UPPER BOUND OF SECOND HANKEL DETERMINANT FOR $k$-BI-SUBORDINATE FUNCTIONS

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In this work, we determine an upper bound of the functional $H_2(2) = a_2a_4 - a_3^2$ for functions belonging to a subclass of analytic bi-univalent functions which is defined by subordination conditions in the open unit disk $D$. In addition, we get a smaller upper bound and more accurate estimation than the previous results and we correct their mistake.

Keywords: Univalent function, $k$-bi-subordinate functions, second Hankel determinant, subordination.

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1. Introduction

Let $A$ be a class of analytic functions in the open unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in D). \quad (1)$$

A function $f : D \to \mathbb{C}$ is called univalent on $D$ if $f(z_1) \neq f(z_2)$ all $z_1, z_2 \in D$ with $z_1 \neq z_2$. Let $S$ be the class of functions $f \in A$ which are univalent in $D$.

A function $f \in A$ is said to be starlike, if it satisfies the inequality

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in D). \quad (2)$$

We denote the class which consists of all functions $f \in A$ that are starlike by $S^*$.

A function $f \in A$ is said to be convex, if it satisfies the inequality

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in D). \quad (3)$$

We denote the class which consists of all functions $f \in A$ that are convex by $C$.

For two functions $f$ and $g$ which are analytic in $D$, we say that the function $f$ is subordinate to $g$, and write $f(z) \prec g(z)$, if there exists a Schwarz function $w$, that is a function analytic in $D$ with $w(0) = 0$ and $|w(z)| < 1$ in $D$, such that $f(z) = g(w(z))$ for all $z \in D$.

In particular, if the function $g$ is univalent in $D$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subseteq g(D)$, [7].

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By means of the subordination, the conditions (2) and (3) are, respectively, equivalent to
\[
\frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + z}{1 - z}.
\]
Ma and Minda [11] gave a unified presentation of various subclasses of starlike and convex functions by replacing the subordinate function \(\frac{zf'(z)}{f(z)}\) by a more general analytic function \(\varphi\) with positive real part in the unit disk \(\mathbb{D}\), symmetric with respect to the real axis and starlike with respect to \(\varphi(0) = 1\), and \(\varphi'(0) > 0\).

One of the important tools in the theory of univalent functions are the Hankel determinants which are used, for example, in showing that a function of bounded characteristic
\[
\begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}
\]
where the Hankel determinants \(H_2(1) = a_3 - a_2^2\) and \(H_2(2) = a_4a_2 - a_3^2\) are well-known as Fekete-Szegö and second Hankel determinant functionals, respectively. Further, Fekete and Szegö [8] introduced the generalized functional \(a_3 - \lambda a_2^2\), where \(\lambda\) is some real number. Problems in this field has also been argued by several authors (see for example [1, 4, 6, 9, 14, 15, 16, 20]).

In 1983, Sălăgean [17] introduced differential operator \(D^k : \mathcal{A} \to \mathcal{A}\) defined by
\[
D^0f(z) = f(z), \quad D^1f(z) = Df(z) = zf'(z),
\]
and in general
\[
D^kf(z) = D(D^{k-1}f(z)), \quad k \in \mathbb{N} = \{1, 2, \ldots\}.
\]
We easily find that
\[
D^kf(z) = z + \sum_{n=2}^{\infty} a_nz^n, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},
\]
with \(D^0f(0) = 0\).

The Koebe one-quarter theorem [7] ensures that the image of \(\mathbb{D}\) under every univalent function \(f \in \mathcal{S}\) contains a disk of radius \(1/4\). Thus every function \(f \in \mathcal{S}\) has an inverse \(f^{-1}\), such that
\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{D}), \quad \text{and} \quad f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); \ r_0(f) \geq \frac{1}{4} \right),
\]
where the inverse \(f^{-1}\) has the power series expansion (see [10])
\[
g(w) := f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_4 + a_3)w^4 + \ldots. \tag{4}
\]
A function \(f \in \mathcal{A}\) is said to be bi-univalent in \(\mathbb{D}\) if both \(f\) and \(f^{-1}\) are univalent in \(\mathbb{D}\), in the sense that \(f^{-1}\) has a univalent analytic continuation to \(\mathbb{D}\). Let \(\Sigma\) denote the class of bi-univalent functions in \(\mathbb{D}\). For a brief history of functions in the class \(\Sigma\) and also different other characteristics of these functions see [2, 10, 18, 19, 21] and the references therein.
In this work, we assume that the function \( \varphi \) is an analytic function with positive real part in the unit disk \( \mathbb{D} \), satisfying \( \varphi(0) = 1, \varphi'(0) > 0 \), such that \( \varphi(\mathbb{D}) \) is symmetric with respect to the real axis. Such a function has the power series expansion of the form
\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad z \in \mathbb{D} \quad (B_1 > 0).
\] (5)

By means of the subordination, Bulut [3] defined the class \( \mathcal{B}_{\Sigma}^{m,k}(\gamma; \varphi) \) of analytic bi-univalent functions as follows:

**Definition 1.1.** Let \( m, k \in \mathbb{N}_0 : m > k \) and \( \gamma \in \mathbb{C} \setminus \{0\} \). A function \( f \in \Sigma \) given by (1) is said to be in the class \( \mathcal{B}_{\Sigma}^{m,k}(\gamma; \varphi) \) if the following conditions are satisfied:
\[
1 + \frac{1}{\gamma} \left( \frac{D^m f(z)}{D^k f(z)} - 1 \right) \prec \varphi(z)
\] (6)
and
\[
1 + \frac{1}{\gamma} \left( \frac{D^m g(w)}{D^k g(w)} - 1 \right) \prec \varphi(w),
\] (7)
where \( z, w \in \mathbb{D} \) and the function \( g = f^{-1} \) is defined by (4).

**Remark 1.1.** For \( m = k + 1 \), we get the class \( \mathcal{B}_{\Sigma}^{k+1,k}(\gamma; \varphi) = \mathcal{B}_{\Sigma,k}(\gamma; \varphi) \) of \( k \)-bi-subordinate functions of complex order \( \gamma \in \mathbb{C} \setminus \{0\} \).

**Remark 1.2.** If we set
\[
m = k + 1, \quad \gamma = 1 \quad \text{and} \quad \varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)
\]
in Definition 1.1, then the class \( \mathcal{B}_{\Sigma}^{m,k}(\gamma; \varphi) \) reduces to the class \( \mathcal{S}_{\Sigma,k}(\beta) \) of \( k \)-bi-starlike functions. In other words, a function \( f \in \Sigma \) is said to be in the class \( \mathcal{S}_{\Sigma,k}(\beta) \), if the following conditions are satisfied (see [15]):
\[
\text{Re} \left( \frac{D^{k+1} f(z)}{D^k f(z)} \right) > \beta \quad \text{and} \quad \text{Re} \left( \frac{D^{k+1} g(z)}{D^k g(z)} \right) > \beta.
\]
For \( k = 0 \) and \( k = 1 \), we get the classes
\[
\mathcal{S}_{\Sigma,0}(\beta) = \mathcal{S}_{\Sigma}^{(2)}(\beta) \quad \text{and} \quad \mathcal{S}_{\Sigma,1}(\beta) = \mathcal{K}_{\Sigma}(\beta),
\]
which are the class of bi-starlike functions of order \( \beta \) and bi-convex functions of order \( \beta \), respectively. In particular, we have the classes
\[
\mathcal{S}_{\Sigma,0}(0) = \mathcal{S}_{\Sigma}^{(2)} \quad \text{and} \quad \mathcal{S}_{\Sigma,1}(0) = \mathcal{K}_{\Sigma},
\]
which are the class of bi-starlike functions and bi-convex functions, respectively.

**Example 1.1.** If we set \( f(z) = \frac{z}{1-z} \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) where \( z \in \mathbb{D} \), then both \( f(z) \) and \( g(w) = f^{-1}(w) = \frac{w}{1 + w} \) are univalent in \( \mathbb{D} \) and so \( f \in \Sigma \). On other the hand, conditions (6) and (7) hold for \( k = 1, \ m = 2 \) and \( \gamma = 1 \), that is,
\[
1 + \frac{z f''(z)}{f'(z)} = \frac{1 + z}{1 - z} < \frac{1 + z}{1 - z}, \quad \text{this is equivalent with} \quad \text{Re} \left( \frac{1 + z}{1 - z} \right) > 0
\]
and
\[
1 + \frac{w g''(w)}{g'(w)} = \frac{1 - w}{1 + w} < \frac{1 + w}{1 - w}, \quad \text{this is equivalent with} \quad \text{Re} \left( \frac{1 - w}{1 + w} \right) > 0.
\]
Therefore \( f \in \mathcal{B}_{\Sigma}^{2,1} \left( \frac{1 + z}{1 - z} \right) \), in other words \( f \) is 1-bi-convex function (bi-convex function).

Since every convex function is a starlike function, so also \( f \) is 1-bi-starlike function (bi-starlike function).
then we have

\[ |a_2a_4 - a_3^2| \leq \begin{cases} \frac{4(1-\beta)^2}{3} (4\beta^2 - 8\beta + 5) , & \beta \in \left[0, \frac{29-\sqrt{137}}{32}\right] \\ (1-\beta)^2 \left( \frac{13\beta^2-14\beta-7}{16\beta^2-20\beta+5} \right) , & \beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right) \end{cases} \]

**Corollary 1.1.** [6, Corollary 2.2] Let the function \( f \) given by (1) be in the class \( S_{\Sigma}^\gamma(\beta) \) (0 \( \leq \beta < 1 \)). Then

\[ |a_2a_4 - a_3^2| \leq \frac{20}{3}. \]

**Theorem 1.2.** [6, Theorem 2.3] Let the function \( f \) given by (1) be in the class \( K_{\Sigma}(\beta) \) (0 \( \leq \beta < 1 \)). Then

\[ |a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{24} \left( \frac{5\beta^2 + 8\beta - 32}{3\beta^2 - 3\beta - 4} \right). \]

**Corollary 1.2.** [6, Corollary 2.4] Let the function \( f \) given by (1) be in the class \( K_{\Sigma} \). Then

\[ |a_2a_4 - a_3^2| \leq \frac{1}{3}. \]

The class \( S_{\Sigma,k}(\beta) \) of \( k \)-bi-starlike functions is defined by Orhan et al. [15] and they obtained an upper bound for the second Hankel determinant of functions \( f \in S_{\Sigma,k}(\beta) \) (see [15, Theorem 2.1]). They got for \( \eta, \mu \leq 1 \)

\[ |a_2a_4 - a_3^2| \leq T_1 + (\eta + \mu)T_2 + (\eta^2 + \mu^2)T_3 + (\eta + \mu)^2T_4 = G(\eta, \mu), \]

where

\[
T_1 = T_1(p) = \frac{(1-\beta)^2}{3(2^{3k})} \left( \left(1-\beta^2 \right)^2 \frac{3(k^2)+2^{2k} - \frac{13}{4}^k \frac{1}{3}^{k+1}}{2^{2k}} + \frac{1}{4} \frac{p^4 - \frac{p^3}{2} + 2p}{2^{3k}} \right) \geq 0 \\
T_2 = T_2(p) = \frac{(1-\beta)^2}{2^{2k+1}} \left( \frac{1}{3^{2k}} + \frac{1}{4} \frac{p^4 - \frac{p^3}{2} + 2p}{2^{3k}} \right) \geq 0 \\
T_3 = T_3(p) = \frac{(1-\beta)^2}{2^{3k}} \frac{p(4 - p^2)(p-2)}{24(2^{3k})} \leq 0 \\
T_4 = T_4(p) = \frac{(1-\beta)^2}{16(9^k)} \frac{(4 - p^2)^2}{4} \geq 0.
\]

They claimed that

\[ T_3 + 2T_4 > 0 \quad \text{for} \quad p \in [0, 2), \]

to maximize the function \( G(\eta, \mu) \) on the closed square \([0, 1] \times [0, 1]\). But there is a mistake in their proof. Now we give a counterexample that this inequality is not true:

If we choose

\[ \beta = 0, \quad k = 10 \quad \text{and} \quad p = 0, 9, \]

then we have

\[ T_3 + 2T_4 = -3, 134800373 \times 10^{-11} < 0. \]

The main purpose of this paper is that, by using a different method from the one in [15], to determine the functional \( H_2(\alpha) = a_2a_4 - a_3^2 \) for functions belonging to the subclass of analytic bi-univalent functions \( R_{\Sigma,m,k}(\gamma, \varphi) \) which is defined by subordination principle in the open unit disk \( D \). In addition, we get more accurate estimation than the previous results and we give the correction of [15, Theorem 2.1].

In order to prove our main results, we need the following lemmas.
Lemma 1.1. [7, p. 190] Let \( u \) be analytic function in the unit disk \( \mathbb{D} \), with \( u(0) = 0 \), and \( |u(z)| < 1 \) for all \( z \in \mathbb{D} \), with the power series expansion
\[
 u(z) = \sum_{n=1}^{\infty} c_n z^n.
\]
Then, \( |c_n| \leq 1 \) for all \( n \in \mathbb{N} \). Furthermore, \( |c_n| = 1 \) for some \( n \in \mathbb{N} \) if and only if \( u(z) = e^{i\theta} z^n \), \( \theta \in \mathbb{R} \).

Lemma 1.2. [9] If \( \psi(z) = \sum_{n=1}^{\infty} \psi_n z^n \), \( z \in \mathbb{D} \), is a Schwarz function, then
\[
 \psi_2 = x \left(1 - \psi_2^2\right),
\]
\[
 \psi_3 = (1 - \psi_2^2) (1 - |x|^2) s - \psi_1 (1 - \psi_2^2) x^2,
\]
for some \( x, s \), with \( |x| \leq 1 \) and \( |s| \leq 1 \).

2. Main Results

Whilst Lemma 1.1 holds for complex-valued \( c_n \) \((n \in \mathbb{N})\), in this paper we restrict our attention to the case of real valued \( c_1 \).

Theorem 2.1. Let the function \( f \) given by (1) be in the class \( \mathcal{B}_{\Sigma}^{m,k}(\gamma; \varphi) \). Then
\[
 |a_2 a_4 - a_3^2| \leq B_1 |\gamma|^2 \times \begin{cases} 
 R & \text{if } Q \leq 0, P \leq -Q \\
 P + Q + R & \text{if } (Q \geq 0, P \geq -Q), \text{ or, } (Q \leq 0, P \leq -Q) \\
 \frac{4P R - Q^2}{4P} & \text{if } Q > 0, P \leq -Q,
\end{cases}
\]
where
\[
 P = \frac{\left\lvert [2(2^{m-2}) (2^{2k} - 3^{k}) - 2^{k} (2^m - 3^k) + (4^m - 4^k)] \gamma^2 B_3^3 \right\rvert (4^m - 4^k) (2^m - 2^k)^4}{(4^m - 4^k) (2^m - 2^k)^4} + \frac{B_3}{(4^m - 4^k) (2^m - 2^k)} - \frac{B_1}{(4^m - 4^k) (2^m - 2^k)} + \frac{B_1}{(3^m - 3^k)^2},
\]
\[
 Q = \frac{\left\lvert [2(2^{m-2}) (2^{2k} - 3^{k}) - 2^{k} (2^m - 3^k) + (4^m - 4^k)] \gamma^2 B_3^3 \right\rvert (4^m - 4^k) (2^m - 2^k)^4}{(4^m - 4^k) (2^m - 2^k)^4} + \frac{|B_2|}{(4^m - 4^k) (2^m - 2^k)} + \frac{B_1}{(4^m - 4^k) (2^m - 2^k)} - \frac{2B_1}{(3^m - 3^k)^2},
\]
\[
 R = \frac{B_1}{(3^m - 3^k)^2}.
\]

Proof. Let \( f \in \mathcal{B}_{\Sigma}^{m,k}(\gamma; \varphi) \). Then by definition of subordination and Lemma 1.1, there exist two Schwarz functions \( u \) and \( v \), of the form \( u(z) = \sum_{n=1}^{\infty} c_n z^n \) and \( v(z) = \sum_{n=1}^{\infty} d_n z^n \), \( z \in \mathbb{D} \) such that
\[
 1 + \frac{1}{\gamma} \left( \frac{D^m f}{D^k f}(z) - 1 \right) = \varphi(u(z))
\]
and
\[
 1 + \frac{1}{\gamma} \left( \frac{D^m g}{D^k g}(w) - 1 \right) = \varphi(v(w)),
\]
where
\[
 \varphi(u(z)) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + (B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^2) z^3 + \cdots
\]
and
\[ \varphi(v(w)) = 1 + B_1 d_1 w + (B_1 d_2 + B_2 d_2^2) w^2 + (B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_3^2) w^3 + \cdots. \] (11)

From (8), (10) and (9), (11), we have
\[ (2^m - 2^k) a_2 = \gamma B_1 c_1 \] (12)
\[ (3^m - 3^k) a_3 - 2^k (2^m - 2^k) a_2^2 = \gamma (B_1 c_2 + B_2 c_2^2) \] (13)
\[ (4^m - 4^k) a_4 - \left[ 3^k (2^m - 2^k) + 2^k (3^m - 3^k) \right] a_2 a_3 + 2^{2k} (2^m - 2^k) a_3^2 = \gamma (B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_3^2) \] (14)

and
\[ -(2^m - 2^k) a_2 = \gamma B_1 d_1 \] (15)
\[ (3^m - 3^k) (2a_2^2 - a_3) - 2^k (2^m - 2^k) a_2^2 = \gamma (B_1 d_2 + B_2 d_2^2) \] (16)
\[ -(4^m - 4^k) \left( 5a_2^2 - 5a_2 a_3 + a_4 \right) + \left[ 3^k (2^m - 2^k) + 2^k (3^m - 3^k) \right] a_2 (2a_2^2 - a_3) - 2^{2k} (2^m - 2^k) a_3^2 = \gamma (B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_3^2), \] (17)

respectively. From (12) and (15), we get that
\[ c_1 = -d_1 \] (18)

and
\[ a_2 = \frac{\gamma B_1 c_1}{2^m - 2^k}. \] (19)

Nevertheless, from (13) and (16), we get
\[ a_3 = \frac{\gamma^2 B_1^2 c_1 c_2^2}{(2^m - 2^k)^2} + \frac{\gamma B_1 (c_2 - d_2)}{2(3^m - 3^k)}. \] (20)

Furthermore, from (14) and (17), we obtain
\[ a_4 = \frac{\left[ (2^m - 2^k) (3^k - 2^{2k}) + 2^k (3^m - 3^k) \right] \gamma^2 B_1^2 c_1^2}{(4^m - 4^k)(2^m - 2^k)^3} + \frac{5 \gamma^2 B_1^2 c_1 (c_2 - d_2)}{4(2^m - 2^k)(3^m - 3^k)} \]
\[ + \frac{\gamma B_1 (c_3 - d_3)}{2(4^m - 4^k)} + \frac{\gamma B_2 c_1 (c_2 + d_2)}{(4^m - 4^k)} + \frac{\gamma B_3 c_3^2}{(4^m - 4^k)}. \] (21)

Therefore, after calculations we have
\[ \left| a_2 a_4 - a_3^2 \right| = \left| \frac{\gamma^2 B_1^2 c_1^2 (c_2 - d_2)}{4(2^m - 2^k)(3^m - 3^k)^2} + \frac{\gamma^2 B_1 B_2 c_2^2 (c_2 + d_2)}{(4^m - 4^k)(2^m - 2^k)^2} \right| \]
\[ + \frac{\gamma^2 B_1 B_3 c_3 (c_3 - d_3)}{(4^m - 4^k)(2^m - 2^k)} + \frac{\gamma^2 B_2^2 c_1^2 (c_2 - d_2)}{4(3^m - 3^k)^2} \] (22)

According to Lemma 1.2 and (18), we find that
\[ c_2 - d_2 = (1 - c_1^2) \left( x - y \right) \quad \text{and} \quad c_2 + d_2 = (1 - c_1^2) \left( x + y \right) \] (23)

and
\[ c_3 = (1 - c_1^2) \left( 1 - |x|^2 \right) s - c_1 \left( 1 - c_1^2 \right) x^2 \quad \text{and} \]
\[ d_3 = (1 - d_1^2) \left( 1 - |y|^2 \right) t - d_1 \left( 1 - d_1^2 \right) y^2, \]

where
\[ c_3 - d_3 = (1 - c_1^2) \left[ (1 - |x|^2) s - (1 - |y|^2) t \right] - c_1 (1 - c_1^2)(x^2 + y^2). \] (24)
for some $x$, $y$, $s$, $t$ with $|x| \leq 1$, $|y| \leq 1$, $|s| \leq 1$ and $|t| \leq 1$. Applying (23) and (24) in (22), it follows that

$$|a_2a_4 - a_3^2| = B_1|\gamma|^2 \left[ \frac{-(2m-2k^2)(2^k - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2k)} \frac{\gamma^2 B_3^3}{(4^m - 4^k)(2^m - 2k)^4} + B_3 \right] \frac{c^3}{4(3^m - 3^k)^2} \left( x^2 + y^2 \right) \right] B_1 \right] \left( 1 - c^2 \right) \left( \frac{1 - c^2}{2(4^m - 4^k)(2^m - 2k)} \right) \left( (1 - |x|^2) s - (1 - |y|^2) t \right) \left. \right|.

Since $|c_1| \leq 1$, we assume that $c_1 = c \in [0, 1]$. So we have

$$|a_2a_4 - a_3^2| \leq B_1|\gamma|^2 \left[ \frac{-(2m-2k^2)(2^k - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2k)} \frac{\gamma^2 B_3^3}{(4^m - 4^k)(2^m - 2k)^4} + B_3 \right] \frac{c^3}{4(3^m - 3^k)^2} \left( x^2 + y^2 \right) \right] B_1 \right] \left( 1 - c^2 \right) \left( \frac{1 - c^2}{2(4^m - 4^k)(2^m - 2k)} \right) \left( (1 - |x|^2) s - (1 - |y|^2) t \right) \left. \right|.

Now, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq B_1|\gamma|^2 \left[ T_1 + (\lambda + \mu)T_2 + (\lambda^2 + \mu^2)T_3 + (\lambda + \mu)T_4 \right] = B_1|\gamma|^2 F(\lambda, \mu),$$
where
\[ T_1 = T_1(c) = \left[ -\frac{(2m - 2k)(2^{k} - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2^k)^2} \gamma^2 B_1^3 + \frac{B_3}{(4^m - 4^k)(2^m - 2^k)} \right] c^4 + \frac{2B_1c(1 - c^2)}{2(4^m - 4^k)(2^m - 2^k)} \geq 0 \]
\[ T_2 = T_2(c) = \left[ \frac{\gamma B_1^2}{4(2^m - 2^k)(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)} \right] c^2 (1 - c^2) \geq 0 \]
\[ T_3 = T_3(c) = \frac{B_1c(c-1)(1 - c^2)}{2(4^m - 4^k)(2^m - 2^k)} \leq 0 \]
\[ T_4 = T_4(c) = \frac{B_1(1 - c^2)^2}{4(3^m - 3^k)^2} \geq 0. \]

We now need to maximize the function \( F(\lambda, \mu) \) on the closed square \([0, 1] \times [0, 1]\) for \( c \in [0, 1] \).

With regards to \( F(\lambda, \mu) = F(\mu, \lambda) \), it is sufficient that we investigate the maximum of
\[ G(\lambda) = F(\lambda, \lambda) = T_1 + 2\lambda T_2 + 2\lambda^2(T_3 + 2T_4), \quad (25) \]
on \( \lambda \in [0, 1] \) according to \( c \in (0, 1), c = 0 \) and \( c = 1 \).

Firstly, if we let \( c = 1 \), then we obtain
\[
\max \{ G(\lambda) : \lambda \in [0, 1] \} = \left[ -\frac{(2^m - 2k)(2^{k} - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2^k)^2} \gamma^2 B_1^3 + \frac{B_3}{(4^m - 4^k)(2^m - 2^k)} \right] \]
\[
= \frac{B_3}{(4^m - 4^k)(2^m - 2^k)} \]

Secondly, letting \( c = 0 \), so we get
\[ G(\lambda) = \frac{4B_1}{4(3^m - 3^k)^2} \lambda^2, \]
hence we can see that
\[ \max \{ G(\lambda) : \lambda \in [0, 1] \} = G(1) = \frac{B_1}{(3^m - 3^k)^2}. \]

Finally, we let \( c \in (0, 1) \). Considering equation (25) for \( 0 \leq \lambda \leq 1 \) we get
(i) If \( T_3 + 2T_4 \geq 0 \), it is clear that
\[ G'(\lambda) = 4(T_3 + 2T_4)\lambda + 2T_2 \geq 0 \]
for \( 0 < \lambda < 1 \) and any fixed \( c \in (0, 1) \), that is \( G(\lambda) \) is an increasing function. Hence
\[ \max \{ G(\lambda) : \lambda \in [0, 1] \} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4. \]
(ii) If \( T_3 + 2T_4 < 0 \), then we consider for critical point
\[ \lambda_0 = \frac{-T_2}{2(T_3 + 2T_4)} = \frac{T_2}{2k} \]
for any fixed \( c \in (0, 1) \), where \( k = -(T_3 + 2T_4) > 0 \), the following two cases:

**Case 1.** For \( \lambda_0 = \frac{T_2}{2k} > 1 \), it follows that \( k \leq \frac{T_2}{2} \leq T_2 \), and so \( T_2 + T_3 + 2T_4 \geq 0 \).
Therefore,
\[ G(0) = T_1 \leq T_1 + 2(T_2 + T_3 + 2T_4) = G(1). \]

**Case 2.** For \( \lambda_0 = \frac{T_2}{2k} \leq 1 \), since \( T_2 \geq 0 \), we get that \( \frac{T_2}{2k} \leq T_2 \). Therefore,
\[ G(0) = T_1 \leq T_1 + \frac{T_2}{2k} = G(\lambda_0) \leq T_1 + T_2. \]
Considering the above cases for point of \( c \), it follows that the function \( G(\lambda) \) gets its maximum when \( T_3 + 2T_4 \geq 0 \), it means
\[
\max \{ G(\lambda) : \lambda \in [0, 1] \} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]
Therefore, \( \max F(\lambda, \mu) = F(1, 1) \) on the boundary of the square.
Let \( K : [0, 1] \to \mathbb{R} \),
\[
K(c) = B_1|\gamma|^2 \max F(\lambda, \mu) = B_1|\gamma|^2 F(1, 1) = B_1|\gamma|^2(T_1 + 2T_2 + 2T_3 + 4T_4).
\]
By replacing the values of \( T_1, T_2, T_3 \) and \( T_4 \) in the above function \( K \), we have
\[
K(c) = B_1|\gamma|^2 \left\{ \left[ -\frac{(2^m - 2^k)(2^{2k} - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2^k)^4} \right] \frac{\gamma^2 B_1}{(4^m - 4^k)(2^m - 2^k)} + \frac{B_3}{(4^m - 4^k)(2^m - 2^k)} \right. \\
- 2 \left( \frac{|\gamma|B_1^2}{(4^m - 4^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)} \right) - \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} + \frac{B_1}{(3^m - 3^k)^2} \right\} c^4 \\
+ \left[ 2 \left( \frac{|\gamma|B_1^2}{(4^m - 4^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)} \right) + \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} - \frac{2B_1}{(3^m - 3^k)^2} \right\} c^2 \\
+ \frac{B_1}{(3^m - 3^k)^2}
\]
Suppose \( c^2 = u \) and for the simplicity, set
\[
P = \left[ -\frac{(2^m - 2^k)(2^{2k} - 3^k) - 2^k(3^m - 3^k) + (4^m - 4^k)}{(4^m - 4^k)(2^m - 2^k)^4} \right] \frac{\gamma^2 B_1^3}{(4^m - 4^k)(2^m - 2^k)} + \frac{B_3}{(4^m - 4^k)(2^m - 2^k)} \\
- 2 \left( \frac{|\gamma|B_1^2}{(4^m - 4^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)} \right) - \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} + \frac{B_1}{(3^m - 3^k)^2}
\]
\[
Q = 2 \left( \frac{|\gamma|B_1^2}{(4^m - 4^k)^2(3^m - 3^k)} + \frac{|B_2|}{(4^m - 4^k)(2^m - 2^k)} \right) + \frac{B_1}{(4^m - 4^k)(2^m - 2^k)} - \frac{2B_1}{(3^m - 3^k)^2}
\]
\[
R = \frac{B_1}{(3^m - 3^k)^2}
\]
According to
\[
\max(Pu^2 + Qu + R)_{0 \leq u \leq 1} = \begin{cases} 
R & \text{if } Q \leq 0, P \leq -Q \\
P + Q + R & \text{if } (Q \geq 0, P \geq -Q), \text{ or, } (Q \leq 0, P \geq -Q) \\
\frac{4PR-Q^2}{4P} & \text{if } Q > 0, P \leq -\frac{Q}{2}
\end{cases}
\]
it follows that
\[
|a_2a_4 - a_3^2| \leq B_1|\gamma|^2 \times \begin{cases} 
R & \text{if } Q \leq 0, P \leq -Q \\
P + Q + R & \text{if } (Q \geq 0, P \geq -Q), \text{ or, } (Q \leq 0, P \geq -Q) \\
\frac{4PR-Q^2}{4P} & \text{if } Q > 0, P \leq -\frac{Q}{2}
\end{cases}
\]
where \( P, Q \) and \( R \) are given by (27). This completes the proof. □
For 
\[ m = k + 1, \gamma = 1 \quad \text{and} \quad \varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1) \]
in Theorem 2.1, we get the following correction of the estimates in [15, Theorem 2.1]:

**Corollary 2.1.** Let the function \( f \) given by \((1)\) be in the class \( S_{\Sigma_k}^{\ast} (\beta) \) \((0 \leq \beta < 1)\). Then

\[ |a_{2a4} - a_{3}^2| \leq 2(1 - \beta) \times \begin{cases} R & \text{if} \quad Q \leq 0, P \leq -Q \\ P + Q + R & \text{if} \quad (Q \geq 0, P \geq -\frac{Q}{2}), \text{ or, } (Q \leq 0, P \geq -Q) \\ \frac{4PR - Q^2}{4P} & \text{if} \quad Q > 0, P \leq -\frac{Q}{2}, \end{cases} \]

where

\[ P = (1 - \beta) \left\{ -\frac{2^{2k} + 3(2^k) - 3^{k+1}}{3(2^{2k-3})} (1 - \beta)^2 + \frac{1}{3(2^{2k-1})} \right\} - \frac{1 - \beta}{(2^{2k})(3^k)} \]
\[ Q = (1 - \beta) \left[ \frac{1 - \beta}{(2^{2k})(3^k)} + \frac{1}{2^{2k-1}} - \frac{1}{3^{2k}} \right], \]
\[ R = \frac{1 - \beta}{2(3^{2k})}. \]

For \( k = 0 \) in Corollary 2.1, we get the following result that is an improvement of the estimates which in Theorem 1.1.

**Corollary 2.2.** Let the function \( f \) given by \((1)\) be in the class \( S_{\Sigma}^{\ast} (\beta) \) \((0 \leq \beta < 1)\). Then

\[ |a_{2a4} - a_{3}^2| \leq 2(1 - \beta)^2 \begin{cases} \frac{2}{3} (4\beta^2 - 8\beta + 3) & , \quad 0 \leq \beta \leq \frac{29 - \sqrt{549}}{32} \\ \frac{13\beta^2 - 14\beta - 15}{12\beta^2 - 52\beta - 6} & , \quad \frac{29 - \sqrt{549}}{32} \leq \beta \leq \frac{1}{2} \\ \frac{19\beta^2 - 50\beta + 39}{52\beta^2 - 76\beta + 34} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases} \]

For \( \beta = 0 \), Corollary 2.2 yields the following coefficient estimates for bi-starlike functions. This result is an improvement of the estimates obtained in Corollary 1.1.

**Corollary 2.3.** Let the function \( f \) given by \((1)\) be in the class \( S_{\Sigma}^{\ast} \). Then

\[ |a_{2a4} - a_{3}^2| \leq 4. \]

For \( k = 1 \) in Corollary 2.1, we get the following result that is an improvement of the estimates in Theorem 1.2.

**Corollary 2.4.** Let the function \( f \) given by \((1)\) be in the class \( K_{\Sigma} (\beta) \) \((0 \leq \beta < 1)\). Then

\[ |a_{2a4} - a_{3}^2| \leq \frac{(1 - \beta)^2}{24} \begin{cases} 11\beta^2 - 40\beta + 48 & , \quad 0 \leq \beta < 1 \end{cases} \]

For \( \beta = 0 \), Corollary 2.4 yields the following coefficient estimates for bi-convex functions. This result is an improvement of the estimates obtained in Corollary 1.2.

**Corollary 2.5.** Let the function \( f \) given by \((1)\) be in the class \( K_{\Sigma} \). Then

\[ |a_{2a4} - a_{3}^2| \leq \frac{1}{5}. \]
3. Conclusion

In the final section, we found improved upper bounds for the functional $|H_2(2)|$ for functions in the class $S_{\Sigma}^{m,k}(\gamma;\varphi)$. The technique of proof for Theorem 2.1 can be extended to other classes of functions similar to $S_{\Sigma}^{m,k}(\gamma;\varphi)$ as for example $M_{\Sigma}(\varphi,\beta)$ introduced in Definition 1.1 of [12], in order to improve previous estimates by their Theorem 2.1. Sharp estimates for $|H_2(2)|$ are for now open problems.

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