AN EFFICIENT APPROACH FOR SOLUTION OF MODIFIED CAMASSA-HOLM AND DEGASPERIS-PROCESI EQUATIONS

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In this article, optimal homotopy asymptotic method (OHAM) has been introduced for finding the approximate solutions of modified Camassa-Holm (mCH) and Degasperis-Procesi (mDP) equations. The obtained results give higher accuracy than that of Variational Homotopy Perturbation Method (VHPM) [1]. It is shown that the proposed technique is effective, suitable, and reliable for solving these types of equations.

Keywords: Optimal Homotopy Asymptotic Method, Modified Camassa-Holm Equation, Modified Degasperis-Procesi Equation, Approximate solution.

1. Introduction:

Most of the problems in nature can be expressed in terms of nonlinear partial differential equations. In such situation, it is very difficult to achieve the exact solution for these types of equations. Therefore analytical methods have been used to find approximate solutions. Recently, variety of analytical methods such as the Adomian decomposition method(ADM) [2-3], the Homotopy Analysis Method(HAM) [4-5], the Variational Iteration Method (VIM) [6-7], the Homotopy Perturbation Method (HPM) [8-11], and Variational Homotopy Perturbation Method (VHPM)[12-13] have been tested to solve linear and nonlinear partial differential equations. Optimal Homotopy Asymptotic Method (OHAM) is one of the powerful techniques which was introduced by Marinca and Herisanu et al. for solving Non-linear Differential equations and for steady flow of a fourth-grade fluid past a porous plate [14-18]. By means of the more elastic supporter function called the auxiliary function the proposed technique give us more precise results.
Our aim in this paper is to find accurate approximate solution of mCH and mDP equations by using OHAM. They are the special cases of the modified$b$ –equation [19]

$$u_t - u_{xxt} + (b+1)u^2 u_x - b u_x u_{xx} - u u_{xx} = 0, \tag{1}$$

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where b is a positive integer. For b = 2, Eq. (1.1) reduces to mCH equation, while for b = 3 mDP equation is obtained.
Different techniques have been used in the literature to find the approximate solution of mCH and mDP equations. Zhang et al. employed HPM for finding the solution of mCH and mDP equations [20]. Behera et al. found approximate solution of mCH and mDP equations using wavelet optimized finite difference method [21]. Yıldırım employed VIM for solving mCH and mDP equations. Yousif et al. found the solitary wave solutions of mCH and mDPby VHPM [1].

2. Basic Mathematical Theory of OHAM

Consider the general form of the partial differential equation as:

$$\ell\left(u\left(x,t\right)\right)+g\left(x,t\right)+\eta\left(u\left(x,t\right)\right)=0,\ \upsilon\in\Omega \tag{2.1}$$

$$\beta\left(u,\frac{\partial u}{\partial t}\right)=0, \tag{2.2}$$

where ℓ , η and β are linear ,nonlinear and boundary operator respectively. $u\left(x,t\right)$ is a unknown function and $g\left(x,t\right)$ is a known function , t and x denote temporal and spatial variables, respectively, δ is the domain of the problem.

According to homotopy $\gamma\left(x,t;q\right):\delta\times\left[0,1\right]\rightarrow R$ which satisfied:

$$(1-q)\left\{\ell\left(\delta\left(x,t;q\right)\right)+g(x,t)\right\}=H(q)\left\{\ell\left(\delta\left(x,t;q\right)\right)+g(x,t)+\eta\left(\delta\left(x,t;q\right)\right)\right\}, \tag{2.3}$$

where $q\in\left[0,1\right]$ is an embedding parameter, $H(q)$ is an auxiliary function. When $q=0$, or $q=1$ then clearly, we have:

$$q=0\Rightarrow H\left(\delta\left(x,t;0\right),0\right)=\ell(\delta(x,t;0))+g(x,t)=0, \tag{2.4}$$

$$q=1\Rightarrow H\left(\delta\left(x,t;1\right),1\right)=H(1)\left\{\ell(\delta(x,t;q))+g(x,t)+\eta(\delta(x,t;q))\right\}=0. \tag{2.5}$$

Obviously, when $q=0$ and $q=1$. It keeps that $\delta\left(x,t;0\right)=u_0\left(x,t\right)$ and $\delta\left(x,t;1\right)=u\left(x,t\right)$ respectively. So as q varies from 0 to 1, the result $\delta\left(x,t;q\right)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is achieved from Eq.(3.1). For $q=0$:

$$\ell\left(u_0(x,t)\right)+g(x,t)=0, \qquad \beta\left(u_0,\partial u_0/\partial t\right)=0\ . \tag{2.6}$$

By choosing auxiliary function $H(q)$ as:

$$H\left(q\right)=qC_1+q^2C_2+\ldots \tag{2.7}$$

Here $C_1, C_{2,}\ldots$ are constants to be determined. by expanding $\delta\left(x,t;q,C_i\right)$ in Taylor's series about q as

$$\delta\left(x,t;q,C_i\right)=u_0\left(x,t\right)+\sum_{\kappa=1}^{\infty}u_\kappa\left(x,t;C_i\right)q^\kappa, \quad i=1,2,\ldots \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.2) and equating the coefficient of q, we get zeroth order equation, given by Eq. (2.4), the first and second order equations are given by Eqs. (2.9-2.10) respectively and $u_\kappa(x,t)$ are given by Eq. (2.11):

$$\ell\left(u_1(x,t)\right)=C_1\eta_0\left(u_0(x,t)\right), \quad \beta\left(u_1,\partial u_1/\partial t\right)=0 \tag{2.9}$$

$$\ell\left(u_2(x,t)\right)-\ell\left(u_1(x,t)\right)=C_2\,\eta_0\left(u_0(x,t)\right)+C_1\left[\ell\left(u_1(x,t)\right)+\eta_1\left(u_0(x,t),u_1(x,t)\right)\right],$$
$$\beta\left(u_2,\partial u_2/\partial t\right)=0 \tag{2.10}$$

$$\begin{aligned}&\ell\left(u_\kappa(x,t)\right)-\ell\left(u_{\kappa-1}(x,t)\right)=C_\kappa\eta_0\left(u_0(\upsilon,t)\right)+\\&\sum_{i=1}^{\kappa-1}C_i\left[\ell\left(u_{\kappa-i}(x,t)\right)+\eta_{\kappa-i}\left(u_0(x,t),u_1(x,t),\ldots,u_{\kappa-i}(x,t)\right)\right],\\&\beta\left(u_\kappa,\partial u_\kappa/\partial t\right)=0. \quad \kappa=2,3,\ldots.\end{aligned} \tag{2.11}$$

Here $\eta_{\kappa-i}\left(u_0(x,t),u_1(x,t),\ldots,u_{\kappa-i}(x,t)\right)$ is the coefficient of $q^{\kappa-i}$ in the expansion of $\eta\left(\delta\left(x,t;q\right)\right)$ about the embedding parameter σ.

$$\eta\left(\delta\left(x,t;q,C_i\right)\right)=\eta_0\left(u_0(x,t)\right)+\sum_{\kappa\geq1}\eta_\kappa\left(u_0,u_1,u_2,\ldots,u_\kappa\right)q^\kappa. \tag{2.12}$$

The convergence of the series in Eq. (2.8) depends upon the convergence control parameters $C_1, C_{2,}\ldots$, If it is convergent at $q=1$, one has:

$$u\left(x,t;C_i\right)=u_0(x,t)+\sum_{\kappa\geq1}u_\kappa\left(x,t;C_i\right). \tag{2.13}$$

Substituting Eq. (2.13) into Eq. (2.1), we gained residual:

$$R\left(x,t;C_i\right)=\ell(u(x,t;C_i))+g(x,t)+\eta(u(x,t;C_i)). \tag{2.14}$$

If $R\left(x,t;C_i\right)=0$, then $u\left(x,t;C_i\right)$ will be the exact solution. For calculating the of convergence control parameters, C_i, $i=1,2,...,m$ there are many methods. We used Least Squares method as

$$j(C_i)=\int_0^t\int_\delta R^2(x,t,C_i)dx\,dt, \tag{2.15}$$

where R is the residual,

$$R\left(x,t;C_i\right)=\ell\left(u\left(x,t;C_i\right)\right)+g(x,t)+\eta\left(u\left(x,t;C_i\right)\right),$$

and

$$\frac{\partial j}{\partial C_1}=\frac{\partial j}{\partial C_2}=...=\frac{\partial j}{\partial C_m}=0. \tag{2.16}$$

To determine the convergence control parameters C_i we used another method as.

$$R\left(\hbar_1;C_i\right)=R\left(\hbar_2;C_i\right)=...=R\left(\hbar_m;C_i\right)=0,\ i=1,2,...,m. \tag{2.17}$$

at any time t, where $\hbar_i\in\delta$.

3. Application of OHAM

3.1: Application of OHAM to mCH Equation:

Consider mCH equation with initial condition as follow:

$$\begin{aligned}&u_t-u_{xxt}+3u^2u_x-2u_xu_{xx}-uu_{xx}=0,\\&u(x,0)=-2\,\text{sech}^2\left(\frac{1}{2}x\right).\end{aligned} \tag{3.1.1}$$

The exact solution of eq. (3.1.1) is [19]

$$u(x,t)=-2\,\text{sech}^2\left(\frac{1}{2}x-t\right). \tag{3.1.2}$$

According to OHAM formulation, the zeroth and 1st order problems are given under as:

Zeroth order problem

$$\frac{\partial u_0(x,t)}{\partial tt}=0,\quad u(x,0)=-2\,\text{sech}^2\left(\frac{1}{2}x\right). \tag{3.1.3}$$

Its solution is given under as

$$u_0(x,t)=-2\,\text{sech}^2\left(\frac{x}{2}\right). \tag{3.1.4}$$

1st order problem:

$$\frac{-\partial u_0(x,t)}{\partial t}-C_1\frac{\partial u_0(x,t)}{\partial t}+\frac{\partial u_1(x,t)}{\partial t}-3C_1u^2(x,t)\frac{\partial u_0(x,t)}{\partial x}+2C_1\frac{\partial u_0(x,t)}{\partial x}$$
$$\frac{\partial^2 u_0(x,t)}{\partial x^2}+C_1\frac{\partial^2 u_0(x,t)}{\partial x\partial t}+C_1u_0(x,t)\frac{\partial^2 u_0(x,t)}{\partial x^2}=0,\qquad u_1(x,0)=0. \tag{3.1.5}$$

Its solution is given under as

$$u_1(x,t,C_1)=\left(12\operatorname{sech}^6\left(\frac{x}{2}\right)C_1\tanh\left(\frac{x}{2}\right)+\operatorname{sech}^4\left(\frac{x}{2}\right)C_1\tanh^3\left(\frac{x}{2}\right)\right). \tag{3.1.6}$$

At last we can have obtained the following expression as,

$$u(x,t,C_1)=u_0(x,t)+u_1(x,t,C_1). \tag{3.1.7}$$

The value of convergence control parameter is calculated by using least square method and its optimum value is

$$C_1= -0.16706435582136045.$$

3.2: Application of OHAM to mDP Equation:

Consider mDP equation with initial condition as:

$$u_t-u_{xxt}+4u^2u_x-3u_xu_{xx}-uu_{xx}=0,$$
$$u(x,0)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}x\right). \tag{3.2.1}$$

Exact solution of eq.(3.2.1) is [19],

$$u(x,t)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}x-\frac{5}{4}t\right). \tag{3.2.2}$$

According to OHAM formulation, the zeroth and 1st order problems are given under as:

Zeroth order problem

$$\frac{\partial u_0(x,t)}{\partial tt}=0,\quad u(x,0)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{1}{2}x\right). \tag{3.2.3}$$

Its solution is given under as

$$u_0(x,t)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2}\right).$$

1st order problem:

$$\frac{-\partial u_0(x,t)}{\partial t}-C_1\frac{\partial u_0(x,t)}{\partial t}+\frac{\partial u_1(x,t)}{\partial t}-4C_1u_0{}^2(x,t)\frac{\partial u_0(x,t)}{\partial x}+3C_1\frac{\partial u_0(x,t)}{\partial x}$$
$$\frac{\partial^2 u_0(x,t)}{\partial x^2}+C_1\frac{\partial^2 u_0(x,t)}{\partial x\partial t}+C_1u_0(x,t)\frac{\partial^2 u_0(x,t)}{\partial x^2}=0,\qquad u_1(x,0)=0.$$

(3.2.4)

Its solution is given under as

$$u_1(x,t,C_1)=\frac{225}{16}t\left(\operatorname{sech}^6\left(\frac{x}{2}\right)C_1\tanh\left(\frac{x}{2}\right)+\operatorname{sech}^4\left(\frac{x}{2}\right)C_1\tanh^3\left(\frac{x}{2}\right)\right). \quad (3.2.5)$$

At last we can obtained the following expression as,

$$u(x,t,C_1)=u_0(x,t)+u_1(x,t,C_1),$$

$$u(x,t,C_1)=-\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2}\right)+\frac{225}{16}t\left(\operatorname{sech}^6\left(\frac{x}{2}\right)C_1\tanh\left(\frac{x}{2}\right)+\operatorname{sech}^4\left(\frac{x}{2}\right)C_1\tanh^3\left(\frac{x}{2}\right)\right). \quad (3.2.6)$$

The value of convergence control parameter is calculated by using least square method and its optimum value is

$$C_1 = -0.15288211787484748.$$

4. Tables and Figures

In tables (4.1-4.2) the absolute error of 1st order approximate solution by OHAM are compared with VHPM solution for mCH equation at $t=0.01$ and $t=0.001$ respectively. Table 4.3 and 4.4 shows comparison of 1st order approximate solution by OHAM with VHPM solution for mDP equation at $t=0.01$ and $t=0.001$ respectively. From absolute errors it is clear that 1st order approximate solution by OHAM is more accurate than that of VHPM. Figures (5.1-5.3) show the 3D plots of exact solution, 1st order OHAM solution and VHPM solution for mCH equation respectively. Figure 5.4 shows the comparison of 1st order OHAM and exact solution while Figure 5.5 shows the comparison of VHPM and exact solution for mCH equation at $t=0.1$. Figures (5.6-5.8) show the 3D plots of exact solution, 1st order OHAM solution and VHPM solution for mDP equation respectively. Figure 5.9 shows the comparison of 1st order OHAM and exact solution while figure 5.10 shows the comparison of VHPM and exact solution mDP equation at $t=0.1$. from all these figures we can see that 1st order OHAM solution is in close agreement with exact solution than that of VHPM solution.

Table 4.1

Comparison of absolute errors of 1st order OHAM and VHPM solution [1] for mCH equation at $t=0.01$.

x	OHAM	Exact	Absolute Error VHPM [1]	Absolute Error OHAM
-1.	-1.56717	-1.5583	0.0197059	0.00886242
-0.5	-1.87569	-1.87067	0.0166079	0.00502347
0.	-2.	-1.9998	0.000199987	0.000199987
0.5	-1.88437	-1.88908	0.0169162	0.00471514
1.	-1.57863	-1.58737	0.0198189	0.00874937

Table 4.2

Comparison of absolute errors of 1st order OHAM and VHPM solution [1] for mCH equation at $t = 0.001$.

x	OHAM	Exact	Absolute Error VHPM [1]	Absolute Error OHAM
-1.	-1.57232	-1.57144	0.00197555	0.000881285
-0.5	-1.8796	-1.87911	0.00167455	0.000488583
0.	-2.	-2.	$2.\times10^{-6}$	$2.\times10^{-6}$
0.5	-1.88046	-1.88095	0.00167764	0.000485499
1.	-1.57347	-1.57435	0.00197668	0.000880154

Table 4.3

Comparison of absolute errors of 1st order OHAM and VHPM solution [1] for mDP equation at $t = 0.01$.

x	OHAM	Exact	Absolute error VHPM [1]	Absolute Error OHAM
-1.	-1.46844	-1.45747	0.0230771	0.0109714
-0.5	-1.75788	-1.75151	0.019418	0.00636291
0.	-1.875	-1.87471	0.000292938	0.000292938
0.5	-1.76718	-1.77309	0.0198696	0.00591127
1.	-1.48073	-1.49154	0.0232427	0.0108058

Table 4.4

Comparison of absolute errors of 1st order OHAM and VHPM solution [1] for mDP equation at $t = 0.001$.

x	OHAM	Exact	Absolute Error VHPM [1]	Absolute Error OHAM
-1.	-1.47398	-1.47289	0.00231493	0.00108992
-0.5	-1.76206	-1.76145	0.00196192	0.00061617
0.	-1.875	-1.875	2.92968×10^{-6}	2.92968×10^{-6}
0.5	-1.76299	-1.7636	0.00196643	0.000611654
1.	-1.4752	-1.47629	0.00231658	0.00108827

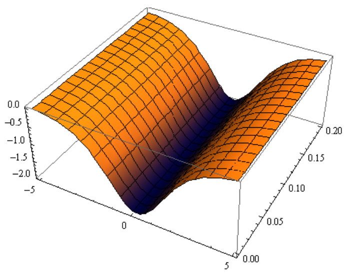

Fig 5.1: 3D plot exact solution equation for mCH eqution

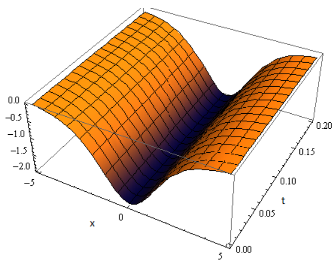

Fig 5.2: 3D plot 1st order OHAM solution for mCH equation

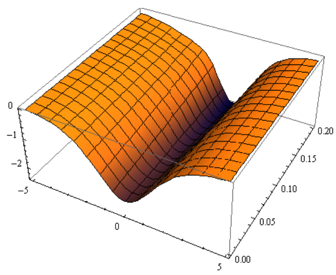

Fig 5.3: 3D plot VHPM for mCH solution mCH equation

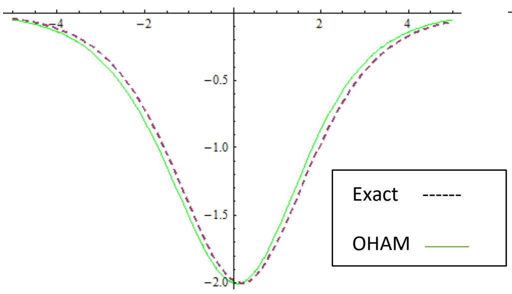

Fig 5.4: 2D plot for exact and 1st order approximate solution by OHAM at $t = 0.1$ for mCH equation

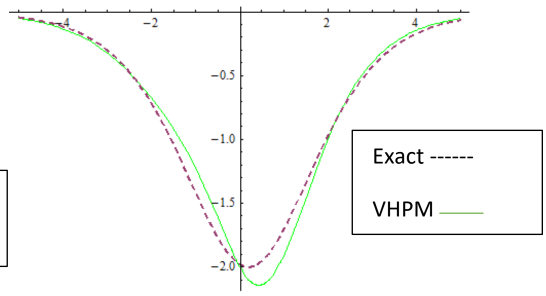

Fig 5.5: 2D plot for exact and VHPM solution at $t = 0.1$ for mCH equation

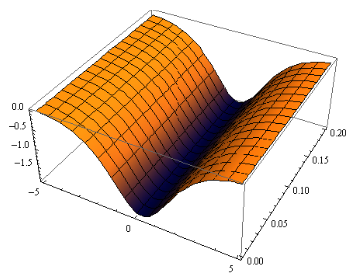

Fig 5.6: 3D plot exact solution of mDP equation

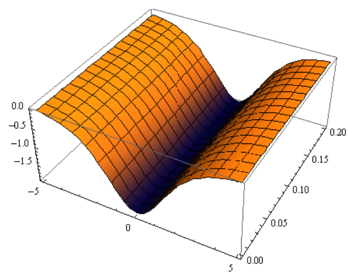

Fig 5.7: 3D plot 1st order OHAM solution for mDP equation

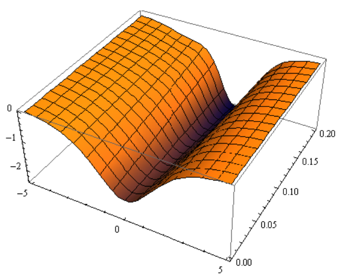

Fig 5.8: 3D plot VHPM solution for mDP equation

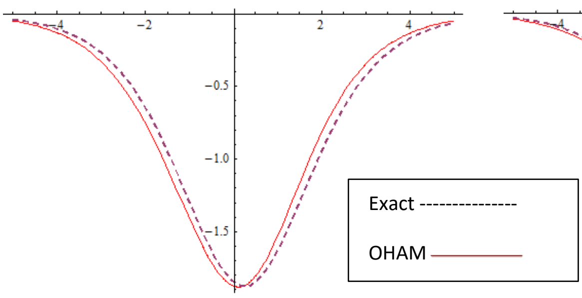

Fig 5.9: 2D plot for exact and 1st order approximate solution by OHAM at $t = 0.1$ for mDP equation

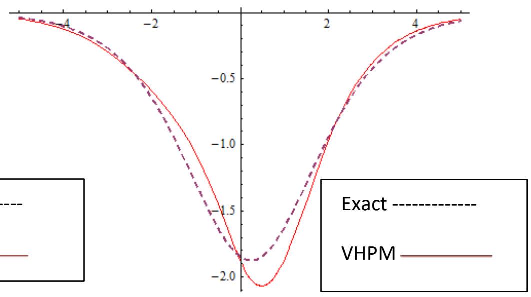

Fig 5.10: 2D plot for exact and VHPM solution at $t = 0.1$ for mDP equation

6. Conclusion

OHAM formulation is tested upon Modified Camassa-Holm and Degasperis-Procesi equations. The solutions obtained by proposed method are compared with the VHPM [1] and exact solution. It is shown that the results obtained by OHAM have good agreement with exact solution than that of VHPM. We concluded that the proposed method is simple, effective and reliable for solving nonlinear equations.

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