integral matrix functional

$$\Gamma^{-1}(A+(kr+1)I) = \frac{1}{2\pi i}\int_C e^t t^{-A-(kr+1)I}dt. \tag{2.3}$$

From (1.7) and (2.3), we get

$$\begin{aligned}
&\frac{\Gamma(A+(kn+1)I)}{n!2\pi i}\int_C (t^k-(\lambda\, x)^k)^n e^t t^{-A-(kn+1)I}dt\\
&= \frac{\Gamma(A+(kn+1)I)}{n!2\pi i}\sum_{r=0}^{n}\frac{(-1)^r n!(\lambda\, x)^{kr}}{r!(n-r)!}\int_C e^t t^{-A-(kr+1)I}dt\\
&= \Gamma(A+(kn+1)I)\sum_{r=0}^{n}\frac{(-1)^r(\lambda\, x)^{kr}}{r!(n-r)!}\Gamma^{-1}(A+(kr+1)I) = Z_n^{(A,\lambda)}(x;k).
\end{aligned}$$

This immediately leads to the proof of the theorem. □

Theorem 2.2. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8). Then the Konhauser matrix polynomials has the following integral representation:*

$$Z_n^{(A,\lambda)}(x;k) = \frac{\Gamma(A+(kn+1)I)}{n!2\pi i}(\lambda x)^{-A}\int_C (u^k-1)^n e^{\lambda\, xu}u^{-A-(kn+1)I}du. \tag{2.4}$$

Proof. If we make the substitution $t = \lambda xu$ in (2.1), we get an integral representation for $Z_n^{(A,\lambda)}(x;k)$. □

In the same way, one can derive the following integrals formulas:

Theorem 2.3. *Let A and B be commutative matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (1.8), the integral representation for the Konhauser matrix polynomials satisfy the following:*

$$\begin{aligned}&\int_0^\infty x^B e^{-x}\ Z_n^{(A,\lambda)}(x;k)dx = \frac{1}{n!}(A+I)_{kn}\Gamma(B+I)\\&\times\ {}_{k+1}F_{\boldsymbol{k}}\Big(-nI,\frac{B+I}{k},\dots,\frac{B+kI}{k};\frac{A+I}{k},\dots,\frac{A+kI}{k};\lambda^k\Big).\end{aligned}\tag{2.5}$$

Proof. Using the formula in [26], we have

$$\begin{aligned}(A+I)_{kr}&=\Gamma(A+(kr+1)I)\Gamma^{-1}(A+I)\\&=k^{kr}\left(\frac{A+I}{k}\right)_r\left(\frac{A+2I}{k}\right)_r\cdots\left(\frac{A+kI}{k}\right)_r,\\(B+I)_{kr}&=\Gamma(B+(kr+1)I)\Gamma^{-1}(B+I)\\&=k^{kr}\left(\frac{B+I}{k}\right)_r\left(\frac{B+2I}{k}\right)_r\cdots\left(\frac{B+kI}{k}\right)_r.\end{aligned}\tag{2.6}$$

Using (1.1), we can write

$$\begin{aligned}&\int_0^\infty x^B e^{-x}\ Z_n^{(A,\lambda)}(x;k)dx=\Gamma(A+(kn+1)I)\\&\times\sum_{r=0}^{n}\frac{(-1)^r(\lambda)^{rk}}{(n-r)!r!}\Gamma^{-1}(A+(kr+1)I)\int_0^\infty e^{-x}x^{B+krI}dx.\end{aligned}$$

More simplification of the above expression, we have

$$\begin{aligned}&\int_0^\infty x^B e^{-x}\ Z_n^{(A,\lambda)}(x;k)dx=\Gamma(A+(kn+1)I)\\&\times\sum_{r=0}^{n}\frac{(-1)^r(\lambda)^{rk}}{(n-r)!r!}\Gamma^{-1}(A+(kr+1)I)\Gamma(B+(kr+1)I).\end{aligned}$$

This formula can be written as:

$$\begin{aligned}&\int_0^\infty x^B e^{-x}\ Z_n^{(A,\lambda)}(x;k)dx=\Gamma(A+(kn+1)I)\\&\times\sum_{r=0}^{n}\frac{(-1)^r(\lambda)^{rk}}{(n-r)!r!}\Gamma^{-1}(A+(kr+1)I)\Gamma(B+(kr+1)I).\end{aligned}$$

Using (2.6), we obtain

$$\begin{aligned}&\int_0^\infty x^B e^{-x}\ Z_n^{(A,\lambda)}(x;k)dx = \frac{1}{n!}(A+I)_{kn}\Gamma(B+I)\\&\times\ {}_{k+1}F_k\Big(-nI,\frac{B+I}{k},\dots,\frac{B+kI}{k};\frac{A+I}{k},\dots,\frac{A+kI}{k};\lambda^k\Big).\end{aligned}$$

This immediately leads to the proof of the theorem. □

Theorem 2.4. *If A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8) and $AB=BA$. Then the integral representation for the Konhauser matrix polynomials satisfies the following:*

$$\begin{aligned}&\int_0^x (x-t)^{B-A-I}t^A\ Z_n^{(A,\lambda)}(t;k)dt=\Gamma(A+(kn+1)I)\\&\times\Gamma(B-A)\Gamma^{-1}(B+(kn+1)I)x^B Z_n^{(B,\lambda)}(x;k),\end{aligned}\tag{2.7}$$

where $Re(\mu)>0$ for all $\mu\in\sigma(B-A)$ and $\Gamma(B+(kn+1)I)$ is an invertible matrix.

Proof. Using (1.7) in the left hand side of (2.7), we have the integral matrix functional

$$\int_0^x (x-t)^{B-A-I} t^A \; Z_n^{(A,\lambda)}(t;k)dt = \Gamma(A+(kn+1)I)$$
$$\sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!}\Gamma^{-1}(A+(kr+1)I)\lambda^{kr}\int_0^x (x-t)^{B-A-I}t^{A+krI}dt.$$

To evaluate the integral matrix functional

$$\int_0^x (x-t)^{B-A-I}t^{A+krI}dt,$$

Putting $t = xu$, we have

$$\begin{aligned}&x^{A+B+krI}\int_0^1 u^{A+krI}(1-u)^{B-A-I}dt\\&=x^{B+krI}\Gamma(A+(kr+1)I)\Gamma(B-A)\Gamma^{-1}(B+(kr+1)I).\end{aligned}\tag{2.8}$$

Using (2.8), we have

$$\begin{aligned}&\Gamma(A+(kn+1)I)\Gamma(B-A)\sum_{r=0}^{n}\frac{(-1)^r}{r!(n-r)!}\Gamma^{-1}(B+(kr+1)I)\lambda^{kr}x^{B+krI}\\&=\Gamma(A+(kn+1)I)\Gamma(B-A)\Gamma^{-1}(B+(kn+1)I)x^B \; Z_n^{(B,\lambda)}(x;k).\end{aligned}$$

Thus we obtain (2.7). □

Here we desire the matrix differential recurrence relations for Konhauser matrix polynomials have been obtained using a new technique discussed is novelty, urgently and originality in the following theorems.

Theorem 2.5. *Let A and $A-mI$ be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8). The following matrix differential recurrence formula for Konhauser matrix polynomials holds*

$$\begin{aligned}&\frac{d^m}{dx^m}\left[x^A \; Z_n^{(A,\lambda)}(x;k)\right]=\Gamma(A+(kn+1)I)\\&\times\Gamma^{-1}(A+(kn-m+1)I)x^{A-mI}Z_n^{(A-mI,\lambda)}(x;k),\end{aligned}\tag{2.9}$$

where $\Gamma(A+(kn-m+1)I)$ is an invertible matrix.

Proof. Now we make use of the differential operator D^m defined by

$$D^m x^{A-I} = \Gamma(A)\Gamma^{-1}(A-mI)x^{A-(m+1)I},\tag{2.10}$$

where $\Gamma(A-mI)$ is an invertible matrix.

Multiply both sides of (1.7) by x^A, and apply the differential operator D^m for equation in (2.10), we get

$$\begin{aligned}
&\frac{d^m}{dx^m}\left[x^A Z_n^{(A,\lambda)}(x;k)\right] = \Gamma(A+(kn+1)I)\\
&\times \sum_{r=0}^{n} \frac{(-1)^r \lambda^{rk}}{(n-r)!r!}\Gamma^{-1}(A+(kr+1)I)\frac{d^m}{dx^m}x^{A+krI}\\
&= \Gamma(A+(kn+1)I)\Gamma^{-1}(A+(kn-m+1)I)\Gamma(A+(kn-m+1)I)\\
&\times \sum_{r=0}^{n} \frac{(-1)^r \lambda^{rk}}{r!(n-r)!}\Gamma^{-1}(A+(kr-m+1)I)x^{A+(kr-m)I}\\
&= x^{A-mI}\Gamma(A+(kn+1)I)\Gamma^{-1}(A+(kn-m+1)I)\Gamma(A+(kn-m+1)I)\\
&\times \sum_{r=0}^{n} \frac{(-1)^r (\lambda\, x)^{rk}}{r!(n-r)!}\Gamma^{-1}(A+(kr-m+1)I)\\
&= x^{A-mI}\Gamma(A+(kn+1)I)\Gamma^{-1}(A+(kn-m+1)I)Z_n^{(A-mI,\lambda)}(x;k),
\end{aligned}$$

which immediately leads to (2.9). This completes the proof of the theorem. $\square$

Corollary 2.1. *Konhauser matrix polynomials satisfy the following matrix differential recurrence relation:*

$$\frac{d^k}{dx^k}\left[x^{A+kI}\ Z_n^{(A+kI,\lambda)}(x;k)\right] = (A+(kn+1)I)_k x^A Z_n^{(A,\lambda)}(x;k). \tag{2.11}$$

Proof. Replacing in m by k and A by $A+kI$ in (2.9), we obtain (2.11). $\square$

Theorem 2.6. *For the matrices A and $A+kI$ in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8). The Konhauser matrix polynomials satisfy the following matrix differential recurrence relation:*

$$\left(\frac{d^k}{dx^k}-\lambda^k\right)\left[x^{A+kI}\ Z_n^{(A+kI,\lambda)}(x;k)\right] = (n+1)x^A Z_{n+1}^{(A,\lambda)}(x;k). \tag{2.12}$$

Proof. Rewrite the equation in (2.4) in the following form

$$\begin{aligned}
&n!(\lambda\, x)^A\Gamma^{-1}(A+(kn+1)I)Z_n^{(A,\lambda)}(x;k)\\
&= \frac{1}{2\pi i}\int_C \exp\left(\lambda\, ux\right)(u^k-1)^n u^{-A-(kn+1)I}du.
\end{aligned}$$

By using $(u^k-1)^n = u^k(u^k-1)^n-(u^k-1)^{n+1}$, we have

$$\begin{aligned}
&\frac{\lambda^k}{2\pi i}\int_C \exp\left(\lambda\, ux\right)(u^k-1)^n u^{-A-(kn+1)I}du\\
&= \frac{\lambda^k}{2\pi i}\int_C \exp\left(\lambda\, ux\right)(u^k-1)^n u^k u^{-A-(kn+1)I}du\\
&- \frac{\lambda^k}{2\pi i}\int_C \exp\left(\lambda\, ux\right)(u^k-1)^{n+1} u^{-(A-kI)-(k(n+1)+1)I}du.
\end{aligned}$$

Differentiating both sides with respect to x at k-times, and after some simplification, we get

$$\left(\frac{d^k}{dx^k}-\lambda^k\right)\left[x^{A}\ Z_n^{(A,\lambda)}(x;k)\right] = (n+1)x^{A-kI} Z_{n+1}^{(A-kI,\lambda)}(x;k),$$

which proves (2.12). $\square$

Theorem 2.7. *For the matrices A and $A+kI$ in $\mathbb{C}^{N\times N}$ satisfying the condition in (1.8). The Konhauser matrix polynomials satisfy the following pure recurrences matrix relation*

$$\begin{aligned}(x\lambda)^k Z_n^{(A+kI,\lambda)}(x;k) =&(A+(kn+1)I)_k Z_n^{(A,\lambda)}(x;k)\\ &-(n+1)Z_{n+1}^{(A,\lambda)}(x;k).\end{aligned} \tag{2.13}$$

Proof. From (2.11) and (2.12), we have the pure recurrence matrix relation in (2.13). Hence the proof is established. □

3. Lie operators associated with Konhauser matrix polynomials

In this section, we define some linear partial differential operators in two independent variables x and y. We will investigate their commutative properties while operating on Konhauser matrix polynomials.

With the help of matrix differential recurrence relations given in [26] and [27], we obtain matrix differential recurrence relations for $Z_n^{(A,\lambda)}(x;k)$ as follows

$$x^{1-k}D\, Z_n^{(A,\lambda)}(x;k) = -k\lambda^k\, Z_{n-1}^{(A+kI,\lambda)}(x;k); n\geq 1, \tag{3.1}$$

which gives the equation (4.10) in [27], and

$$\left[x^{-A}D^k x^{A+kI} - x^k\lambda^k I\right] Z_n^{(A,\lambda)}(x;k) = (n+1)\lambda^k Z_{n+1}^{(A-kI,\lambda)}(x;k). \tag{3.2}$$

From (3.1) and (3.2), we get the following matrix differential equation for $Z_n^{(A,\lambda)}(x;k)$

$$\left(\frac{D}{\lambda}\right)^k\left[x^{A+I}DZ_n^{(A,\lambda)}(x;k)\right] - x^A\left(xD-kn\right)Z_n^{(A,\lambda)}(x;k) = \mathbf{0}, \tag{3.3}$$

which is equivalent to

$$\left[x^{1-k}D\left[x^{kI-A}\left(\frac{D}{\lambda}\right)^k - I\right]x^A Z_n^{(A,\lambda)}(x;k)\right] + (n+1)Z_{n+1}^{(A,\lambda)}(x;k) = \mathbf{0}.$$

Replacing n by $y\frac{\partial}{\partial y}$ and D by $\frac{\partial}{\partial x}$, we get the matrix partial differential equation satisfied by $Z_n^{(A,\lambda)}(x,y;k) = y^n Z_n^{(A,\lambda)}(x;k)$ as

$$\begin{aligned}&\mathbb{L}Z_n^{(A,\lambda)}(x,y;k) = \mathbf{0}.\\ &\frac{\partial^k}{\lambda^k\partial x^k}\left[x^{A+I}\frac{\partial}{\partial x}Z_n^{(A,\lambda)}(x;k)\right] - x^A\left(x\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y}\right)Z_n^{(A,\lambda)}(x;k) = \mathbf{0}.\end{aligned} \tag{3.4}$$

First we consider the following linear partial differential operators of the Lie group

$$\begin{aligned}\mathbb{A} =&y\frac{\partial}{\partial y}I,\\ \mathbb{B} =&\frac{x^{1-k}}{y}\frac{\partial}{\partial x}I; y\neq 0,\\ \mathbb{C} =&x^{-A}y\frac{\partial^k}{\lambda^k\partial x^k}x^{A+kI} - yx^kI.\end{aligned} \tag{3.5}$$

Then

$$\lambda^{-A}x^{-A}\mathbb{L} = \mathbb{C}\mathbb{B} + k\mathbb{A}. \tag{3.6}$$

According to the differential operators properties, we have the following rules

$$\begin{aligned}\mathbb{A}\Big[Z_n^{(A,\lambda)}(x;k)\times y^n\Big] =&nZ_n^{(A,\lambda)}(x;k)y^n,\\ \mathbb{B}\Big[Z_n^{(A,\lambda)}(x;k)\times y^n\Big] =&-kZ_{n-1}^{(A+kI,\lambda)}(x;k)y^{n-1};n\geq 1,\\ \mathbb{C}\Big[Z_n^{(A,\lambda)}(x;k)\times y^n\Big] =&(n+1)Z_{n+1}^{(A-kI,\lambda)}(x;k)y^{n+1},\end{aligned} \tag{3.7}$$

where A, $A+kI$ and $A-kI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8), and the commutator relations satisfied by using the differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are

$$\begin{aligned}[\mathbb{A},\mathbb{B}] =&-\mathbb{B},\\ [\mathbb{A},\mathbb{C}] =&\mathbb{C},\\ [\mathbb{B},\mathbb{C}] =&-k\mathbb{I},\end{aligned} \tag{3.8}$$

where $[\mathbb{A},\mathbb{B}]u=(\mathbb{A}\mathbb{B}-\mathbb{B}\mathbb{A})u$ and $\mathbb{I}$ stands for the identity operator.

These commutator relations show that $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ generate a Lie group transformations. We express the extended forms of the transformation group generated by each of the differential operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ as follows:

$$e^{a\mathbb{A}}f^{(A,\lambda)}(x,y)=f^{(A,\lambda)}\Big(x,ye^a\Big), \tag{3.9}$$

in which the differential operator $\mathbb{A}$ is defined as in (3.5) and where $f^{(A,\lambda)}(x,y)$ is an arbitrary matrix function,

$$e^{b\mathbb{B}}\Big[y^nf^{(A,\lambda)}(x)\Big]=y^ne^{b\mathbb{B}}f^{(A,\lambda)}(x)=y^nf^{(A,\lambda)}\bigg(\Big(\frac{kb}{y}+x^k\Big)^{\frac{1}{k}}\bigg);y\neq 0 \tag{3.10}$$

and

$$e^{c\mathbb{C}}\Big[y^nf^{(A,\lambda)}(x)\Big]=y^ne^{-cyx^k}\exp\bigg(cy\Big(\frac{D_x}{\lambda}I+\frac{1}{\lambda x}A\Big)^kx^k\bigg)f^{(A,\lambda)}(x). \tag{3.11}$$

4. **Generating matrix functions cancelled by conjugates of** $(\mathbb{A}-n\mathbb{I})$

In this section, we extend the differential operators $\mathbb{B}$ and $\mathbb{C}$, which we defined in the previous section to the exponential form. Consider an arbitrary matrix polynomials $Z_n^{(A,\lambda)}(x,y;k)=y^nZ_n^{(A,\lambda)}(x;k)$ in two independent variables. Also, we consider the arbitrary constants b and c. The exponential operators $\exp(b\mathbb{B})$ and $\exp(c\mathbb{C})$ are called the extended form of the transformation groups generated by $\mathbb{B}$ and $\mathbb{C}$, respectively.

Here, we show how readily new generating matrix functions for the Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x,y;k)$ can be derived from the operational representations of the Konhauser matrix polynomials.

From this discussion, we see that $Z_n^{(A,\lambda)}(x,y;k)=y^nZ_n^{(A,\lambda)}(x;k)$ is a solution of the following matrix differential equation

$$\mathbb{L}Z_n^{(A,\lambda)}(x,y;k)=\mathbf{0}$$

and

$$\mathbb{A}Z_n^{(A,\lambda)}(x,y;k)=nZ_n^{(A,\lambda)}(x,y;k)$$

for arbitrary n. With the help of (3.10) and (3.11), we get

$$
\begin{aligned}
&e^{c\mathbb{C}}e^{b\mathbb{B}}[Z_n^{(A,\lambda)}(x;k)y^n] = e^{c\mathbb{C}}y^n Z_n^{(A,\lambda)}\left(\left(x^k+\frac{kb}{y}\right)^{\frac{1}{k}};k\right)\\
&= y^n \exp\left(-cy\left(x^k+\frac{kb}{y}\right)\right)\exp\left[cy\left(\frac{D}{\lambda}I+\frac{1}{\lambda}\left(x^k+\frac{kb}{y}\right)^{-\frac{1}{k}}A\right)^k\right. \qquad (4.1)\\
&\left.\times\left(x^k+\frac{kb}{y}\right)\right]Z_n^{(A,\lambda)}\left(\left(x^k+\frac{kb}{y}\right)^{\frac{1}{k}};k\right) = F(x,y,A); y\neq 0.
\end{aligned}
$$

Put $\mathbb{S} = e^{b\mathbb{B}+c\mathbb{C}}$ then $\mathbb{S}\mathbb{A}\mathbb{S}^{-1}$ is conjugate of $\mathbb{A}$ and $F(x,y,A)$ is cancelled by $\mathbb{L}$ and $\mathbb{S}(\mathbb{A}-n\mathbb{I})\mathbb{S}^{-1}$.

Now we consider the following cases :

Case 1. Putting $c=0$ and $b=1$, then (4.1) reduces to

$$
e^{\mathbb{B}}[Z_n^{(A,\lambda)}(x;k)y^n] = y^n Z_n^{(A,\lambda)}\left(\left(x^k+\frac{k}{y}\right)^{\frac{1}{k}};k\right); y\neq 0. \qquad (4.2)
$$

Separately, we consider the left hand side of (4.2) and we write exponential operators in a series form so that we have the following relation

$$
\begin{aligned}
&e^{\mathbb{B}}[Z_n^{(A,\lambda)}(x;k)y^n] = \sum_{m=0}^{\infty}\frac{\mathbb{B}^m}{m!}[Z_n^{(A,\lambda)}(x;k)y^n]\\
&= \sum_{m=0}^{\infty}\frac{\mathbb{B}^{m-1}}{m!}(-k)Z_{n-1}^{(A+kI,\lambda)}(x;k)y^{n-1}\\
&\ldots\\
&\ldots\\
&\ldots\\
&= \sum_{m=0}^{\infty}\frac{\mathbb{B}^{m-m}}{m!}\Big[(-k)\times(-k)\times(-k)\ldots\times(-k)\Big]Z_{n-m}^{(A+mkI,\lambda)}(x;k)y^{n-m}\\
&= y^n\sum_{m=0}^{n}\frac{1}{m!}\left(-\frac{k}{y}\right)^m Z_{n-m}^{(A+kmI,\lambda)}(x;k); y\neq 0,
\end{aligned}
$$

where A and $A+mkI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8).

Hence, we obtain

$$
e^{\mathbb{B}}[Z_n^{(A,\lambda)}(x;k)y^n] = y^n\sum_{m=0}^{n}\frac{1}{m!}\left(-\frac{k}{y}\right)^m Z_{n-m}^{(A+kmI,\lambda)}(x;k); Z_{-m}^{(A,\lambda)}(x;k) = \mathbf{0}. \qquad (4.3)
$$

Equating the two values and after minor adjustments, we get a generating matrix relation as

$$
Z_n^{(A,\lambda)}\left((x^k+kt)^{\frac{1}{k}};k\right) = \sum_{m=0}^{\infty}\frac{k^m}{m!}(-t)^m Z_{n-m}^{(A+kmI,\lambda)}(x;k), \qquad (4.4)
$$

where $t=\frac{1}{y}$ and $y\neq 0$.

If we put $k=1$, the $Z_n^{(A,\lambda)}(x;k)$ reduces to the Laguerre matrix polynomials, $L_n^{(A,\lambda)}\big(x+t\big)$. Thus putting $k=1$ in (4.4), we get the following formula on Laguerre matrix polynomials:

$$
L_n^{(A,\lambda)}\big(x+t\big) = \sum_{m=0}^{\infty}\frac{1}{m!}(-t)^m L_{n-m}^{(A+mI,\lambda)}(x), \qquad (4.5)
$$

where A and $A+mI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.5).

Case 2. Putting $b=0$ and $c=1$, then (4.1) reduces to

$$e^{\mathbb{C}}[Z_n^{(A,\lambda)}(x;k)y^n] = y^n e^{-yx^k}\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda x}A\right)^k x^k\right]Z_n^{(A,\lambda)}(x;k). \tag{4.6}$$

But we have

$$\begin{aligned} e^{\mathbb{C}}[Z_n^{(A,\lambda)}(x;k)y^n] &= \sum_{m=0}^{\infty}\frac{\mathbb{C}^m}{m!}[Z_n^{(A,\lambda)}(x;k)y^n]\\ &= \sum_{m=0}^{\infty}\frac{\mathbb{C}^{m-1}}{m!}(n+1)Z_{n+1}^{(A-kI,\lambda)}(x;k)y^{n+1}\\ &\dots\\ &\dots\\ &\dots\\ &= \sum_{m=0}^{\infty}\frac{\mathbb{C}^{m-m}}{m!}\Big[(n+1)\times(n+2)\times\ldots\times(n+m)\Big]Z_{n+m}^{(A-mkI,\lambda)}(x;k)y^{n+m}\\ &= y^n\sum_{m=0}^{\infty}\frac{y^m}{m!}(n+1)_m Z_{n+m}^{(A-mkI,\lambda)}(x;k), \end{aligned} \tag{4.7}$$

where A and $A-mkI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8).

Equating both values and after minor adjustments, we get a generating matrix relation as

$$\begin{aligned} &\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda x}A\right)^k x^k - x^kI)\right]Z_n^{(A,\lambda)}(x;k)\\ &= \sum_{m=0}^{\infty}\frac{y^m}{m!}(n+1)_m Z_{n+m}^{(A-mkI,\lambda)}(x;k). \end{aligned} \tag{4.8}$$

For $k=1$ in (4.8), we give the generating matrix relation on Laguerre matrix polynomials:

$$\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda x}A\right)x - xI)\right]L_n^{(A,\lambda)}(x) = \sum_{m=0}^{\infty}\frac{y^m}{m!}(n+1)_m L_{n+m}^{(A-mI,\lambda)}(x). \tag{4.9}$$

Simplifying more, we have

$$\exp\left(y\frac{D}{\lambda}xI+\frac{y}{\lambda}A-xyI)\right)L_n^{(A,\lambda)}(x) = e^{\frac{y}{\lambda}(A+I)}\left[e^{-xye^y}L_n^{(A,\lambda)}(xe^y)\right]. \tag{4.10}$$

Using $e^{tDx}f(x)=e^t f(xe^t)$, we get

$$e^{\frac{y}{\lambda}(A+I)}\left[e^{-xye^y}L_n^{(A,\lambda)}(xe^y)\right] = \sum_{m=0}^{\infty}\frac{y^m}{m!}(n+1)_m L_{n+m}^{(A-mI,\lambda)}(x), \tag{4.11}$$

where A and $A-mI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.5).

Further, we proceed to determine $e^{b\mathbb{B}}e^{\mathbb{C}}$, where b is an arbitrary constant.

Case 3. If we substitute $c=1$ and $b\neq 0$ in (4.1), then we obtain

$$\begin{aligned} &e^{b\mathbb{B}}e^{\mathbb{C}}[Z_n^{(A,\lambda)}(x;k)y^n] = y^n e^{-y}\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda}\left(x^k+\frac{kb}{y}\right)^{-\frac{1}{k}}A\right)^k\right.\\ &\left.\times\left(x^k+\frac{kb}{y}\right)\right]Z_n^{(A,\lambda)}\left((x^k+\frac{kb}{y})^{\frac{1}{k}};k\right); y\neq 0. \end{aligned} \tag{4.12}$$

Also we get

$$
\begin{aligned}
&e^{b\mathbb{B}}e^{\mathbb{C}}[Z_n^{(A,\lambda)}(x;k)y^n] = e^{\mathbb{C}}e^{b\mathbb{B}}[Z_n^{(A,\lambda)}(x;k)y^n]\\
&= e^{\mathbb{C}}\sum_{m=0}^{\infty}\frac{(-kb)^m}{m!}Z_{n-m}^{(A+mkI,\lambda)}(x;k)y^{n-m}\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!s!}(n-m+1)_s(-kb)^m Z_{n-m+s}^{(A+(m-s)kI,\lambda)}(x;k)y^{n-m+s}.
\end{aligned}
$$

On the other hand, equating both values and after minor adjustments, we get

$$
\begin{aligned}
&e^{-y}\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda}\left(x^k+\frac{kb}{y}\right)^{-\frac{1}{k}}A\right)^k\left(x^k+\frac{kb}{y}\right)\right]Z_n^{(A,\lambda)}\left(\left(x^k+\frac{kb}{y}\right)^{\frac{1}{k}};k\right)\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!s!}(n-m+1)_s(-kb)^m Z_{n-m+s}^{(A+(m-s)kI,\lambda)}(x;k)y^{s-m}; y\neq 0,
\end{aligned}
\tag{4.13}
$$

where A and $A+(m-s)kI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8).

In particular for $k=1$ in (4.13), we get

$$
\begin{aligned}
&e^{-y}\exp\left[y\left(\frac{D}{\lambda}I+\frac{1}{\lambda}\left(x+\frac{b}{y}\right)^{-1}A\right)\left(x+\frac{b}{y}\right)\right]L_n^{(A,\lambda)}\left((x+\frac{b}{y})\right)\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!s!}(n-m+1)_s(-b)^m L_{n-m+s}^{(A+(m-s)I,\lambda)}(x)y^{s-m}; y\neq 0.
\end{aligned}
\tag{4.14}
$$

After simplifications, we give

$$
\begin{aligned}
&e^{-y}e^{\frac{yD}{\lambda}(x+\frac{b}{y})}\exp\left[\frac{y}{\lambda}\left(x+\frac{b}{y}\right)^2A-y\left(x+\frac{b}{y}\right)I\right]L_n^{(A,\lambda)}\left((x+\frac{b}{y})\right)\\
&= e^{-y}e^{\frac{y}{\lambda}}\exp\left[\frac{y}{\lambda}\left(x+\frac{b}{y}\right)^2e^{2y}A-ye^y\left(x+\frac{b}{y}\right)I\right]L_n^{(A,\lambda)}\left((x+\frac{b}{y})e^y\right)\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m!s!}(n-m+1)_s(-b)^m L_{n-m+s}^{(A+(m-s)I,\lambda)}(x)y^{s-m}; y\neq 0,
\end{aligned}
\tag{4.15}
$$

where A and $A+(m-s)I$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.5).

5. Some more generating matrix relations for Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$

As an application of our results, we give some more recurrence matrix relations for Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$

$$
\left(x\frac{\partial}{\partial x}I+A\right)Z_n^{(A,\lambda)}(x;k)=(A+knI)Z_n^{(A-I,\lambda)}(x;k),
\tag{5.1}
$$

where A and $A-I$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8) (see eq. (3.20) [26]).

Let us consider the differential operator (infinite small generator the Lie group)

$$
\mathbb{B}_1=\frac{x}{y}\frac{\partial}{\partial x}I+\frac{1}{y}A; y\neq 0.
\tag{5.2}
$$

Then we observe

$$
\mathbb{B}_1\left[Z_n^{(A,\lambda)}(x;k)\times y^A\right]=(A+knI)Z_{n-1}^{(A-I,\lambda)}(x;k)y^{A-I}; n\geq 1.
\tag{5.3}
$$

Using the same method as used in (4.2), we get the extended form of the transformation Lie group generated by $\mathbb{B}_1$ as

$$e^{b\mathbb{B}_1}\left[Z_n^{(A,\lambda)}(x;k)\times y^A\right] = y^A e^{\frac{bA}{y}} Z_n^{(A,\lambda)}(xe^{\frac{b}{y}};k). \tag{5.4}$$

Also, using (5.3) and (5.4), we get

$$\begin{aligned} e^{b\mathbb{B}_1}\left[Z_n^{(A,\lambda)}(x;k)\times y^A\right] =& y^A\sum_{m=0}^{\infty}\frac{1}{m!}(A+(kn-m+1)I)_m \\ \times\left(\frac{b}{y}\right)^m Z_n^{(A-mI,\lambda)}(x;k);\ y\neq 0, \end{aligned} \tag{5.5}$$

where A and $A-mI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8).

Equating both values of $e^{b\mathbb{B}_1}\left[Z_n^{(A,\lambda)}(x;k)\times y^A\right]$ and making appropriate adjustments, we get

$$e^{At}Z_n^{(A,\lambda)}(xe^t;k) = \sum_{m=0}^{\infty}\frac{1}{m!}(A+(kn-m+1)I)_m t^m Z_n^{(A-mI,\lambda)}(x;k), \tag{5.6}$$

where A and $A-mI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.8) and $t=\frac{b}{y}$ and $y\neq 0$.

For $k=1$ in (5.6), we give

$$e^{At}L_n^{(A,\lambda)}(xe^t) = \sum_{m=0}^{\infty}\frac{1}{m!}(A+(n-m+1)I)_m t^m L_n^{(A-mI,\lambda)}(x), \tag{5.7}$$

where A and $A-mI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfy the condition (1.5).

For $Z_n^{(A,\lambda)}(x;k)$, the recurrence matrix relation is given by [27]

$$\left(kn-xD\right)Z_n^{(A,\lambda)}(x;k) = k(A+(kn-k+1)I)_k Z_{n-1}^{(A,\lambda)}(x;k); n\geq 1. \tag{5.8}$$

Let us define the differential operator $\mathbb{B}_2$

$$\mathbb{B}_2 = k\frac{\partial}{\partial y}-\frac{x}{y}\frac{\partial}{\partial x}; y\neq 0, \tag{5.9}$$

Then we have

$$\mathbb{B}_2\left[Z_n^{(A,\lambda)}(x;k)\times y^n\right] = k(A+(kn-k+1)I)Z_{n-1}^{(A,\lambda)}(x;k)y^{n-1}; n\geq 1. \tag{5.10}$$

The extended form of the transformation group generated by the infinite generator $\mathbb{B}_2$ is given as

$$e^{b\mathbb{B}_2}\left[Z_n^{(A,\lambda)}(x;k)\times y^n\right] = (bk+y)^n Z_n^{(A,\lambda)}\left(\frac{xy^{\frac{1}{k}}}{(bk+y)^{\frac{1}{k}}};k\right); \left|\frac{y}{bk}\right|<1. \tag{5.11}$$

It has been obtained by using the similar method as

$$\begin{aligned} e^{b\mathbb{B}_2}\left[Z_n^{(A,\lambda)}(x;k)\times y^n\right] &= \sum_{m=0}^{\infty}\frac{1}{m!}b^m\mathbb{B}_2^m\left[Z_n^{(A,\lambda)}(x;k)\times y^n\right] \\ &= y^n\sum_{m=0}^{\infty}\frac{1}{m!}(A+(kn-km+1)I)_{mk}\left(\frac{bk}{y}\right)^m Z_{n-m}^{(A,\lambda)}(x;k); y\neq 0. \end{aligned} \tag{5.12}$$

Hence equating both values of $e^{b\mathbb{B}_2}\left[Z_n^{(A,\lambda)}(x;k)\times y^n\right]$ in (5.11) and (5.12), and making appropriate adjustments, we get

$$\begin{aligned}(1+t)^n Z_n^{(A,\lambda)}\left(\frac{x}{(1+t)^{\frac{1}{k}}};k\right) = \sum_{m=0}^{\infty}\frac{t^m}{m!}(A+(kn-km+1)I)_{mk}\\ \times Z_{n-m}^{(A,\lambda)}(x;k); |t|<1,\end{aligned} \tag{5.13}$$

where $t=\frac{bk}{y}$ and $y\neq 0$.

For $k=1$ in (5.13), we give

$$(1+t)^n L_n^{(A,\lambda)}\left(\frac{x}{1+t}\right) = \sum_{m=0}^{\infty}\frac{t^m}{m!}(A+(n-m+1)I)_m L_{n-m}^{(A,\lambda)}(x); |t|<1. \tag{5.14}$$

6. Conclusion

In this paper a new approach has been introduced for studying some important properties of Konhauser matrix polynomials $Z_n^{(A,\lambda)}(x;k)$ viz matrix recurrence relations, matrix differential recurrence relations, matrix differential equation and certain generating matrix relations. The method developed can also be used to study some other Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k)$ which play vital role in Mathematical Physics, Chemistry and Mechanics. In a forthcoming paper, we propose to extend the present investigation to Konhauser matrix polynomials $Y_n^{(A,\lambda)}(x;k)$ and to the biorthogonal matrix polynomials with a view to showing how their theories can be developed within a unifying framework.

Acknowledgements

The author is grateful to the anonymous referees for useful comments and suggestions towards the improvement of this paper.

REFERENCES

[1] *R. Agarwal and S. Jain*, Certain properties of some special matrix functions via Lie algebra, Internat. Bull. Math. Res., **2** (2015), No. 1, 9-15.

[2] *A. K. Chongdar*, Group theoretic study of certain generating functions, Bull. Calcutta Math. Soc., **77** (1985), No. 77, 151-157.

[3] *A. K. Chongdar*, A Lie algebra and generating functions of modified Laguerre polynomials, Bull. Calcutta Math. Soc., **78** (1986), 219-226.

[4] *A. K. Chongdar*, A note on the derivation of some mixed trilateral generating relations of modified Laguerre polynomials by group theoretic method, J. Orissa Math. Soc., **6, No. 6** (1987), 55-59.

[5] *A. K. Chongdar and B. K. Guhathakurta*, Group theoretic study of certain generating functions Hypergeometric polynomials, Tamkang J. Math., **16, No. 3**, (1985), 1-10.

[6] *N. Dunford and J. T. Schwartz*, Linear Operators, part I, General Theory, Interscience Publishers, INC. New York, 1957.

[7] *E. Erkuş-Duman and B. Çekim*, New generating functions for Konhauser matrix polynomials, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., **63, No. 1** (2014), 35-41.

[8] *L. Jódar, R. Company and E. Navarro*, Laguerre matrix polynomials and systems of second order differential equations, Appl. Math. Lett., **15** (1994), 53-63.

[9] *L. Jódar and J. C. Cortés*, Some properties of gamma and beta matrix functions, Appl. Math. Lett., **11** (1998), 89-93.

[10] *L. Jódar and J. C. Cortés*, On the hypergeometric matrix function, J. Comput. Appl. Math., **99** (1998), 205-217.

[11] *S. Khan and N. A. M. Hassan*, 2-Variable Laguerre matrix polynomials and Lie algebraic techniques, J. Phys. A: Math. Theor, **43** (2010), No. 23, 235204 (21pp).

[12] *S. Khan and M. A. Pathan,* Lie theoretic generating relations of Hermite $2D$ polynomials, J. Comput. Appl. Math., **160** (2003), 139-146.

[13] *S. Khan, M. A. Pathan and G. Yasmin*, Representation of a Lie algebra $G(0,1)$ and three variables generalized Hermite polynomials, $H_n(x,y,z)$, Integral Transforms Spec. Funct., **13** (2002), 59-64.

[14] *S. Khan and N. Raza*, 2-Variable generalized Hermite matrix polynomials and Lie algebra representation, Rep. Math. Phys., **66** (2010), No. 2, 159-174.

[15] *S. Khan and G. Yasmin*, Lie theoretic generating relations of two variables Laguerre polynomials, Rep. Math. Phys., **51** (2003), 1-7.

[16] *J. D. E. Konhauser*, Some properties of biorthogonal polynomials, J. Math. Anal. Appl., **11** (1965), 242-260.

[17] *J. D. E. Konhauser*, Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math., **21** (1967), 303-314.

[18] *N. N. Lebedev*, Special Functions and Their Applications, Dover Publications Inc., New York, 1972.

[19] *E. D. McBride*, Obtaining Generating Functions, Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[20] *W. Jr. Miller*, Lie Theory and Special Functions, Academic Press, New York and London, 1968.

[21] *N. K. Rana*, Lie theoretic origin of some generating functions for the Laguerre polynomials, Bull. Calcutta Math. Soc., **94** (2002), 291-296.

[22] *M. J. S. Shahwan and M. A. Pathan*, Origin of certain generating relations of Hermite matrix functions from the view point of Lie algebra, Integral Transforms Spec. Funct., **17** (2006), No. 10, 743-747.

[23] *M. J. S. Shahwan and M. A. Pathan*, Generating relations of Hermite matrix polynomials by Lie algebraic method. Ital. J. Pure Appl. Math., **25** (2009), 187-192.

[24] *A. Shehata*, Some relations on Laguerre matrix polynomials, Malays. J. Math. Sci., **9** (2015), No. 3, 443-462.

[25] *A. Shehata*, On modified Laguerre matrix polynomials, J. Nat. Sci. Math., **8** (2015), No. 2, 153-166.

[26] *A. Shehata*, Some relations on Konhauser matrix polynomials, *Miskolc Math. Notes*, **17** (2016), No. 1, 605-633.

[27] *S. Varma, B. Çekim and F. Taşdelen*, Konhauser matrix polynomials, Ars Combin., **100** (2011), 193-204.

[28] *S. Varma and F. Taşdelen*, Some properties of Konhauser matrix polynomials, Gazi Univ. J. Sci., **29** (2016), No. 3, 703-709.

[29] *L. Weisner*, Group theoretic origin of certain generating functions, Pacific J. Math., **5** (1955), 1033-1039.