WEIGHTED SEMIGROUP MEASURE ALGEBRA AS A WAP-ALGEBRA

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A Banach algebra $\mathfrak{A}$ for which the natural embedding from $\mathfrak{A}$ into $WAP(\mathfrak{A})^*$ is bounded below is called a WAP-algebra. We study those conditions under which the weighted semigroup measure algebra $M_b(S, \omega)$ is a WAP-algebra or a dual Banach algebra. In particular, we show that the semigroup measure algebra $M_b(S)$ is a WAP-algebra (resp. dual Banach algebra) if and only if $wap(S)$ separates the points of $S$ (resp. $S$ is compactly cancellative semigroup). Some older results, in the case where $S$ is discrete, are also improved.

Keywords: WAP-algebra, dual Banach algebra, Arens regularity, weak almost periodicity.


1. Introduction and Preliminaries

The dual $\mathfrak{A}^*$ of a Banach algebra $\mathfrak{A}$ can be turned into a Banach $\mathfrak{A}$-module equipped with the natural module operations

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle \quad (a, b \in \mathfrak{A}, f \in \mathfrak{A}^*).$$

A dual Banach algebra is a Banach algebra $\mathfrak{A}$ enjoying a predual $\mathfrak{A}_*$ such that $\mathfrak{A}_*$, as a Banach space is a closed $\mathfrak{A}$-submodule of $\mathfrak{A}^*$; or equivalently, the multiplication on $\mathfrak{A}$ is separately weak*-continuous. It should be remarked that the predual of a dual Banach algebra need not be unique, in general (see [5, 10]); so we usually point to the involved predual of a dual Banach algebra.

A functional $f \in \mathfrak{A}^*$ is said to be weakly almost periodic if $\{f \cdot a : \|a\| \leq 1\}$ is relatively weakly compact in $\mathfrak{A}^*$. We denote by $WAP(\mathfrak{A})$ the set of all weakly...
almost periodic elements of $\mathfrak{A}^\ast$. It is easy to verify that, $WAP(\mathfrak{A})$ is a (norm) closed subspace of $\mathfrak{A}^\ast$. It is known that the multiplication of a Banach algebra $\mathfrak{A}$ has two natural but, in general, different extensions (called Arens products) to the second dual $\mathfrak{A}^{\ast\ast}$ each turning $\mathfrak{A}^{\ast\ast}$ into a Banach algebra. When these products are equal, $\mathfrak{A}$ is said to be (Arens) regular. It can be verified that $\mathfrak{A}$ is Arens regular if and only if $WAP(\mathfrak{A}) = \mathfrak{A}^\ast$. Further information for the Arens regularity of Banach algebras can be found in [5, 6].

WAP-algebras, as a generalization of the Arens regular algebras, have been introduced and intensively studied in [9]. A Banach algebra $\mathfrak{A}$ for which the natural embedding $x \mapsto \hat{x}$ of $\mathfrak{A}$ into $WAP(\mathfrak{A})^\ast$, where $\hat{x}(f) = f(x)$ for $f \in WAP(\mathfrak{A})$, is bounded below, is called a WAP-algebra. When $\mathfrak{A}$ is either Arens regular or a dual Banach algebra, then natural embedding of $\mathfrak{A}$ into $WAP(\mathfrak{A})^\ast$ is an isometry [16, Corollary 4.6]. It has also known that $\mathfrak{A}$ is a WAP-algebra if and only if it admits an isomorphic representation on a reflexive Banach space. Convolution group algebras are the main examples of WAP-algebras; however; they are neither dual nor Arens regular in general, see [17]. For more information about WAP-algebras one may consult to the impressive paper [9].

The main aim of this paper is to investigate those conditions under which the weighted measure algebra $M_b(S, \omega)$ is either a WAP-algebra or a dual Banach algebra, where $\omega$ is a weight on a locally compact semigroup $S$.

First we recall some preliminaries about the (weighted) measure algebras. Let $S$ be a locally compact semitopological semigroup. Let $M_b(S)$ be the space of all complex regular Borel measures on $S$, which is known as a Banach algebra under the convolution product $*$ defined by the equation $\langle \mu * \nu, f \rangle = \int_S \int_S f(xy)d\mu(x)d\nu(y)$ ($f \in C_0(S)$). Our mean by a weight $\omega$ on $S$ is a Borel measurable function $\omega : S \rightarrow (0, \infty)$ such that $\omega(st) \leq \omega(s)\omega(t)$, $(s, t \in S)$. For $\mu \in M_b(S)$ we define $(\mu \omega)(E) = \int_E \omega d\mu$, $(E \subseteq S$ is Borel set). If $\omega \geq 1$, then $M_b(S, \omega) = \{ \mu \in M_b(S) : \mu \omega \in M_b(S) \}$ is known as a Banach algebra which is called the weighted semigroup measure algebra (see [6, 12, 13, 14]). In the case where $S$ is discrete we write $\ell_1(S, \omega)$ instead of $M_b(S, \omega)$ and $C_0(S, 1/\omega)$ instead of $C_0(S, 1/\omega)$. Then the Banach algebra $\ell_1(S, \omega) = \{ f : f = \sum_{s \in S} f(s)\delta_s, \quad \|f\|_{1, \omega} = \sum_{s \in S} |f(s)|\omega(s) < \infty \}$ (where, $\delta_s \in \ell_1(S, \omega)$ is the point mass at $s$) equipped with the convolution product is called a weighted semigroup algebra. We also suppress 1 from the notation whenever $w = 1$.

Let $B(S)$ denote the space of all bounded Borel measurable functions on $S$. Set $B(S, 1/\omega) = \{ f : S \rightarrow \mathbb{C} : f/\omega \in B(S) \}$. Let $f \in C(S, 1/\omega)$ then $f$ is called $\omega$-weakly almost periodic if the set $\{ R_{1/\omega_s} f : s \in S \}$ is relatively weakly compact in $C(S)$. The set of all $\omega$-weakly almost periodic functions on $S$ is denoted by
The space \( \text{wap}(S) \) of \( 1 \)-weakly almost periodic functions on \( S \) is a \( C^* \)-subalgebra of \( C(S) \) and its character space \( S^{\text{wap}} \), endowed with the Gelfand topology, enjoys a (Arens type) multiplication that turns it into a compact semitopological semigroup. Many other properties of \( \text{wap}(S) \) and its inclusion relations among other function algebras are completely explored in [3].

The paper is organized as follows. In section 2 we study the weighted measure algebra \( M_b(S, \omega) \) from the dual Banach algebra point of view. In this respect, we shall show that, \( M_b(S, \omega) \) is a dual Banach algebra with respect to the predual \( C_0(S, 1/\omega) \) if and only if for all compact subsets \( F \) and \( K \) of \( S \), the maps \( \frac{\chi_{F^{-1}K}}{\omega} \) and \( \frac{\chi_{KF^{-1}}}{\omega} \) vanishes at infinity. This extends an earliear result of Abolghasemi, Rejali, and Ebrahimi Vishki [1]. We also conclude that, the measure algebra \( M_b(S) \) is a dual Banach algebra with respect to \( C_0(S) \) if and only if \( S \) is a compactly cancellative semigroup. The later result is an extension of a known result of Dales, Lau and Strauss [7, Theorem 4.6] stating that, \( \ell_1(S) \) is a dual Banach algebra with respect to \( c_0(S) \) if and only if \( S \) is a weakly cancellative semigroup.

Section 3 is devoted to the study of \( M_b(S, \omega) \) from the \( \text{WAP} \)-algebra point of view. We shall prove that, \( M_b(S, \omega) \) is a \( \text{WAP} \)-algebra if and only if the evaluation map \( \epsilon : S \to \tilde{X} \) is one to one, where \( \tilde{X} = M_M(\text{wap}(S, 1/\omega)) \). The main result of this section is that \( M_b(S) \) is \( \text{WAP} \)-algebra if and only if \( \text{wap}(S) \) separate the points of \( S \). We conclude the paper with some illuminating examples.

2. Semigroup Measure Algebras as Dual Banach Algebras

It is known that the (discrete) semigroup algebra \( \ell_1(S) \) is a dual Banach algebra with respect to \( c_0(S) \) if and only if \( S \) is a weakly cancellative semigroup, see [7, Theorem 4.6]. This result has been extended to the weighted semigroup algebras; [1, 8]. In this section we extend the aforementioned results to the non-discrete case. More precisely, we provide some necessary and sufficient conditions that the measure algebra \( M_b(S, \omega) \) becomes a dual Banach algebra with respect to the predual \( C_0(S, 1/\omega) \).

Let \( F \) and \( K \) be nonempty subsets of a semigroup \( S \) and \( s \in S \). We set \( s^{-1}F = \{ t \in S : st \in F \} \), and \( Fs^{-1} = \{ t \in S : ts \in F \} \). We also write \( s^{-1}t \) for \( s^{-1}\{t \} \), \( FK^{-1} \) for \( \cup_{s \in K} Fs^{-1} \) and \( K^{-1}F \) for \( \cup_{s \in K} s^{-1}F \).

A semigroup \( S \) is called left (respectively, right) zero semigroup if \( xy = x \) (respectively, \( xy = y \)), for all \( x, y \in S \). A semigroup \( S \) is called zero semigroup if there exist \( z \in S \) such that \( xy = z \) for all \( x, y \in S \). A semigroup \( S \) is said to be left (respectively, right) weakly cancellative semigroup if \( s^{-1}F \) (respectively, \( Fs^{-1} \)) is finite for each \( s \in S \) and each finite subset \( F \) of \( S \). A semigroup \( S \) is said to be weakly cancellative semigroup if it is both left and right weakly cancellative semigroup.
A semi-topological semigroup $S$ is said to be compactly cancellative semigroup if for every compact subsets $F$ and $K$ of $S$ the sets $F^{-1}K$ and $KF^{-1}$ are compact set.

The following lemma needs a routine argument.

**Lemma 2.1.** Let $S$ be a topological semigroup. For every compact subsets $F$ and $K$ of $S$ the sets $F^{-1}K$ and $KF^{-1}$ are closed.

In the next result we study $M_0(S, \omega)$ from the dual Banach algebra point of view.

**Theorem 2.1.** Let $S$ be a locally compact topological semigroup and $\omega$ be a continuous weight on $S$. Then the measure algebra $M_0(S, \omega)$ is a dual Banach algebra with respect to the predual $C_0(S, 1/\omega)$ if and only if for all compact subsets $F$ and $K$ of $S$, the maps $\frac{X_{F^{-1}K}}{\omega}$ and $\frac{X_{KF^{-1}}}{\omega}$ vanishes at infinity.

**Proof.** Suppose that $M_0(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$ and let $\varepsilon > 0$. Let $K, F$ be nonempty compact subsets of $S$ with a net $(x_\alpha)$ in $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$. Let $C_0^+(S)$ denote the non-negative continuous functions with compact support on $S$ and set $C_0^+(S, 1/\omega) = \{f \in C_0(S, 1/\omega) : f/\omega \in C_0^+(S)\}$. Since $\omega$ is continuous we may choose $f \in C_0^+(S, 1/\omega)$ with $f(K) = 1$.

There exists a net $(t_\alpha) \in F$ such that $t_\alpha x_\alpha \in K$ and the compactness of $F$ guarantees the existence of a subnet $(t_\gamma)$ of $(t_\alpha)$ such that $t_\gamma \to t_0$ for some $t_0 \in S$. Indeed, since for each $s \in S$,

$$\lim_{\gamma} \left(\frac{\delta_{t_\gamma} f}{\omega}\right)(s) = \lim_{\gamma} \frac{f(t_\gamma s)}{\omega(s)} = \frac{f(t_0 s)}{\omega(s)} = \delta_{t_0} \frac{f}{\omega}(s),$$

there exists a $\gamma_0$ such that

$$\{t \in \bigcup_{\gamma \geq \gamma_0} t^{-1}_\gamma K : 1/\omega(t) \geq \varepsilon\} \subseteq \bigcup_{\gamma \geq \gamma_0} \{r \in S : \left(\frac{\delta_{t_\gamma} f}{\omega}\right)(r) \geq \varepsilon\} \subseteq \{r \in S : \left(\frac{\delta_{t_\gamma} f}{\omega}\right)(r) \geq \frac{\varepsilon}{2}\}.$$

Let $H = \{t_\gamma : \gamma \geq \gamma_0\} \cup \{t_0\}$. Then

$$\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\} = \{t \in \bigcup_{\gamma \geq \gamma_0} t^{-1}_\gamma K \cup t_0^{-1}K : 1/\omega(t) \geq \varepsilon\}$$

as a closed subset of $\{r \in S : \left(\frac{\delta_{t_\gamma} f}{\omega}\right)(r) \geq \frac{\varepsilon}{2}\}$ is compact. It follows that the net $(x_\gamma)$ in $\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\}$ has a convergent subnet. Thus $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$ is compact and that $\frac{X_{F^{-1}K}}{\omega}$ vanishes at infinity. Similarly $\frac{X_{KF^{-1}}}{\omega}$ vanishes at infinity.

The proof of sufficiency can be adopted from [1, Proposition 3.1]. Let $f \in C_0(S, 1/\omega)$, $\mu \in M_0(S, \omega)$ and $\varepsilon > 0$ be arbitrary. Then there exist compact subsets $F$ and $K$ of $S$ such that $|\frac{f}{\omega}(s)| < \varepsilon$ for all $s \not\in K$ and $|\mu \omega|(S \setminus F) < \varepsilon$. Let
Let \( s \notin \{ t \in F^{-1}K : \omega(t) \leq \frac{1}{2} \} \). Then

\[
\left| \frac{\mu, f}{\omega}(s) \right| = \left| \int_S \frac{f(ts)}{\omega(s)} d\mu(t) \right| \leq \int_F \frac{f(ts)}{\omega(s)} d\mu(t) + \int_{S \setminus F} \frac{f(ts)}{\omega(s)} d\mu(t) \\
\leq \int_F \left| \frac{f(ts)}{\omega(ts)} \right| \omega(t) d\mu(t) + \int_{S \setminus F} \left| \frac{f(ts)}{\omega(ts)} \right| \omega(t) d\mu(t) \\
\leq \varepsilon \int_S \omega(t) d\mu(t) + \| f \|_{\omega, \infty} \int_{S \setminus F} \omega(t) d\mu(t) \leq \varepsilon \| \mu \|_\omega + \varepsilon \| f \|_{\omega, \infty}.
\]

That is, \( \mu, f \in C_0(S, 1/\omega) \) and so \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \).

As immediate consequences of Theorem 2.1 we have the next corollary.

**Corollary 2.1.** Let \( S \) be a locally compact topological semigroup.

1. The measure algebra \( M_b(S) \) is a dual Banach algebra with respect to \( C_0(S) \) if and only if \( S \) is compactly cancellative.
2. If \( M_b(S) \) is a dual Banach algebra with respect to \( C_0(S) \) then \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \).

Applying Theorem 2.1 for a discrete semigroup, we arrive at the next result.

**Corollary 2.2** ([1, Theorem 2.2]). For a semigroup \( S \), the weighted semigroup algebra \( \ell_1(S, \omega) \) is a dual Banach algebra with respect to the predual \( c_0(S, 1/\omega) \) if and only if the maps \( \frac{X_{t^{-1} \omega}}{\omega} \) and \( \frac{X_{s^{-1} \omega}}{\omega} \) are in \( c_0(S) \) for all \( s, t \in S \).

We have also the next result as an application of Theorem 2.1.

**Corollary 2.3.** Let \( S \) be either a left zero, a right zero or a zero locally compact semigroup. Then there exists a weight \( \omega \) on \( S \) such that \( M_b(S, \omega) \) is a dual Banach algebra with respect to \( C_0(S, 1/\omega) \) if and only if \( S \) is \( \sigma \)-compact.

**Proof.** Let \( K \) and \( F \) be compact subsets of \( S \). It can be readily verified that in either cases (being left zero, right zero or zero) the sets \( F^{-1}K \) and \( KF^{-1} \) are either empty or the whole \( S \). For each \( m \in \mathbb{N} \) we set \( S_m = \{ t \in F^{-1}K : \omega(t) \leq m \} = \{ t \in S : \omega(t) \leq m \} \). Then \( S = \bigcup_{m \in \mathbb{N}} S_m \) and so \( S \) is \( \sigma \)-compact. For the converse let \( S = \bigcup_{n \in \mathbb{N}} S_n \) as a disjoint union of compact sets and let \( z \) be a (left or right) zero for \( S \). Define \( \omega(z) = 1 \) and \( \omega(x) = 1 + n \) for \( x \in S_n \) then \( \omega \) is a weight on \( S \) and \( M_b(S, \omega) \) is a dual Banach algebra.

**Examples 2.1.**
1. The set \( S = \mathbb{R}^+ \times \mathbb{R} \) equipped with the multiplication

\[
(x, y)(x', y') = (x + x', y') \quad ((x, y), (x', y') \in S)
\]
and the weight $\omega(x,y) = e^{-x} (1 + |y|)$ is a weighted semigroup. Set $F = [a,b] \times [c,d]$ and $K = [e,f] \times [g,h]$. Then $F^{-1}K = [e-b, f-a] \times [g, h]$ and $KF^{-1} = \begin{cases} [e-b, f-a] \times \mathbb{R} & \text{if } [c,d] \cap [g, h] \neq \emptyset \\ \emptyset & \text{if } [c,d] \cap [g, h] = \emptyset. \end{cases}$

Thus $M_0(S)$ is not a dual Banach algebra by Corollary 2.1 (1). However, for all compact subsets $F$ and $K$ of $S$, the maps $\frac{XF^{-1}K}{\omega}$ and $\frac{KF^{-1}}{\omega}$ vanishes at infinity. So $M_0(S, \omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$. This shows that the converse of Corollary 2.1 (2) may not be valid.

(2) For the semigroup $S = [0, \infty)$ endowed with the zero multiplication, neither $M_0(S)$ nor $\ell_1(S)$ is a dual Banach algebra. In fact, $S$ is neither compactly nor weakly cancellative semigroup.

3. Semigroup Measure Algebras as WAP-Algebras

In this section we study some conditions under which the weighted measure algebra $M_0(S, \omega)$ is a WAP-algebra. First, we provide some preliminaries.

**Definition 3.1.** Let $\tilde{F}$ be a linear subspace of $B(S, 1/\omega)$, and let $\tilde{F}_r$ denote the set of all real-valued members of $\tilde{F}$. A mean on $\tilde{F}$ is a linear functional $\tilde{\mu}$ on $\tilde{F}$ with the property that $\inf_{s \in S} \overline{\mathcal{L}}(s) \leq \tilde{\mu}(f) \leq \sup_{s \in S} \overline{\mathcal{L}}(s) \quad (f \in \tilde{F}_r)$. The set of all means on $\tilde{F}$ is denoted by $M(\tilde{F})$. If $\tilde{F}$ is also an algebra with the multiplication given by $f \circ g := (f,g)/\omega \quad (f, g \in \tilde{F})$ and if $\tilde{\mu} \in M(\tilde{F})$ satisfies $\tilde{\mu}(f \circ g) = \tilde{\mu}(f)\tilde{\mu}(g) \quad (f, g \in \tilde{F})$, then $\tilde{\mu}$ is said to be multiplicative. The set of all multiplicative means on $\tilde{F}$ will be denoted by $MM(\tilde{F})$.

Let $\tilde{F}$ be a conjugate closed, linear subspace of $B(S, 1/\omega)$ such that $\omega \in \tilde{F}$.

(i) For each $s \in S$ define $\epsilon(s) \in M(\tilde{F})$ by $\epsilon(s)(f) = (f/\omega)(s) \quad (f \in \tilde{F})$. The mapping $\epsilon : S \longrightarrow M(\tilde{F})$ is called the evaluation mapping. If $\tilde{F}$ is also an algebra, then $\epsilon(S) \subseteq MM(\tilde{F})$.

(ii) Let $\tilde{X} = M(\tilde{F})$ (resp. $\tilde{X} = MM(\tilde{F})$, if $\tilde{F}$ is a subalgebra) be endowed with the relative weak$^*$ topology. For each $f \in \tilde{F}$ the function $\hat{f} \in C(\tilde{X})$ is defined by $\hat{f}(\tilde{\mu}) := \tilde{\mu}(f) \quad (\tilde{\mu} \in \tilde{X})$.

Furthermore, we define $\hat{\tilde{F}} := \{ \hat{f} : f \in \tilde{F} \}$.

**Remark 3.1.** (i) The mapping $f \longrightarrow \hat{f} : \tilde{F} \longrightarrow C(\tilde{X})$ is clearly linear and multiplicative if $\tilde{F}$ is an algebra and $\tilde{X} = MM(\tilde{F})$. Also it preserves complex
conjugation, and is an isometry, since for any \( f \in \widetilde{\mathcal{F}} \)

\[
\| \hat{f} \| = \sup \{ |\mu(f) / \omega| : \mu \in X \} \leq \sup \{ |\mu f / \omega| : \mu \in C(X)^* \}, \| \mu \| \leq 1 \}
\]

\[
= \| f / \omega \| = \sup \{ f(s) / \omega : s \in S \} = \sup \{ |\epsilon(s)(f)| : s \in S \}
\]

\[
= \sup \{ |f(\epsilon(s))| : s \in S \} \leq \| \hat{f} \|,
\]

where \( X = M(\mathcal{F}) \) and \( \mathcal{F} = \{ f/\omega : f \in \widetilde{\mathcal{F}} \} \). Note that \( \hat{f}(\epsilon(s)) = \epsilon(s)(f) = (\frac{f}{\omega})(f \in \widetilde{F}, s \in S) \). This identity may be written in terms of dual map \( \epsilon^* : C(X) \rightarrow C(S, \omega) \) as \( \epsilon^*(\hat{f}) = f \) for \( f \in \widetilde{F} \).

(ii) Let \( \mathcal{F} \) be a conjugate closed linear subspace of \( B(S, 1/\omega) \), containing \( \omega \). Then \( M(\mathcal{F}) \) is convex and weak* compact, \( co(\epsilon(S)) \) is weak* dense in \( M(\mathcal{F}) \). \( \mathcal{F}^* \) is the weak* closed linear span of \( \epsilon(S) \), \( \epsilon : S \rightarrow M(\mathcal{F}) \) is weak* continuous, and if \( \mathcal{F} \) is also an algebra, then \( MM(\mathcal{F}) \) is weak* compact and \( \epsilon(S) \) is weak* dense in \( MM(\mathcal{F}) \).

(iii) Let \( \mathcal{F} \) be a \( C^* \)-subalgebra of \( B(S, 1/\omega) \), containing \( \omega \). If \( \tilde{X} \) denotes the space \( MM(\mathcal{F}) \) with the relative weak* topology, and if \( \epsilon : S \rightarrow \tilde{X} \) denotes the evaluation mapping, then the mapping \( f \mapsto \tilde{f} : \mathcal{F} \rightarrow \tilde{X} \) is an isometric isomorphism with the inverse \( \epsilon^* : C(\tilde{X}) \rightarrow \mathcal{F} \).

Let \( \mathcal{F} = wap(S, 1/\omega) \). Then \( \mathcal{F} \) is a \( C^* \)-subalgebra of \( WAP(M_b(S, \omega)) \), see \[11, Theorem 1.6, Theorem 3.3\]. Set \( \tilde{X} = MM(\mathcal{F}) \). By the above remark \( wap(S, 1/\omega) \cong C(\tilde{X}) \) and so

\[
M_b(\tilde{X}) \cong C(\tilde{X})^* \cong wap(S, 1/\omega)^* \subset WAP(M_b(S, \omega))^*.
\]

Let \( \epsilon : S \rightarrow \tilde{X} \) be the evaluation mapping. We also define

\[
\tilde{\epsilon} : M_b(S, \omega) \rightarrow M_b(\tilde{X}) \text{ by } \langle \tilde{\epsilon}(\mu), f \rangle = \int_S f \omega d\mu
\]

for \( f \in wap(S, 1/\omega) \cong C(\tilde{X}) \). Then for every Borel set \( B \) in \( \tilde{X} \), we have \( \tilde{\epsilon}(\mu)(B) = (\mu \omega)(\epsilon^{-1}(B)) \). In particular, \( \tilde{\epsilon}(\frac{\delta}{\omega(x)}) = \delta_{\epsilon(x)} \).

The next theorem is the main result of this section.

**Theorem 3.1.** For every weighted locally compact semi-topological semigroup \((S, \omega)\) the following statements are equivalent:

1. The map \( \epsilon : S \rightarrow \tilde{X} \) is one to one, where \( \tilde{X} = MM(wap(S, 1/\omega)) \);
2. \( \epsilon : M_b(S, \omega) \rightarrow M_b(\tilde{X}) \) is an isometric isomorphism;
3. \( M_b(S, \omega) \) is a WAP-algebra.

**Proof.** (1) \( \Rightarrow \) (2). Take \( \mu \in M_b(S, \omega) \), say \( \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \), where \( \mu_j \in M_b(S, \omega)^+ \) for each \( j = 1, 2, 3, 4 \). Set \( \nu_j = \epsilon(\mu_j) \in M_b(\tilde{X})^+ \), and \( \nu = \epsilon(\mu) = \nu_1 - \nu_2 + i(\nu_3 - \nu_4) \). Take \( \delta > 0 \). For each \( j \), there exists a Borel set \( B_j \) in \( \tilde{X} \) such that \( \nu_j(B) \geq 0 \) for each Borel subset \( B \) of \( B_j \) with \( \sum_{j=1}^4 \nu_j(B_j) > ||\nu|| - \delta \). In
For a locally compact semi-topological semigroup $\nu - 1$ if and only if $\nu$ is a WAP-algebra if and only if $\ell$ maps the extreme points of the unit ball onto the extreme points of the unit ball, $\{\epsilon \}$. Since $||| - \delta \leq \sum_{j=1}^{4} |||\nu_j||| - \delta \leq \sum_{j=1}^{4} \nu_j(D_j \cap K) = \sum_{j=1}^{4} \nu_j(D_j \cap K)$. Set $B_j = D_j \cap K$. Then the sets $B_1, B_2, B_3, B_4$ are pairwise disjoint.

For each $j$, set $C_j = (\epsilon)^{-1}(B_j)$, a Borel set in $S$. Then $\delta_j(\omega)(C_j) = \nu_j(B_j)$. Since $\epsilon$ is injection, the sets $C_1, C_2, C_3, C_4$ are pairwise disjoint, and so $||\mu||_\omega \geq \sum_{j=1}^{4} \mu(\omega(C_j)) \geq \sum_{j=1}^{4} (\mu_j(\omega)(C_j) = \sum_{j=1}^{4} \nu_j(B_j) > ||\nu|| - \delta$. This holds for each $\delta > 0$, so $||\mu||_\omega \geq ||\nu||$. A similar argument shows that $||\mu||_\omega \leq ||\nu||$. Thus $||\mu||_\omega = ||\nu||$.

$(2) \Rightarrow (1)$. Let $P(S, \omega)$ denote the subspace of all probability measures of $M_b(S, \omega)$ and $ext(P(S, \omega))$ the extreme points of unit ball of $P(S, \omega)$. Then $ext(P(S, \omega)) = \{\frac{\delta_x}{\omega(x)} : x \in S\} \cong S$ and $ext(P(\bar{X})) \cong \bar{X}$, see [4, p.151]. By the injectivity of $\bar{\epsilon}$, it maps the extreme points of the unit ball onto the extreme points of the unit ball, thus $\epsilon : S \to \bar{X}$ is one to one.

$(2) \Rightarrow (3)$. Since $\bar{X}$ is compact, $M_b(\bar{X})$ is a dual Banach algebra with respect to $C(\bar{X})$, so it has an isometric representation $\psi$ on a reflexive Banach space $E$, see [9]. In the following commutative diagram,

$$M_b(S, \omega) \xrightarrow{\xi} M_b(\bar{X}) \xrightarrow{\phi} B(E) \xrightarrow{\psi}$$

If $\bar{\epsilon}$ is isometric, then so is $\phi$. Thus $M_b(S, \omega)$ has an isometric representation on a reflexive Banach space $E$ if $\bar{\epsilon}$ is an isometric isomorphism. So $M_b(S, \omega)$ is a WAP-algebra if $\bar{\epsilon}$ is an isometric isomorphism.

$(3) \Rightarrow (1)$. Let $M_b(S, \omega)$ be a WAP-algebra. Since $\ell_1(S, \omega)$ is a norm closed sub-algebra of $M_b(S, \omega)$, the weighted semigroup algebra $\ell_1(S, \omega)$ is a WAP-algebra. Using the double limit criterion, it is easy to check that $wap(S, 1/\omega) = WAP(\ell_1(S, \omega))$ (see also [11, Theorem 3.7]) where we treat $\ell^\infty(S, 1/\omega)$ as an $\ell_1(S, \omega)$-bimodule. Then $\bar{\epsilon} : \ell_1(S, \omega) \to wap(S, 1/\omega)^*$ is an isometric isomorphism. Since $wap(S, 1/\omega)$ is a $C^*$-algebra, as $(2) \Rightarrow (1)$, $\epsilon : S \to \bar{X}$ is one to one.

**Corollary 3.1.** For a locally compact semi-topological semigroup $S$, $M_b(S, \omega)$ is a WAP-algebra if and only if $\ell_1(S, \omega)$ is a WAP-algebra.

For $\omega = 1$, it is clear that $\bar{X} = S^{wap}$, and the map $\epsilon : S \to S^{wap}$ is one to one if and only if $wap(S)$ separates the points of $S$, see [3].
Corollary 3.2. For a locally compact semi-topological semigroup $S$, the following statements are equivalent:

1. $M_b(S)$ is a WAP-algebra;
2. $\ell_1(S)$ is a WAP-algebra;
3. The evaluation map $\epsilon : S \rightarrow S^{wap}$ is one to one;
4. $wap(S)$ separates the points of $S$.

Illustrating our results, we conclude the paper with the following examples.

Examples 3.1.

(i) We examine the semigroup algebra $\ell_1(S)$ for $S = \mathbb{N}$ equipped with various multiplications. When $S$ is equipped with the min multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra, while, is not neither Arens regular nor a dual Banach algebra. If we furnish $S$ with the max multiplication, then $\ell_1(S)$ is a dual Banach algebra (and so a WAP-algebra) which is not Arens regular. If we change the multiplication of $S$ to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

(ii) Let $S$ be the set of all sequences with $0, 1$ values. We equip $S$ with pointwise multiplication. We denote by $e_n$ the characteristic of $n$. Let $s = \{x_n\} \in S$, and let $F_w(S)$ be the set of all elements of $S$ such that $x_k = 0$ for only finitely index $k$. It is easy to see that $F_w(S)$ is countable. Let $\{s_i\} = \{s_1, s_2, \cdots \}$ be an infinite set. Put $a_n = s + \sum_{j=1}^{n} e_{k_j}$ and $b_m = s + \sum_{i=m}^{\infty} e_{k_i}$. Then

$$a_n b_m = \begin{cases} \sum_{j=m}^{n} e_{k_j} + s & \text{if } m \leq n \\ s & \text{if } m > n. \end{cases}$$

Thus

$$g(s) = \lim_{n} \lim_{m} g(a_n b_m) = \lim_{m} \lim_{n} g(a_n b_m) = \lim_{m} g(s + \sum_{i=m}^{\infty} e_{k_i}) = 0.$$ 

Indeed,

$$wap(S) = \{ f \in \ell^\infty(S) : f = \sum_{i=1}^{\infty} f(s_i) \chi_{s_i}, \ s_i \in F_w(S) \} \oplus \mathbb{C}$$

It is also clear that $F_w(S)$ is the subsemigroup of $S$ with $wap(F_w(S)) = \ell^\infty(F_w(S))$. So $\ell_1(F_w(S))$ is Arens regular. Let $T$ consist of those sequences $s = \{x_n\} \in S$. 

such that \( x_i = 0 \) for infinitely index \( i \), then \( T \) is a subsemigroup of \( S \) and \( \text{wap}(T) = \mathbb{C} \). Since \( \epsilon_1 : T \rightarrow \text{S^{wap}} \) is not one to one, \( \ell_1(S) \) is not a WAP-algebra. This shows that \( \ell_1(S) \) need not be a WAP-algebra.

(iii) If we equip \( S = \mathbb{R}^2 \) with the multiplication \( (x,y),(x',y') = (xx', xy + y') \), then \( M_0(S) \) is not a WAP-algebra. Indeed, every non-constant function \( f \) over \( x \)-axis is not in \( \text{wap}(S) \). Let \( f(0,z_1) \neq f(0,z_2) \) and \( \{ x_m \}, \{ y_m \}, \{ \beta_n \} \) be sequences with distinct elements satisfying the recursive equation

\[
\beta_n x_m + y_m = \frac{m z_1 + n z_2}{m + n}.
\]

Then

\[
\lim_{n} \lim_{m} f((0, \beta_n) \cdot (x_m, y_m)) = \lim_{n} \lim_{m} f(0, \beta_n x_m + y_m) = \lim_{n} \lim_{m} f(0, \frac{m z_1 + n z_2}{m + n}) = f(0, z_1),
\]

and similarly

\[
\lim_{n} \lim_{m} f((0, \beta_n) \cdot (x_m, y_m)) = f(0, z_2).
\]

Thus the map \( \epsilon : S \rightarrow \text{S^{wap}} \) is not one to one, so \( M_0(S) \) is not a WAP-algebra.

(iv) Let \( S \) be the interval \([\frac{1}{2}, 1]\) with the multiplication \( x \cdot y = \max\{ \frac{1}{2}, xy \} \), where \( xy \) is the ordinary multiplication on \( \mathbb{R} \). Then for each \( s \in S \setminus \{ \frac{1}{2} \} \), \( x \in S \), the set \( x^{-1}s \) is finite. But \( x^{-1} = \left( \frac{1}{2}, \frac{1}{2x} \right) \). Let \( B = \left[ \frac{1}{2}, \frac{3}{4} \right] \). Then for every finite subset \( F \) of \( B \),

\[
\bigcap_{x \in F} x^{-1} \frac{1}{2} \bigcap_{x \in B[F]} x^{-1} \frac{1}{2} = \left[ \frac{2}{3}, \frac{1}{2x_F} \right],
\]

where \( x_F = \max F \). By [15, Theorem 4], \( \chi_{\frac{1}{2}} \notin \text{wap}(S) \). So \( c_0(S \setminus \{ \frac{1}{2} \}) \oplus \mathbb{C} \subseteq \text{wap}(S) \). It can be readily verified that \( \epsilon : \mathbb{S}^{\frac{3}{2}} \rightarrow \text{S^{wap}} \) is one to one, so \( \ell_1(S) \) is a WAP-algebra but \( c_0(S) \not\subseteq \text{wap}(S) \).

(v) Take \( T = (\mathbb{N} \cup \{ 0 \},.) \) with 0 as zero of \( T \) and the multiplication defined by

\[
n.m = \begin{cases} 
n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}
\]

Set \( S = T \times T \) equipped with the pointwise product. Now let \( X = \{(k,0) : k \in T\} \), \( Y = \{(0,k) : k \in T\} \) and \( Z = X \cup Y \). We use the Ruppert criterion [15] to show that \( \chi_z \notin \text{wap}(S) \), for each \( z \in Z \). Let \( B = \{(k,n) : k,n \in T\} \), then \( (k,n)^{-1}(k,0) = \{(k,m) : m \neq n\} = B \setminus \{(k,n)\} \). Thus for each finite subsets
$F$ of $B$, 
\[
\left(\cap\{(k,n)^{-1}(k,0) : (k,n) \in F\}\right) \setminus \left(\cap\{(k,0)(k,n)^{-1} : (k,n) \in F\}\right) = (B \setminus F) \setminus F = B \setminus F
\]
and the last set is infinite. This means that $\chi_{(k,0)} \notin \operatorname{wap}(S)$. Similarly $\chi_{(0,k)} \notin \operatorname{wap}(S)$. Let 
\[
f = \sum_{n=0}^{\infty} f(0,n) \chi_{(0,n)} + \sum_{m=1}^{\infty} f(m,0) \chi_{(m,0)}
\]
be in $\operatorname{wap}(S)$. Then for each fixed $n$ and the sequence $\{(n,k)\}$ in $S$, we have 
\[
\lim_k f(n,k) = \lim_k \lim_l f(n,l,k) = \lim_l \lim_k f(n,l,k) = f(n,0),
\]
which implies that $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. Since $\operatorname{wap}(S)$ can not separate the points of $S$ so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n,m) = 2^n 3^m$ for $(n,m) \in S$. Then $\omega$ is a weight on $S$ such that $\omega \in \operatorname{wap}(S,1/\omega)$. Then the evaluation mapping $\epsilon : S \to \tilde{X}$ is one to one. This means that $\ell_1(S,\omega)$ is a WAP-algebra while $\ell_1(S)$ is not!

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