CONFORMAL ANTI-ININVARIANT Riemannian Maps to Kähler Manifolds

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We introduce conformal anti-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds and show that they include both anti-invariant submanifolds and anti-invariant Riemannian maps. We give non-trivial examples, investigate the geometry of certain distributions and obtain decomposition theorems for the base manifold. The harmonicity and totally geodesicity of conformal anti-invariant Riemannian maps are also obtained. Moreover, we study weakly umbilical conformal Riemannian maps and obtain a classification theorem for umbilical conformal anti-invariant Riemannian maps.

Keywords: Anti-invariant submanifold, anti-invariant Riemannian map, conformal anti-invariant Riemannian map, totally geodesic map, harmonic map.

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1. Introduction

Let \((M, J_M)\) be an almost complex manifold with almost complex structure \(J_M\). A totally real submanifold (anti-invariant submanifold) \(M\) is a submanifold such that the almost complex structure \(J_M\) of the ambient manifold \(M\) carries a tangent space of \(M\) into the corresponding normal space of \(M\). A totally real submanifold is called Lagrangian if \(\dim \mathbb{R}M = \dim \mathbb{C}M\). Real curves of Kähler manifolds are examples of totally real submanifolds. The first contribution to the geometry of totally real submanifolds was given in the early 1970's [3]. For details, see [13].

As a generalization of isometric immersions and Riemannian submersions, Riemannian maps were introduced in [4] as follows. Let \(F : (M, g_M) \rightarrow (N, g_N)\) be a smooth map between Riemannian manifolds such that \(0 < \text{rank} F < \min \{m, n\}\), where \(\dim M = m\) and \(\dim N = n\). Then we denote the kernel space of \(F\) by \(\ker F\), and consider the orthogonal complementary space \(\mathcal{H} = (\ker F)^{\perp}\) to \(\ker F\) in \(TM\). Then the tangent bundle of \(M\) has the following decomposition

\[
TM = \ker F \oplus \mathcal{H}.
\]

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We denote the range of \( F \) by \( \text{range} F \) and consider the orthogonal complementary space \((\text{range} F)^\perp\) to \(\text{range} F\) in the tangent bundle \(TN\) of \(N\). Since \(\text{rank} F < \min\{m, n\}\), we always have \((\text{range} F)^\perp\). Thus the tangent bundle \(TN\) of \(N\) has the following decomposition
\[
F^*(TN) = (\text{range} F) \oplus (\text{range} F)^\perp.
\]
Now, a smooth map \( F : (M^m, g_M) \rightarrow (M^n, g_N) \) is called Riemannian map at \(p_1 \in M\) if the horizontal restriction \( F_{\text{h}}^{p_1} : (\ker F_{p_1})^\perp \rightarrow (\text{range} F_{p_1}) \) is a linear isometry between the inner product spaces \(((\ker F_{p_1})^\perp, g_M(p_1) |_{(\ker F_{p_1})^\perp})\) and \((\text{range} F_{p_1}, g_N(p_2) |_{(\text{range} F_{p_1})})\), \(p_2 = F(p_1)\). Thus \( F \) satisfies the equation
\[
g_N(F_*\hat{X}, F_*\hat{Y}) = g_M(\hat{X}, \hat{Y}) \tag{1}
\]
for \(\hat{X}, \hat{Y}\) vector fields tangent to \(H\). Indeed, it follows that isometric immersions and Riemannian submersions are particular Riemannian maps with \(\ker F = \{0\}\) and \((\text{range} F)^\perp = \{0\}\). It is known that a Riemannian map is a submersion \([4]\) and this fact implies that the rank of the linear map \( F_{\text{p}} : T_pM \rightarrow T_{F(p)}N \) is constant for \(p\) in each connected component of \(M\), \([1]\) and \([4]\). It is also important to note that Riemannian maps satisfy the eikonal equation. Different properties of Riemannian maps have been studied widely by many authors, see: \([5]\), \([6]\), \([8]\), and \([9]\). Recently, conformal Riemannian maps as a generalization of Riemannian maps have been defined in \([12]\) and the harmonicity of such maps have been also obtained.

On the other hand, as a generalization of totally real submanifolds, anti-invariant Riemannian maps from Riemannian manifolds to almost complex manifolds were defined and studied in \([11]\). In this paper, we are going to introduce and study conformal anti-invariant Riemannian maps from Riemannian manifolds to almost complex manifolds as a generalization of totally real submanifolds and anti-invariant Riemannian maps.

2. Preliminaries

In this section, we recall some basic materials from \([2, 14]\). A \(2n\)-dimensional Riemannian manifold \((M, g, J)\) is called an almost Hermitian manifold if there exists a tensor field \(J\) of type \((1, 1)\) on \(M\) such that \(J^2 = -I\) and
\[
g(\hat{X}, \hat{Y}) = g(J\hat{X}, J\hat{Y}), \quad \forall \hat{X}, \hat{Y} \in \Gamma(TM), \tag{2}
\]
where \(I\) denotes the identity transformation of \(T_pM\). Consider an almost Hermitian manifold \((M, g, J)\) and denote by \(\nabla\) the Levi-Civita connection on \(M\) with respect to \(g\). Then \(M\) is called a Kähler manifold \([14]\) if \(J\) is parallel with respect to \(\nabla\), i.e.
\[
(\nabla_{\hat{X}}J)\hat{Y} = 0, \tag{3}
\]
\(\forall \hat{X}, \hat{Y} \in \Gamma(TM)\).

We now recall the notion of harmonic maps between Riemannian manifolds. Let \((M, g_M)\) and \((N, g_N)\) be Riemannian manifolds and suppose that \(\varphi : M \rightarrow N\) is
a smooth map between them. Then the second fundamental form of \( \varphi \) is given by

\[
(\nabla \varphi_\ast)(\tilde{X}, \tilde{Y}) = \nabla^\ast_X \varphi_\ast(\tilde{Y}) - \varphi_\ast(\nabla^M_X \tilde{Y})
\]

(4)

for \( \tilde{X}, \tilde{Y} \in \Gamma(TM) \), where \( \nabla^\varphi \) is the pullback connection. It is known that the second fundamental form is symmetric. The tension field of \( \varphi \) is the section \( \tau(\varphi) \) of the pullback bundle \( \Gamma(\varphi^{-1}TN) \) defined by \( \tau(\varphi) = \text{div} \varphi_\ast = \sum_{i=1}^m (\nabla \varphi_\ast)(e_i, e_i) \), where \( \{e_1, \ldots, e_m\} \) is the orthonormal frame on \( M \). A smooth map \( \varphi \) satisfying \( \tau(\varphi) = 0 \) is called a harmonic map, see [2].

We denote by \( \nabla^2 \) both the Levi-Civita connection of \( (N, g_N) \) and its pullback along \( F \). Then according to [7], for any vector field \( \tilde{X} \) on \( M \) and any section \( V \) of \( (\text{range}F_\ast)\perp \), where \( (\text{range}F_\ast)\perp \) is the subbundle of \( F^{-1}(TN) \) with fiber \( (F_\ast(T_pM))\perp \) — orthogonal projection of \( F_\ast(T_pM) \) for \( g_N \) over \( p \), we have \( \nabla^2 \tilde{X} V \) which is the orthogonal projection of \( \nabla^2_X V \) on \( (F_\ast(T_pM))\perp \) — such that \( \nabla^2 \tilde{X} \tilde{Y} \). We now define \( A_V \) as

\[
\nabla^2_X V = -A_V F_\ast \tilde{X} + \nabla^2 \tilde{X} V
\]

(5)

where \( A_V F_\ast \tilde{X} \) is tangential component (a vector field along \( F \)) of \( \nabla^2 \tilde{X} V \). It is easy to see that \( A_V F_\ast \tilde{X} \) is bilinear in \( V \) and \( F_\ast \) and \( A_V F_\ast \tilde{X} \) at \( p \) depends only on \( V_p \) and \( F_{sp} \tilde{X}_p \). By direct computations, we obtain

\[
g_2(A_V F_\ast \tilde{X}, F_\ast \tilde{Y}) = g_2(V, (\nabla F_\ast)(\tilde{X}, \tilde{Y}))
\]

(6)

for \( \tilde{X}, \tilde{Y} \in \Gamma((\ker F_\ast)\perp) \) and \( V \in \Gamma((\text{range}F_\ast)\perp) \). Since \( (\nabla F_\ast) \) is symmetric, it follows that \( A_V \) is a symmetric linear transformation of \( \text{range}F_\ast \).

3. Conformal anti-invariant Riemannian maps

In this section, we define and study conformal anti-invariant Riemannian maps, give examples, investigate the geometry of leaves of the distributions which are defined on the target manifolds. We also give a decomposition theorem and obtain necessary and sufficient conditions for such conformal Riemannian maps to be totally geodesic. We first recall that, in [12], the second author of the present paper showed that the second fundamental form \( (\nabla F_\ast)(\tilde{X}, \tilde{Y}), \forall \tilde{X}, \tilde{Y} \in \Gamma((\ker F_\ast)\perp) \), of a conformal Riemannian map is in the following form

\[
(\nabla F_\ast)(\tilde{X}, \tilde{Y})^{\text{range}F_\ast} = \tilde{X}(\ln \lambda) F_\ast \tilde{Y} + \tilde{Y}(\ln \lambda) F_\ast \tilde{X} - g_1(\tilde{X}, \tilde{Y}) F_\ast(\text{grad} \ln \lambda).
\]

(7)

Thus if we denote the \( (\text{range}F_\ast)\perp \) — component of \( (\nabla F_\ast)(\tilde{X}, \tilde{Y}) \) by \( (\nabla F_\ast)(\tilde{X}, \tilde{Y})^{\text{range}F_\ast} \), we can write \( (\nabla F_\ast)(\tilde{X}, \tilde{Y}) \) as

\[
(\nabla F_\ast)(\tilde{X}, \tilde{Y}) = (\nabla F_\ast)(\tilde{X}, \tilde{Y})^{\text{range}F_\ast} + (\nabla F_\ast)(\tilde{X}, \tilde{Y})^{(\text{range}F_\ast)\perp},
\]

(8)

for \( \tilde{X}, \tilde{Y} \in \Gamma((\ker F_\ast)\perp) \). Hence we have

\[
(\nabla F_\ast)(\tilde{X}, \tilde{Y}) = \tilde{X}(\ln \lambda) F_\ast \tilde{Y} + \tilde{Y}(\ln \lambda) F_\ast \tilde{X} - g_1(\tilde{X}, \tilde{Y}) F_\ast(\text{grad} \ln \lambda)
\]

(9)
We now present the following definition for conformal anti-invariant Riemannian maps as a generalization of totally real submanifolds and anti-invariant Riemannian maps.

**Definition 3.1.** Let $F$ be a conformal Riemannian map from a Riemannian manifold $(M_1, g_1)$ to an almost Hermitian manifold $(M_2, g_2, J)$. Then we say that $F$ is a conformal anti-invariant Riemannian map at $p \in M$ if $J(\text{range } F_p) \subseteq (\text{range } F_p)^\perp$. If $F$ is a conformal anti-invariant Riemannian map for any $p \in M$, then $F$ is called a conformal anti-invariant Riemannian map.

We are going to give some examples of conformal anti-invariant Riemannian maps.

**Example 3.1.** [13] Every anti-invariant submanifold of an almost Hermitian manifold is a conformal anti-invariant Riemannian map with $\lambda = 1$ and $\ker F = \{0\}$.

**Example 3.2.** [11] Every anti-invariant Riemannian map from a Riemannian manifold to an almost Hermitian manifold is a conformal anti-invariant Riemannian map with $\lambda = 1$.

We say that a conformal anti-invariant Riemannian map is proper if $\lambda \neq I$. We now present an example of a proper conformal anti-invariant Riemannian map.

In the following $\mathbb{R}^{2m}$ denotes the Euclidean $2m$-space with the standard metric. An almost complex structure $J$ on $\mathbb{R}^{2m}$ is said to be compatible if $(\mathbb{R}^{2m}, J)$ is complex analytically isometric to the complex number space $\mathbb{C}^m$ with the standard Kählerian metric. We denote by $J$ the compatible almost complex structure on $\mathbb{R}^{2m}$ defined by

$$J(\overline{u}_1, \ldots, \overline{u}_{2m}) = (-\overline{u}_2, \overline{u}_1, \ldots, -\overline{u}_{2m}, \overline{u}_{2m-1}).$$

**Example 3.3.** Consider the following map defined by

$$F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

$$(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4) \mapsto (e^{x_1} \sin \mathcal{T}_2, 0, e^{x_1} \cos \mathcal{T}_2, 0).$$

We have

$$\ker F = \text{span}\{Z_1 = \partial \mathcal{T}_3, Z_2 = \partial \mathcal{T}_4\}$$

and

$$(\ker F)^\perp = \text{span}\{H_1 = e^{x_1} \sin \mathcal{T}_2 \partial \mathcal{T}_1 + e^{x_1} \cos \mathcal{T}_2 \partial \mathcal{T}_2, H_2 = e^{x_1} \cos \mathcal{T}_2 \partial \mathcal{T}_1 - e^{x_1} \sin \mathcal{T}_2 \partial \mathcal{T}_2\}.$$
which show that $F$ is a conformal Riemannian map with $\lambda = e^{2\tau_1}$. Moreover, it is easy to see that $JF,H_1 = e^{2\tau_1} \frac{\partial}{\partial y_2}$ and $JF,H_2 = e^{2\tau_1} \frac{\partial}{\partial y_4}$, where $J$ is the canonical complex structure of $\mathbb{R}^4$ defined by

$$J(y_1,y_2,y_3,y_4) = (-y_2,y_1,-y_3,y_4).$$

As a result, $F$ is a conformal anti-invariant Riemannian map.

Let $F$ be a conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1,g_1)$ to an almost Hermitian manifold $(M_2,g_2,J)$. First of all, from Definition 3.1, we have $J(rangeF_\ast) \cap (rangeF_\ast) \perp \neq \{0\}$. We denote the complementary orthogonal distribution to $J(rangeF_\ast)$ in $(rangeF_\ast) \perp$ by $\mu$. Then we have

$$(rangeF_\ast) \perp = J(rangeF_\ast) \oplus \mu. \tag{10}$$

It is easy to see that $\mu$ is an invariant distribution of $(rangeF_\ast) \perp$, under the endomorphism $J_2$. Thus, for $V \in \Gamma((rangeF_\ast) \perp)$, we have

$$JV = BV + CV \tag{11}$$

where $BV \in \Gamma(rangeF_\ast)$ and $CV \in \Gamma((rangeF_\ast) \perp)$.

We now investigate the geometry of the leaves of $(rangeF_\ast)$ and $(rangeF_\ast) \perp$. First, we give the following result.

**Theorem 3.1.** Let $F$ be a conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1,g_1)$ to a Kähler manifold $(M_2,g_2,J)$. Then $(rangeF_\ast)$ defines a totally geodesic foliation on $M_2$ if and only if

$$g_2((\nabla F_\ast)(\tilde{X},\tilde{Y})^{(rangeF_\ast) \perp},JF_\ast\tilde{Y}) = g_2(\nabla_{\tilde{X}}^F JF_\ast\tilde{Y},CW) \tag{12}$$

for any $W \in \Gamma((rangeF_\ast) \perp)$ and $\tilde{X},\tilde{Y},\tilde{Y}' \in \Gamma((kerF_\ast) \perp)$, such that $F_\ast\tilde{Y}' = BV$.

**Proof.** For $\tilde{X},\tilde{Y} \in \Gamma((kerF_\ast) \perp)$ and $W \in \Gamma((rangeF_\ast) \perp)$, using (2) we have

$$g_2(\nabla_{\tilde{X}}^2 F_\ast\tilde{Y},W) = g_2(\nabla_{\tilde{X}}^2 JF_\ast\tilde{Y},JW).$$

Thus from (11) we obtain

$$g_2(\nabla_{\tilde{X}}^2 F_\ast\tilde{Y},W) = -g_2(\nabla_{\tilde{X}}^2 F_\ast\tilde{Y}',JF_\ast\tilde{Y}) + g_2(\nabla_{\tilde{X}}^2 JF_\ast\tilde{Y},CW),$$

where $F_\ast\tilde{Y}' = BV$ for $\tilde{Y}' \in \Gamma((kerF_\ast) \perp)$. Since $F$ is a conformal Riemannian map, using (4), (5) and (8) we obtain

$$g_2(\nabla_{\tilde{X}}^2 F_\ast\tilde{Y},W) = -g_2((\nabla F_\ast)(\tilde{X},\tilde{Y})^{(rangeF_\ast) \perp} + (\nabla F_\ast)(\tilde{X},\tilde{Y})^{(rangeF_\ast) \perp} + F_\ast(\nabla_{\tilde{X}}^M \tilde{Y}'),JF_\ast\tilde{Y})$$

$$+ g_2(-A_{JF_\ast\tilde{Y}}\tilde{X} + \nabla_{\tilde{X}}^F JF_\ast\tilde{Y},CW).$$

Hence, we arrive at

$$g_2(\nabla_{\tilde{X}}^2 F_\ast\tilde{Y},W) = -g_2((\nabla F_\ast)(\tilde{X},\tilde{Y})^{(rangeF_\ast) \perp},JF_\ast\tilde{Y}) + g_2(\nabla_{\tilde{X}}^F JF_\ast\tilde{Y},CW).$$

From above equation, $(rangeF_\ast)$ defines a totally geodesic foliation on $M_2$ if and only if (12) is satisfied. \qed

In a similar way, we obtain the following Theorem:
Theorem 3.2. Let $F$ be a conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1, g_1)$ to a Kähler manifold $(M_2, g_2, J)$. Then $(\text{range} F_*)^\perp$ defines a totally geodesic foliation on $M_2$ if and only if

(i) $(\text{range} F_*)^\perp$ defines a totally geodesic foliation on $M_2$.
(ii) $F$ is a horizontally homothetic conformal Riemannian map.
(iii) $g_2(\mathcal{B} V, A_{\text{ev}} F_* \tilde{X} + F_*(\nabla_X^M Z')) = -g_2(\mathcal{E} W, (\nabla F_*)(\tilde{X}, Z')^{(\text{range} F_*)^\perp} + \nabla_X^{\perp} \mathcal{E} V)
\quad - g_2(W, [V, F_* \tilde{X}])$

for any $V, W \in \Gamma((\text{range} F_*)^\perp)$ and $\tilde{X}, Z' \in \Gamma((\text{ker} F_*)^\perp)$ such that $F_* Z' = \mathcal{B} V$.

Proof. For $\tilde{X} \in \Gamma((\text{ker} F_*)^\perp)$ and $V, W \in \Gamma((\text{range} F_*)^\perp)$, since $M_2$ is a Kähler manifold, using (2) we have

$$g_2(\nabla^2_W F_* \tilde{X}) = g_2(\nabla^2_W [V, F_* \tilde{X}]) - g_2(JW, \nabla_{F_* \tilde{X}} J V).$$

Then using from (11), (4) and (5) we obtain

$$g_2(\nabla^2_W F_* \tilde{X}) = g_2(W, [V, F_* \tilde{X}]) - g_2(\mathcal{B} W, (\nabla F_*)(\tilde{X}, Z') + F_*(\nabla^M_X Z'))
\quad - g_2(\mathcal{B} W, -A_{\text{ev}} F_* \tilde{X} + \nabla^\perp_X \mathcal{E} V)\quad - g_2(\mathcal{E} W, (\nabla F_*)(\tilde{X}, Z') + F_*(\nabla^M_X Z'))
\quad - g_2(\mathcal{E} W, -A_{\text{ev}} F_* \tilde{X} + \nabla^\perp_X \mathcal{E} V),$$

where $F_* Z' = \mathcal{B} V \in \Gamma(\text{range} F_*)$ for $Z' \in \Gamma((\text{ker} F_*)^\perp)$. Since $F$ is a conformal Riemannian map, using (8), we arrive at

$$g_2(\nabla^2_W F_* \tilde{X}) = g_2(W, [V, F_* \tilde{X}]) - g_2(\mathcal{B} W, (\nabla F_*)(\tilde{X}, Z')^{\text{range} F_*} - g_2(\mathcal{B} W, F_*(\nabla^M_X Z'))
\quad + g_2(\mathcal{B} W, A_{\text{ev}} F_* \tilde{X}) - g_2(\mathcal{E} W, (\nabla F_*)(\tilde{X}, Z')^{\text{range} F_*} - g_2(\mathcal{E} W, \nabla^\perp_X \mathcal{E} V)$$

Then from (9), we get

$$g_2(\nabla^2_W F_* \tilde{X}) = g_2(W, [V, F_* \tilde{X}]) - g_2(\mathcal{B} W, F_*(\nabla^M_X Z')) + g_2(\mathcal{B} W, A_{\text{ev}} F_* \tilde{X})
\quad - g_2(\mathcal{E} W, (\nabla F_*)(\tilde{X}, Y')^{\text{range} F_*} - g_2(\mathcal{E} W, \nabla^\perp_X \mathcal{E} V)
\quad - g_2(\mathcal{B} W, \tilde{X} (\ln \lambda) F_* Z' + Z'(\ln \lambda) F_* \tilde{X} - g_1(\tilde{X}, Z') F_*(\text{grad} \ln \lambda))$$

or

$$g_2(\nabla^2_W F_* \tilde{X}) = g_2(W, [V, F_* \tilde{X}]) - g_2(\mathcal{B} W, F_*(\nabla^M_X Z')) + g_2(\mathcal{B} W, A_{\text{ev}} F_* \tilde{X})
\quad - g_2(\mathcal{E} W, (\nabla F_*)(\tilde{X}, Y')^{\text{range} F_*} - g_2(\mathcal{E} W, \nabla^\perp_X \mathcal{E} V)
\quad - g_1(\tilde{X}, \text{grad} \ln \lambda) g_2(\mathcal{B} W, F_* Z' - g_1(Z', \text{grad} \ln \lambda) g_2(\mathcal{B} W, F_* \tilde{X})
\quad + g_1(\tilde{X}, Z') g_2(\mathcal{B} W, F_*(\text{grad} \ln \lambda))$$
Hence we have

g_2(∇^e_X W, F_\ast \tilde{X}) = -g_2(W, [V, F_\ast \tilde{X}]) - g_2(\mathcal{B} W, F_\ast (\nabla^M_X Z')) + g_2(\mathcal{B} W, A_{C V} F_\ast \tilde{X})
- g_2(\mathcal{C} W, (\nabla F_\ast)(\tilde{X}, \tilde{Y'})^{(range F_\ast)\perp}) - g_2(\mathcal{C} W, \nabla^F_{\perp} c V)
- g_1(\tilde{X}, \mathcal{H}(\text{grad} \ln \lambda) g_2(\mathcal{B} W, F_\ast Z') - g_1(Z', \mathcal{H}(\text{grad} \ln \lambda) g_2(\mathcal{B} W, F_\ast \tilde{X})
+ g_1(\tilde{X}, Z') g_2(\mathcal{B} W, F_\ast (\text{grad} \ln \lambda))

From above equation, we can conclude that the two assertions in Theorem 3.2 imply the third.

We now recall the following characterization for locally (usual) product Riemannian manifold from [10]. Let \( g \) be a Riemannian metric tensor on the manifold \( \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \) and assume that the canonical foliations \( D_{\mathcal{M}_1} \) and \( D_{\mathcal{M}_2} \) intersect perpendicularly everywhere. Then \( g \) is the metric tensor of a usual product of Riemannian manifolds if and only if \( D_{\mathcal{M}_1} \) and \( D_{\mathcal{M}_2} \) are totally geodesic foliations. From Theorem 3.1 and Theorem 3.2, we have the following theorem;

**Theorem 3.3.** Let \( F \) be a horizontally homothetic conformal anti-invariant Riemannian map from a Riemannian manifold \((\mathcal{M}_1, g_1)\) to a Kähler manifold \((\mathcal{M}_2, g_2, J)\). Then the base manifold is a locally product manifold \( \mathcal{M}_2(\text{range } F_\ast) \times \mathcal{M}_2(\text{range } F_\ast) \) if and only if

\[
g_2((\nabla F_\ast)(\tilde{X}, \tilde{Y'})^{(\text{range } F_\ast)\perp}), J F_\ast \tilde{Y}) = g_2(\nabla^F_X J F_\ast \tilde{Y}, c V)
\]

and

\[
g_2(\mathcal{B} V, A_{C V} F_\ast \tilde{X} + F_\ast (\nabla^M_X Z')) = -g_2(\mathcal{C} W, (\nabla F_\ast)(\tilde{X}, Z')^{(\text{range } F_\ast)\perp} + \nabla^F_{\perp} c V)
- g_2(W, [V, F_\ast \tilde{X}])
\]

for any \( V, W \in \Gamma((\text{range } F_\ast)\perp) \) and \( \tilde{X}, \tilde{Y}, \tilde{Y'}, Z' \in \Gamma((\text{ker } F_\ast)\perp) \) such that \( F_\ast \tilde{Y}' = \mathcal{B} W \) and \( F_\ast Z' = \mathcal{B} V \).

In the sequel we are going to investigate the harmonicity of conformal anti-invariant Riemannian map. We first have the following general result.

**Theorem 3.4.** Let \( F \) be a conformal anti-invariant Riemannian map from a Riemannian manifold \((\mathcal{M}_1, g_1)\) to a Kähler manifold \((\mathcal{M}_2, g_2, J)\). Then \( F \) is harmonic if and only if the following conditions are satisfied;

(a) the fibres are minimal,
(b) \( \text{trace } B \nabla^{(\perp)}_{\ast} J F_\ast (\ast) - F_\ast (\nabla^{M_1}_X (\ast)) = 0 \),
(c) \( \text{trace } J A_{J F_\ast (\ast)} (\ast) - c \nabla^{(\perp)}_{\ast} J F_\ast (\ast) = 0 \).

**Proof.** For \( U \in \Gamma(\text{ker } F_\ast) \), using (4), we have

\[
(\nabla F_\ast)(U, U) = -F_\ast (\nabla^U_U). \tag{13}
\]

For \( \tilde{X} \in \Gamma((\text{ker } F_\ast)\perp) \), using (4) and (3), we have

\[
(\nabla F_\ast)(\tilde{X}, \tilde{X}) = \nabla^X_X F_\ast \tilde{X} - F_\ast (\nabla^{M_1}_X \tilde{X}) = -J \nabla^X_X J F_\ast \tilde{X} - F_\ast (\nabla^{M_1}_X \tilde{X}).
\]
From (5),(8) and (11) we obtain
\[(\nabla F_*) (\tilde{X}, \tilde{X}) (range F_*) + (\nabla F_*) (\tilde{X}, \tilde{X}) (range F_*) = JA_{JF_* \tilde{X}, \tilde{X}} - \mathcal{B} \nabla_{\tilde{X}}^\perp JF_* \tilde{X} - \mathcal{C} \nabla_{\tilde{X}}^\perp JF_* \tilde{X} - F_* (\nabla_{\tilde{X}}^\perp \tilde{X}). \]  
(14)

Then taking the \((range F_*)\)-components and \((range F_*)^\perp\)-components of above expression (14), we arrive at
\[(\nabla F_*) (\tilde{X}, \tilde{X}) (range F_*) = -\mathcal{B} \nabla_{\tilde{X}}^\perp JF_* \tilde{X} - F_* (\nabla_{\tilde{X}}^\perp \tilde{X}) \]  
(15)

and
\[(\nabla F_*) (\tilde{X}, \tilde{X}) (range F_*)^\perp = JA_{JF_* \tilde{X}, \tilde{X}} - \mathcal{C} \nabla_{\tilde{X}}^\perp JF_* \tilde{X}. \]  
(16)

Then proof follows from (13), (15) and (16).

**Definition 3.2.** Let \(F\) be a conformal Riemannian map from a Riemannian manifold \((M_1, g_1)\) to a Riemannian manifold \((M_2, g_2)\). Then we say that \(F\) is a horizontally homothetic conformal Riemannian map if the gradient of its dilation \(\lambda\) is vertical, i.e., \(\mathcal{H}(\text{grad} \lambda) = 0\).

From Theorem 3.4, we have the following result.

**Corollary 3.1.** Let \(F : (M_1, g_1) \rightarrow (M_2, g_2, J)\) be a conformal anti-invariant Riemannian map such that \(n \neq \frac{2}{3}\), where \((M_1, g_1)\) is a Riemannian manifold and \((M_2, g_2, J)\) is a Kähler manifold. If \(F\) satisfies
\[\text{trace} \mathcal{B} \nabla_{(\cdot)}^\perp JF_* (\cdot) - F_* (\nabla_{(\cdot)}^\perp (\cdot)) = 0,\]
then \(F\) is a horizontally homothetic conformal Riemannian map.

We recall that a differentiable map \(F\) between Riemannian manifold \((M_1, g_1)\) and \((M_2, g_2)\) is called a totally geodesic map if \((\nabla F_*) (\tilde{X}, \tilde{Y}) = 0\) for all \(\tilde{X}, \tilde{Y} \in \Gamma(TM_1)\). We have the following theorem.

**Theorem 3.5.** Let \(F\) be a conformal anti-invariant Riemannian map from a Riemannian manifold \((M_1, g_1)\) to a Kähler manifold \((M_2, g_2, J)\). Then \(F\) is totally geodesic if and only if
\[(a) \ g_2 (B \nabla_{\tilde{X}}^\perp JF_* \tilde{Y}_2, F_* (Z)) = \lambda^2 g_1 (\nabla_{\tilde{X}}^\perp \tilde{Y}, Z)\]
\[(b) \ JA_{JF_* \tilde{Y}_2, \tilde{X}} = C \nabla_{\tilde{X}}^\perp JF_* \tilde{Y}_2\]
for any \(\tilde{X}, \tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2, Z \in \Gamma(TM_1)\), where \(\tilde{Y}_1 \in \Gamma(\text{ker} F_*), \tilde{Y}_2 \in \Gamma((\text{ker} F_*)^\perp)\).

**Proof.** For \(\tilde{X}, \tilde{Y} \in \Gamma(TM_1)\) and \(\tilde{Y}_1 \in \Gamma(\text{ker} F_*), \tilde{Y}_2 \in \Gamma((\text{ker} F_*)^\perp)\), using (4), (3) and (5), we have
\[(\nabla F_*) (\tilde{X}, \tilde{Y}) = -J (-A_{JF_* \tilde{Y}_2, \tilde{X}} + \nabla_{\tilde{X}}^\perp JF_* \tilde{Y}_2) - F_* (\nabla_{\tilde{X}}^\perp \tilde{Y}).\]
Then from (11) we get
\[(\nabla F_*) (\tilde{X}, \tilde{Y}) = JA_{JF_* \tilde{Y}_2, \tilde{X}} - B \nabla_{\tilde{X}}^\perp JF_* \tilde{Y}_2 - C \nabla_{\tilde{X}}^\perp JF_* \tilde{Y}_2 - F_* (\nabla_{\tilde{X}}^\perp \tilde{Y}).\]
Since $F$ is conformal Riemannian map, using (8), we get
\begin{align*}
(\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*} + (\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*\perp} &= JA_{\tilde{F}_*\tilde{Y}_2} \tilde{X} - B\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2 \\
&\quad - C\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2 - F_*(\nabla^{M_1} \tilde{Y}).
\end{align*}
Then taking the $(range F_*)$ and $((range F_*)\perp)$ components we arrive at
\begin{align*}
(\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*} &= B\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2 + F_*(\nabla^{M_1} \tilde{Y})
\end{align*}
and
\begin{align*}
(\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*\perp} &= JA_{\tilde{F}_*\tilde{Y}_2} \tilde{X} - C\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2.
\end{align*}
Thus $(\nabla F_*)(\tilde{X}, \tilde{Y}) = 0$ if and only if $(\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*} = 0$ and $(\nabla F_*)(\tilde{X}, \tilde{Y})^{range F_*\perp} = 0$. Hence we have
\begin{align*}
g_2(B\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2, F_*(Z)) = -\lambda^2 g_1(\nabla^{M_1} \tilde{Y}, Z)
\end{align*}
and
\begin{align*}
JA_{\tilde{F}_*\tilde{Y}_2} \tilde{X} - C\nabla_{\tilde{X}}^{F_*\perp} \tilde{F}_* \tilde{Y}_2 = 0,
\end{align*}
which complete the proof. \qed

We also have the following result for totally geodesic conformal anti-invariant Riemannian maps.

**Theorem 3.6.** Let $F$ be a conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1, g_1)$ to a Kähler manifold $(M_2, g_2, J)$. Then $F$ is totally geodesic if and only if
\begin{itemize}
  \item[(a)] The horizontal distribution $(ker F_*)^{\perp}$ defines a totally geodesic foliation on $M_1$.
  \item[(b)] all the fibres $F^{-1}(y)$ are totally geodesic for $y \in M_2$.
  \item[(c)] $(range F_*)^{\perp}$ defines a totally geodesic foliation on $M_2$.
\end{itemize}
for any $\tilde{X}, \tilde{Y} \in \Gamma(ker F_*)^{\perp}$ and $V \in \Gamma(range F_*)$.

**Proof.** For $\tilde{X}, \tilde{Y} \in \Gamma(ker F_*)^{\perp}$ and $U \in \Gamma(ker F_*)$, using (4), we have
\begin{align*}
g_2((\nabla F_*)(\tilde{X}, U), F_*(\tilde{Y})) = -\lambda^2 g_1(\nabla^{M_1} U, \tilde{Y}).
\end{align*}
Since $\nabla^{M_1}$ is a Levi-Civita connection, we obtain
\begin{align*}
g_2((\nabla F_*)(\tilde{X}, U), F_*(\tilde{Y})) = \lambda^2 g_1(U, \nabla^{M_1} \tilde{Y}), \quad (\lambda \neq 0).
\end{align*}
Hence $(\nabla F_*)(\tilde{X}, U) = 0$ for $\tilde{X} \in \Gamma(ker F_*)^{\perp}$ and $U \in \Gamma(ker F_*)$ if and only if (a).

For $U, V \in \Gamma(ker F_*)$ and $\tilde{X} \in \Gamma(ker F_*)^{\perp}$, we have
\begin{align*}
g_2((\nabla F_*)(U, V), F_*(\tilde{X})) = -\lambda^2 g_1(\nabla^{M_1} V, \tilde{Y}), \quad (\lambda \neq 0)
\end{align*}
Thus $(\nabla F_*)(U, V) = 0$ for $U, V \in \Gamma(ker F_*)$ if and only if (b).

For $\tilde{X}, \tilde{Y} \in \Gamma(ker F_*)^{\perp}$ and $V \in \Gamma(range F_*)$, since $M_2$ is a Kähler manifold, using (2), (4), (11) we have
\begin{align*}
g_2((\nabla F_*)(\tilde{X}, \tilde{Y}), V) - g_2(\nabla^{M_1}_X F_* \tilde{Y}, J F_* \tilde{Y}) + g_2(\nabla^{M_1}_X J F_* \tilde{Y}, CV),
\end{align*}
where \( F_\ast \tilde{Y}' = BV \) for \( \tilde{Y}' \in \Gamma((kerF_\ast)^\perp) \). Since \( F \) is a conformal Riemannian map, using (4), (5) and (8) we obtain
\[
g_2((\nabla F_\ast)(\tilde{X}, \tilde{Y}), V) = -g_2((\nabla F_\ast)(\tilde{X}, \tilde{Y}'^\prime)^\ast(F_\ast)^\perp, JF_\ast \tilde{Y}) + g_2(\nabla X^F \ast JF_\ast \tilde{Y}, \mathbb{C}V).
\]
Thus, \((\nabla F_\ast)(\tilde{X}, \tilde{Y}) = 0 \) for \( \tilde{X}, \tilde{Y} \in \Gamma((kerF_\ast)^\perp) \) if and only if (c).

\[
\text{4. Umbilical conformal anti-invariant Riemannian maps}
\]

In this section, we investigate the umbilical case for the conformal anti-invariant Riemannian maps. We first recall the following definition.

**Definition 4.1.** [7] Let \( F \) be a map from a Riemannian manifold \((M_1, g_1)\) to a Riemannian manifold \((M_2, g_2)\). Then \( F \) is called a weakly g\(_1\)-umbilical if there exist

1. a field \( \xi \) along \( F \), nowhere 0, with values in \((kerF_\ast)^\perp\),
2. a field \( Z \) on \( M \) such that for every \( \tilde{X}, \tilde{Y} \) on \( \Gamma(TM) \) we have
\[
(\nabla F_\ast)(\tilde{X}, \tilde{Y}) = g_1(\tilde{X}, \tilde{Y})[F_\ast(Z) + \xi].
\]

\( F \) is called strong g\(_1\)-umbilical if \( Z = 0 \).

Using the above definition, we can give the following theorem.

**Theorem 4.1.** Let \( F \) be a g\(_1\)-umbilical conformal Riemannian map from a Riemannian manifold \((M_1, g_1)\) to a Riemannian manifold \((M_2, g_2)\) such that \( \text{dim}(\mathcal{H}) \geq 2 \). Then \( F \) is a totally geodesic map.

**Proof.** We suppose that \( F \) is a weakly g\(_1\)-umbilical conformal Riemannian map such that \( \text{dim}(\mathcal{H}) \geq 2 \). Then from (9) and (17) we have
\[
\tilde{X}(\ln \lambda)F_\ast \tilde{Y} + \tilde{Y}(\ln \lambda)F_\ast \tilde{X} - g_1(\tilde{X}, \tilde{Y})F_\ast(\text{grad} \ln \lambda) = g_1(\tilde{X}, \tilde{Y})F_\ast Z \tag{18}
\]
and
\[
(\nabla F_\ast)(\tilde{X}, \tilde{Y})^{(\ast F_\ast)^\perp} = g_1(\tilde{X}, \tilde{Y})\xi. \tag{19}
\]
Since \( \text{dim}(\mathcal{H}) \geq 2 \), we can choose \( \tilde{X} \) and \( \tilde{Y} \) such that \( g_1(\tilde{X}, \tilde{Y}) = 0 \). Then we get
\[
\tilde{X}(\ln \lambda)F_\ast \tilde{Y} + \tilde{Y}(\ln \lambda)F_\ast \tilde{X} = 0.
\]
Since \( \tilde{X} \) and \( \tilde{Y} \) are orthogonal and \( F \) is a conformal Riemannian map, we have
\[
g_2(F_\ast \tilde{X}, F_\ast \tilde{Y}) = \lambda^2 g_1(\tilde{X}, \tilde{Y}) = 0.
\]
\( F_\ast \tilde{X} \) and \( F_\ast \tilde{Y} \) are also orthogonal. Then we get
\[
\tilde{X}(\ln \lambda)F_\ast \tilde{Y} = 0, \quad \tilde{Y}(\ln \lambda)F_\ast \tilde{X} = 0.
\]
Thus \( F \) is a horizontally homothetic Riemannian map. Since \( F \) is horizontally homothetic, from (18), we get \( Z = 0 \). Thus \((\nabla F_\ast)(\tilde{X}, \tilde{Y}) = g_1(\tilde{X}, \tilde{Y})\xi \) for \( \tilde{X}, \tilde{Y} \in \Gamma(TM) \). In particular, for \( U, V \in \Gamma(kerF_\ast) \), we get
\[
-F_\ast(\nabla U V) = g_1(U, V)\xi.
\]
The right side of this equation belongs to $\Gamma((\text{range}F_\ast)^\perp)$ while the left side of this equation belongs to $\Gamma(\text{range}F_\ast)$. Hence $F_\ast(\nabla_V V) = 0$ and $\xi = 0$ which proves our assertion.

From Theorem 3.6 and Theorem 4.1, we have the following result.

**Corollary 4.1.** Let $F$ be a $g_1$–umbilical conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1, g_1)$ to a Kähler manifold $(M_2, g_2, J)$ such that $\dim(\mathfrak{H}) \geq 2$. Then we have the following assertions:

(a) The horizontal distribution $(\text{ker}F_\ast)^\perp$ defines a totally geodesic foliation on $M_1$.

(b) all the fibres $F^{-1}(y)$ are totally geodesic for $y \in M_2$.

(c) $(\text{range}F_\ast)^\perp$ defines a totally geodesic foliation on $M_2$.

From the above Theorem 4.1, we can give the following:

**Theorem 4.2.** Let $F : (M_1, g_1) \rightarrow (M_2, J, g_2)$ be a $g_1$–umbilical conformal anti-invariant Riemannian map from a Riemannian manifold $(M_1, g_1)$ to a Kähler manifold $(M_2, g_2, J)$. Then at least one of the following is satisfied:

(a) The horizontal distribution $(\text{ker}F_\ast)^\perp$ is 1 dimensional distribution.

(b) $F$ is a totally geodesic conformal Riemannian map.

**Proof.** We suppose that $F$ is not a totally geodesic $g_1$–umbilical conformal Riemannian map. Then for $w_1, w_2 \in \Gamma((\text{ker}F_\ast)^\perp)$, since $M_2$ is a Kähler manifold, using (6), (4) and (17) we obtain

$$-A_{JF_\ast(w_1)}F_\ast(w_2) + \nabla_{F_\ast(w_2)}^\perp JF_\ast(w_1) = g_1(w_1, w_2)J\xi + g_1(w_1, w_2)JF_\ast(Z) + JF_\ast(\nabla^\perp_{w_2} w_1).$$

Taking inner product with $F_\ast(w_2)$ in the above equation, we get

$$-g_2(A_{JF_\ast(w_1)}F_\ast(w_2), F_\ast(w_2)) = -g_1(w_1, w_2)g_2(\xi, JF_\ast(w_2)). \quad (20)$$

From (6), (17) and (20), we get

$$g_1(w_2, w_2)g_2(\xi, JF_\ast(w_1)) = g_1(w_1, w_2)g_2(\xi, JF_\ast(w_2)). \quad (21)$$

Interchanging the role of $w_1$ and $w_2$ in (21), we obtain

$$g_1(w_1, w_1)g_2(\xi, JF_\ast(w_2)) = g_1(w_1, w_2)g_2(\xi, JF_\ast(w_1)). \quad (22)$$

From (21) and (22), we get

$$g_2(\xi, JF_\ast(w_2)) = \frac{g_1(w_1, w_2)^2}{g_1(w_1, w_1)g_1(w_2, w_2)}g_2(\xi, JF_\ast(w_2)). \quad (23)$$

From (23), $w_1$ and $w_2$ are linear dependent, which gives the proof.
5. Conclusions

In this paper, we just introduce a general Riemannian map from a Riemannian manifold to an almost Hermitian manifold. From the theory of submanifolds of almost Hermitian manifolds, one can see that there are many new research problems to be investigated.

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