ON EDGE IRREGULARITY STRENGTH OF TOEPLITZ GRAPHS

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An edge irregular k-labeling of a graph G is a labeling of the vertices of G with labels from the set \{1, 2, ..., k\} in such a way that for any two different edges xy and x’y’ their weights w(xy) and w(x’y’) are distinct. The weight w(xy) of an edge xy in G is the sum of the labels of the end vertices x and y. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G).

In this paper, we study the edge irregular k-labeling for Toeplitz graphs and determine the exact value for several classes of Toeplitz graphs.

Keywords: irregular assignment, irregularity strength, edge irregularity strength, Toeplitz graphs

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1. Introduction

Let G be a connected, simple and undirected graph with vertex set V and edge set E. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex labelings or edge labelings. If the domain is V(G)∪E(G) then we call the labeling total labeling. Thus, for an edge k-labeling δ : E(G) → \{1, 2, ..., k\} the associated weight of a vertex x ∈ V(G) is

\[ w_δ(x) = \sum \delta(xy), \]

where the sum is over all vertices y adjacent to x.

Chartrand et al. in [10] introduced edge k-labeling δ of a graph G such that \( w_δ(x) \neq w_δ(y) \) for all vertices \( x, y \in V(G) \) with \( x \neq y \). Such labelings were called irregular assignments and the irregularity strength s(G) of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength s(G) can be interpreted as the

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smallest integer \( k \) for which \( G \) can be turned into a multigraph \( G' \) by replacing each edge by a set of at most \( k \) parallel edges, such that the degrees of the vertices in \( G' \) are all different. This parameter has attracted much attention [5, 6, 9, 16].

Motivated by these papers, Ahmad et al. in [1] started to investigate an edge irregularity strength. A vertex \( k \)-labeling \( \phi : V(G) \to \{1, 2, \ldots, k\} \) is called an edge irregular \( k \)-labeling of the graph \( G \) if for every two different edges \( xy \) and \( x'y' \) there is \( w_\phi(xy) \neq w_\phi(x'y') \), where the weight of an edge \( xy \in E(G) \) is \( w_\phi(xy) = \phi(x) + \phi(y) \). The minimum \( k \) for which the graph \( G \) has an edge irregular \( k \)-labeling is called the edge irregularity strength of \( G \), denoted by \( es(G) \). The notion of the edge irregularity strength was defined in [1]. There is estimated the lower bound of the edge irregularity strength as follows

\[ es(G) \geq \max \left\{ \left[ \left| E(G) \right| + 1 \right]/2, \Delta(G) \right\}. \]

In [1] it is proved that for path \( P_n \), \( n \geq 2 \), \( es(P_n) = \lceil n/2 \rceil \), for star \( K_{1,n} \), \( n \geq 1 \), \( es(K_{1,n}) = n \), for double star \( S_{m,n} \), \( 3 \leq m \leq n \), \( es(S_{m,n}) = n \) and for Cartesian product of two paths \( P_m \) and \( P_n \), \( m, n \geq 2 \), \( es(P_m \square P_n) = \lceil (2mn - m - n + 1)/2 \rceil \). Al-Mushayt [4] determined the edge irregularity strength of products of certain families of graphs with path \( P_2 \).

2. Toeplitz graph

A simple undirected graph \( T \) of order \( p \) is called Toeplitz graph if its adjacency matrix \( A(T) \) is Toeplitz. A Toeplitz matrix \( A(T) = (a_{i,j}) \), is a \((p \times p)\) symmetric matrix which has constant values along all diagonals parallel to the main diagonal, i.e. \( a_{i,j} = a_{i+1,j+1} \) for each \( i, j = 1, 2, \ldots, p - 1 \). The \( p \) distinct diagonals of a \((p \times p)\) symmetric Toeplitz adjacency matrix will be labeled \( 0, 1, 2, \ldots, p - 1 \). Diagonal \( 0 \) is the main diagonal and it contains only zeros, i.e. \( a_{ii} = 0 \) for all \( i = 1, 2, \ldots, p \) so that there are no loops in the Toeplitz graph. A Toeplitz graph \( T \) is uniquely defined by the first row of \( A(T) \), a \((0 - 1)\)-sequence. Let \( t_1, t_2, \ldots, t_s \) be the diagonals containing ones, \( 0 < t_1 < t_2 < \cdots < t_s < p \). Then, the corresponding Toeplitz graph will be denoted by \( T_p(t_1, \ldots, t_s) \). That is, \( T_p(t_1, \ldots, t_s) \) is the graph with the vertex set \( V(T) = \{v_i : i = 1, 2, \ldots, p\} \) in which two vertices \( u, v \) of \( T \) being connected by an edge if and only if \( |u - v| \in \{t_1, t_2, \ldots, t_s\} \). If \( t_j, j = 1, 2, \ldots, s \), is the diagonal containing ones then the diagonal elements \( a_{it_{j+i}}, i = 1, 2, \ldots, p - t_j \), determine edges \( v_iv_{t_{j+i}} \) in the Toeplitz graph. Thus the edge set is \( E(T) = \bigcup_{j=1}^{s} \{v_iv_{t_{j+i}} : i = 1, 2, \ldots, p - t_j\} \), \( |V(T)| = p \) and \( |E(T)| = ps - \sum_{j=1}^{s} t_j \).

Toeplitz graphs have been introduced by Sierksma and first been investigated by van Dal et al. [11] with respect to their hamiltonicity. Later
Heuberger [18] has extended this study in 2002. The properties of Toeplitz graphs; such as bipartitiveness, planarity and colourability, have been studied in [12, 13, 14, 15]. For more recent works on Toeplitz graphs see [8, 21, 22, 25]. A Toeplitz graph is not necessarily connected, see Figure 1.

Figure 1. Toeplitz graphs $T_7\langle 1, 2 \rangle$, $T_7\langle 2, 4 \rangle$ and $T_7\langle 1, 2, 3 \rangle$

The following result proved by van Dal et al. [11], provides a lower bound on the number of components of a Toeplitz graph.

**Theorem 2.1.** [11] $T_p\langle t_1, \ldots, t_s \rangle$ has at least $\gcd(t_1, \ldots, t_s)$ components.

In the paper we investigate the existence of the edge irregularity strength for Toeplitz graphs.

3. Results

Next theorem gives the exact value of the edge irregularity strength of Toeplitz graph $T_n\langle 1, 2 \rangle$ which is bigger than the lower bound in Theorem 1.1.

**Theorem 3.1.** Let $T_n\langle 1, 2 \rangle$ be a Toeplitz graph on $n \geq 3$ vertices. Then $es(T_n\langle 1, 2 \rangle) = n$.

**Proof.** Let $T_n\langle 1, 2 \rangle$ be a Toeplitz graph with the vertex set $V(T_n\langle 1, 2 \rangle) = \{v_i : 1 \leq i \leq n\}$ and the edge set $E(T_n\langle 1, 2 \rangle) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+2} : 1 \leq i \leq n-2\}$. According to Theorem 1.1 we have that $es(T_n\langle 1, 2 \rangle) \geq n-1$. Since every two adjacent vertices in $T_n\langle 1, 2 \rangle$ are a part of complete graph $K_3$, therefore under every edge irregular labeling the smallest edge weight has to be at least 3 and the largest edge weight has to be at least $2n+2-t_1-t_2 = 2n-1$. Since the edge weight $2n-1$ is the sum of two labels, so at least one label is at least $\lceil (2n-1)/2 \rceil = n$. Therefore $es(T_n\langle 1, 2 \rangle) \geq n$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $n$-labeling.

Let $\phi_1 : V(T_n\langle 1, 2 \rangle) \rightarrow \{1, 2, \ldots, n\}$ be the vertex labeling such that $\phi_1(v_i) = i$, for $1 \leq i \leq n$.

Since $w_{\phi_1}(v_i v_{i+1}) = \phi_1(v_i) + \phi_1(v_{i+1}) = 2i + 1$, for $1 \leq i \leq n-1$ and $w_{\phi_1}(v_i v_{i+2}) = \phi_1(v_i) + \phi_1(v_{i+2}) = 2i + 2$, for $1 \leq i \leq n-2$, so the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling $\phi_1$ is an optimal edge irregular $n$-labeling. This completes the proof.  

\[\square\]
Next theorem proves that the lower bound in Theorem 1.1 is tight.

**Theorem 3.2.** Let \( T_n(1, 3) \) be a Toeplitz graph on \( n \geq 4 \) vertices. Then \( es(T_n(1, 3)) = n - 1 \).

**Proof.** Let \( T_n(1, 3) \) be a Toeplitz graph with the vertex set \( V(T_n(1, 3)) = \{v_i : 1 \leq i \leq n\} \) and the edge set \( E(T_n(1, 3)) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+3} : 1 \leq i \leq n-3\} \). According to Theorem 1.1 we have that \( es(T_n(1, 3)) \geq \lceil (2n+1-4)/2 \rceil = n - 1 \). For the converse, we define a suitable edge irregular labeling \( \phi_2 : V(T_n(1, 3)) \to \{1, 2, \ldots, n-1\} \) in the following way.

For \( n \equiv 1 \pmod{4} \)

\[
\phi_2(v_i) = \begin{cases} 
i, & \text{if } i \equiv 1 \pmod{4} \text{ and } 1 \leq i < n - 1 \\
i - 1, & \text{if } i \equiv 2, 3 \pmod{4} \\
i - 1, & \text{if } i \equiv 0 \pmod{4} \text{ and } 1 \leq i \leq n - 2 \\
i, & \text{if } i = n - 1 \\
i - 2, & \text{if } i = n
\end{cases}
\]

For \( n \equiv 0, 2, 3 \pmod{4} \)

\[
\phi_2(v_i) = \begin{cases} 
i, & \text{if } i \equiv 1 \pmod{4} \\
i - 1, & \text{if } i \equiv 0, 2, 3 \pmod{4}
\end{cases}
\]

The edge weights are as follows

If \( n \equiv 1 \pmod{4} \)

\[
w_{\phi_2}(v_i v_{i+1}) = \begin{cases} 
2i, & \text{if } i \equiv 0, 1 \pmod{4} \text{ and } 1 \leq i < n - 2 \\
2i - 1, & \text{if } i \equiv 2, 3 \pmod{4} \text{ and } 1 \leq i < n - 2 \\
2i, & \text{if } i = n - 2 \\
2i - 1, & \text{if } i = n - 1
\end{cases}
\]

If \( n \equiv 0, 2, 3 \pmod{4} \)

\[
w_{\phi_2}(v_i v_{i+1}) = \begin{cases} 
2i, & \text{if } i \equiv 0, 1 \pmod{4} \\
2i - 1, & \text{if } i \equiv 2, 3 \pmod{4}
\end{cases}
\]

If \( n \equiv 1 \pmod{4} \)
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\[
\begin{align*}
w_{\phi_2}(v_iv_{i+3}) &= \begin{cases}
2i + 2, & \text{if } i \equiv 1, 2 \pmod{4} \text{ and } i < n - 4 \\
2i + 1, & \text{if } i \equiv 0, 3 \pmod{4} \\
2i + 3, & \text{if } i = n - 4 \\
2i, & \text{if } i = n - 3
\end{cases}
\end{align*}
\]

If \( n \equiv 0, 2, 3 \pmod{4} \)

\[
\begin{align*}
w_{\phi_2}(v_iv_{i+3}) &= \begin{cases}
2i + 2, & \text{if } i \equiv 1, 2 \pmod{4} \\
2i + 1, & \text{if } i \equiv 0, 3 \pmod{4}.
\end{cases}
\end{align*}
\]

Since, the edge weights are distinct for all pairs of distinct edges, the vertex labeling \( \phi_2 \) is a suitable edge irregular \((n - 1)\)-labeling. Hence, we have \( es(T_n(1, 3)) = n - 1 \). □

Next theorem gives the exact value of the edge irregularity strength for \( T_n(2, 4) \) and show that this value is bigger than the lower bound in Theorem 1.1.

**Theorem 3.3.** Let \( T_n(2, 4), n \geq 5 \), be a Toeplitz graph. Then \( es(T_n(2, 4)) = n - 1 \).

**Proof.** Let \( T_n(2, 4) \) be a Toeplitz graph with the vertex set \( V(T_n(2, 4)) = \{v_i : 1 \leq i \leq n\} \) and the edge set \( E(T_n(2, 4)) = \{v_iv_{i+2} : 1 \leq i \leq n - 2\} \cup \{v_iv_{i+4} : 1 \leq i \leq n - 4\} \). Let \( \phi_3 : V(T_n(2, 4)) \to \{1, 2, \ldots, n - 1\} \) be the vertex labeling such that

\[
\phi_3(v_i) = \begin{cases}
\frac{i + 1}{2}, & \text{if } i \text{ is odd} \\
\left\lceil \frac{n + i - 2}{2} \right\rceil, & \text{if } i \text{ is even}.
\end{cases}
\]

The edge weights are as follows:

\[
\begin{align*}
w_{\phi_3}(v_iv_{i+2}) &= \begin{cases}
i + 2, & \text{if } i \text{ is odd} \\
n + i - 1, & \text{if } i \text{ and } n \text{ are even} \\
n + i, & \text{if } i \text{ is even and } n \text{ is odd}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
w_{\phi_3}(v_iv_{i+4}) &= \begin{cases}
i + 3, & \text{if } i \text{ is odd} \\
n + i, & \text{if } i \text{ and } n \text{ are even} \\
n + i + 1, & \text{if } i \text{ is even and } n \text{ is odd}
\end{cases}
\end{align*}
\]

We can see that all vertex labels are at most \( n - 1 \). The edge weights under the labeling \( \phi_3 \) successively attain values \( 3, 4, \ldots, n - 1, n, n + 2, n + 3, \ldots, 2n - 3 \) for \( n \) odd and \( 3, 4, \ldots, n - 2, n - 1, n + 1, n + 2, \ldots, 2n - 3 \) for \( n \) even. Thus the edge weights are distinct for all pairs of distinct edges and the labeling \( \phi_3 \) provides the upper bound on \( es(T_n(2, 4)) \), i.e \( es(T_n(2, 4)) \leq n - 1 \).
Since every edge of $T_n(2, 4)$ belongs to $K_3$, then under every edge irregular labeling the smallest possible edge weight is obtained as sum of the vertex labels 1 and 2. Then the largest edge weight has to be at least $|E(T_n(2, 4))| + 2 = 2n - 4$ and obtained as the sum of different vertex labels. Thus the largest edge weight is at least $2n - 3$ and $es(T_n(2, 4)) \geq \lceil (2n - 3)/2 \rceil = n - 1$. This provides the lower bound on $es(T_n(2, 4))$. Combining with previous upper bound, we get that $es(T_n(2, 4)) = n - 1$.

The following theorem gives the exact value of the edge irregularity strength for Toeplitz graph $T_n(1, 2, 3)$ for $n \not\equiv 1 \pmod{4}$.

**Theorem 3.4.** Let $T_n(1, 2, 3)$, $n \geq 4$, be a Toeplitz graph. Then

$$es(T_n(1, 2, 3)) = \begin{cases} \frac{3n}{2} - 1, & \text{if } n \text{ is even} \\ \frac{3n-3}{2}, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** Let $V(T_n(1, 2, 3)) = \{v_i : 1 \leq i \leq n\}$ be the vertex set and $E(T_n(1, 2, 3)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+2} : 1 \leq i \leq n-2\} \cup \{v_iv_{i+3} : 1 \leq i \leq n-3\}$ be the edge set of $T_n(1, 2, 3)$ with $|E(T_n(1, 2, 3))| = 3n - 6$. According to Theorem 1.1 we have $es(T_n(1, 2, 3)) \geq \max\{\lfloor (3n - 4)/2 \rfloor, 6\} = \lfloor (3n - 4)/2 \rfloor$. Since every four consecutive vertices in $T_n(1, 2, 3)$ form a complete graph $K_4$, therefore under every edge irregular labeling, no couple of adjacent vertices can be assigned by the same label. This implies that the smallest edge weight 2 is not possible. So if the smallest edge weight is 3 then the largest edge weight is at least $3n - 4$. Since each edge weight is a sum of two labels, at least one label is at least $\lceil (3n - 4)/2 \rceil$. Thus for $n = 4t + 3$, $t \geq 1$, we have

$$es(T_n(1, 2, 3)) \geq \left\lfloor \frac{3n - 4}{2} \right\rfloor = \left\lfloor 6k + 2 + \frac{1}{2} \right\rfloor = \frac{3n - 3}{2}.$$

(1)

For $n$ even the edge weight $3n - 4$ is the sum of two the same labels $3n/2 - 2$ assigned to the adjacent vertices. Since it is not possible, then one label from the sum $3n - 4$ has to be at least $\lceil (3n - 3)/2 \rceil$. Hence we have

$$es(T_n(1, 2, 3)) \geq \left\lfloor \frac{3n - 3}{2} \right\rfloor = \left\lfloor \frac{3n}{2} - 1 - \frac{1}{2} \right\rfloor = \frac{3n}{2} - 1.$$

(2)

For the converse, we define the vertex labeling $\phi_4$ as follows:

$$\phi_4(v_i) = \begin{cases} \frac{3i-2}{2}, & \text{if } i \text{ is even} \\ \frac{3i-1}{2}, & \text{if } i \equiv 1 \pmod{4} \\ \frac{3i-3}{2}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

Observe that under the vertex labeling $\phi_4$ all vertex labels are at most $3n/2 - 1$ for $n$ even and $(3n - 3)/2$ for $n \equiv 3 \pmod{4}$. For $n \equiv 0, 3 \pmod{4}$ the edge weights successively attain values 3, 4, ..., $3n - 4$ and for $n \equiv 2 \pmod{4}$ the edges receive the weights 3, 4, ..., $3n - 6$, $3n - 5$, $3n - 3$. It means that the edge weights are distinct for all pairs of distinct edges and the labeling $\phi_4$ is a
suitable edge irregular \((3n/2 - 1)\)-labeling, respectively \(((3n - 3)/2)\)-labeling. Thus the labeling \(\phi_4\) provides the upper bound on \(es(T_n(1, 2, 3))\). Combining with the lower bounds given by (1) and (2), produces the desired result. □

The following theorem gives the upper bound of the edge irregularity strength for Toeplitz graph \(T_n(1, 2, 3)\) for \(n \equiv 1 \pmod{4}\).

**Theorem 3.5.** Let \(T_n(1, 2, 3)\), be a Toeplitz graph for \(n \equiv 1 \pmod{4}\), \(n \geq 5\). Then

\[
es(T_n(1, 2, 3)) \leq \frac{3n-1}{2}.
\]

**Proof.** In view that \((3n - 1)/2\) is an upper bound on the edge irregularity strength of graph \(T_n(1, 2, 3)\) it suffices to prove the existence of a vertex labeling \(\phi_5 : V(T_n) \to \{1, 2, \ldots, (3n - 1)/2\}\) with edge irregular properties. Define the vertex labels as follows:

\[
\phi_5(v_i) = \phi_4(v_i) \quad \text{for} \quad v_i \in V(T_n(1, 2, 3)).
\]

It is a routine matter to verify that all vertex labels are at most \((3n - 1)/2\) and the edge weights form the set of different integers, namely \(\{3, 4, \ldots, 3n - 6, 3n - 5, 3n - 3\}\). Thus the labeling \(\phi_5\) is desired edge irregular \(((3n - 1)/2)\)-labeling. □

4. Conclusion

In this paper we dealt the existence of the edge irregularity strength for Toeplitz graphs. We determined the exact values of the edge irregularity strength of Toeplitz graphs \(T_n(1, 2)\), \(T_n(1, 3)\) and \(T_n(2, 4)\), namely we proved that \(es(T_n(1, 2)) = n\), \(es(T_n(1, 3)) = es(T_n(2, 4)) = n - 1\). Moreover we proved that \(es(T_n(1, 2, 3)) = 3n/2 - 1\) for \(n\) even and \(es(T_n(1, 2, 3)) = (3n - 3)/2\) for \(n \equiv 3 \pmod{4}\). For \(n \equiv 1 \pmod{4}\) we showed that \(es(T_n(1, 2, 3)) \leq (3n - 1)/2\). We believe that this upper bound is the exact value therefore we propose the following conjecture.

**Conjecture 1.** Let \(n \equiv 1 \pmod{4}\), \(n \geq 5\). Then

\[
es(T_n(1, 2, 3)) = (3n - 1)/2.
\]

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**REFERENCES**


