NEW FIXED POINT RESULTS FOR MULTI-VALUED MAPS VIA MANAGEABLE FUNCTIONS AND AN APPLICATION ON A BOUNDARY VALUE PROBLEM

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In this paper, by using the concepts of α-admissible mappings and manageable functions, we establish some fixed point results for multi-valued-maps in the setting of metric-like spaces. Some examples and an application on a boundary value problem are presented making effective our results.

Keywords: Hausdorff metric-like, fixed point, manageable functions.

1. Introduction and Preliminaries

In 1969, Nadler [19] was the first who generalized the Banach contraction principle for multi-valued mappings. Later, this theorem has been generalized and extended in many directions. The notion of metric-like spaces (also named as dislocated metric spaces) were considered by Hitzler and Seda [13] as a generalization of the notion of partial metric spaces [17]. Many authors proved some (common) fixed point results on (generalized) metric-like spaces. In 2008, Aage and Salunke [1] established some fixed point results in dislocated and dislocated quasi-metric spaces. Recently, Aydi and Karapinar [6] (see also [9]) studied the case of generalized α-ψ-contractions. Later, some best proximity point theorems on metric-like spaces have been presented in [8]. Moreover, Karapinar and Salimi [15] gave some details on dislocated metric spaces to metric spaces. For other related results, see [16, 23, 25]. In what follows, we recall some definitions and results we will need in the sequel.

Definition 1.1. [12, 13] Let $X$ be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a metric-like (or a dislocated metric) on $X$ if for any $x, y, z \in X$, the following conditions hold:

(P1) $\sigma(x, y) = 0 \iff x = y$;
(P2) $\sigma(x, y) = \sigma(y, x)$;
(P3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair $(X, \sigma)$ is then called a metric-like (dislocated metric) space.

Let $(X, \sigma)$ be a metric-like space. A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $\sigma(x, x_n) = \lim_{n \to \infty} \sigma(x, x_n)$. A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if $\lim_{n,m \to \infty} \sigma(x_n, x_m)$ exists and is finite. $(X, \sigma)$ is said to be complete if every Cauchy sequence

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Let \( A \) be a manageable function if it is admissible and satisfies the following conditions:

\[
\lim_{n \to \infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{n, m \to \infty} \sigma(x_n, x_m).
\]

We also have

\[
\sigma(x, x) \leq 2\sigma(x, y) \quad \text{for all } x, y \in X.
\]  

Very recently, Aydi et al. [5] introduced the concept of a Hausdorff metric-like. Let \( CB^\sigma(X) \) be the family of all nonempty, closed and bounded subsets of the metric-like space \((X, \sigma)\), induced by the metric-like \( \sigma \). For \( A, B \in CB^\sigma(X) \) and \( x \in X \), define

\[
\sigma(x, A) = \inf \{ \sigma(x, a) \mid a \in A \}, \quad \delta_\sigma(A, B) = \sup \{ \sigma(a, B) \mid a \in A \}
\]

and

\[
\delta_\sigma(B, A) = \sup \{ \sigma(b, A) \mid b \in B \}.
\]

We have the following results:

**Lemma 1.1.** [5, 7] Let \((X, \sigma)\) be a metric-like space and \( A \) any nonempty set in \((X, \sigma)\), then

\[
\sigma(a, A) = 0 \Rightarrow a \in \bar{A},
\]

where \( \bar{A} \) denotes the closure of \( A \) with respect to the metric-like \( \sigma \). Also, if \( \{x_n\} \) is a sequence in \((X, \sigma)\) that is \( \tau_\sigma \)-convergent to \( x \in X \), then

\[
\lim_{n \to \infty} |\sigma(x_n, A) - \sigma(x, A)| = \sigma(x, x).
\]

Let \((X, \sigma)\) be a metric-like space. For \( A, B \in CB^\sigma(X) \), define

\[
H_\sigma(A, B) = \max \{ \delta_\sigma(A, B), \delta_\sigma(B, A) \}.
\]

We have also some properties of \( H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, \infty) \).

**Proposition 1.1.** [5, 7] Let \((X, \sigma)\) be a metric-like space. For any \( A, B, C \in CB^\sigma(X) \), we have the following:

(i) : \( H_\sigma(A, A) = \delta_\sigma(A, A) = \sup \{ \sigma(a, A) \mid a \in A \} \);

(ii) : \( H_\sigma(A, B) = H_\sigma(B, A) \);

(iii) : \( H_\sigma(A, B) = 0 \) implies that \( A = B \);

(iv) : \( H_\sigma(A, B) \leq H_\sigma(A, C) + H_\sigma(C, B) \).

The mapping \( H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, \infty) \) is called a Hausdorff metric-like induced by \( \sigma \).

The following definition we find it in [2, 18].

**Definition 1.2.** Let \( X \) be a nonempty set and \( T : X \to 2^X \), be a multi-valued mapping. We say that \( T \) is \( \alpha \)-admissible if, for each \( x \in X \) and \( y \in Tx \) with \( \alpha(x, y) \geq 1 \), we have \( \alpha(y, z) \geq 1 \) for all \( z \in Ty \).

We have the following useful lemma.

**Lemma 1.2.** Let \((X, \sigma)\) be a metric-like space, \( B \in CB^\sigma(X) \) and \( c > 0 \). If \( a \in X \) and \( \sigma(a, B) < c \) then there exists \( b = b(a) \in B \) such that \( \sigma(a, b) < c \).

In 2014, Du and Khojasteh [11], introduced a new class of mappings called manageable functions and they obtained some fixed point theorems. Very recently, Hussain et al. [14] established some fixed point theorems for manageable contractions in the setting of metric spaces.

**Definition 1.3.** [11] A manageable function is a mapping \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfying the following conditions:
Let \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the set of manageable functions. We provide the following two examples.

**Example 1.1.** [11] Let \( k \in [0, 1) \). Then \( \eta_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by
\[
\eta_k(t, s) = ks - t
\]
is a manageable function.

**Example 1.2.** Let \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the function defined by
\[
\eta(t, s) = \begin{cases} 
\psi(s) - \varphi(t) & \text{if } (t, s) \in [0, \infty) \times [0, \infty), \\
f(s, t) & \text{otherwise,}
\end{cases}
\]
where \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is any function and \( \psi, \varphi : (0, \infty) \to \mathbb{R} \) are two functions such that \( \psi(t) < t \leq \varphi(t) \) for all \( t > 0 \) and \( \limsup_{r \to t^+} \frac{\psi(r)}{r} < 1 \) for all \( t \in (0, \infty) \). Then, \( \eta \in \widehat{\text{Man}}(\mathbb{R}) \).

Indeed, for any \( s, t > 0 \),
\[
\eta(t, s) = \psi(s) - \varphi(t) < s - t,
\]
so, \((\eta_1)\) holds. Let \( \{t_n\} \) in \((0, \infty)\) be a bounded sequence and \( \{s_n\} \) in \((0, \infty)\) be a non-increasing sequence. Then \( \lim_{n \to \infty} s_n \) exists in \([0, \infty)\). Hence \( \limsup_{n \to \infty} \frac{\psi(s_n)}{s_n} = \limsup_{r \to t^+} \frac{\psi(r)}{r} < 1 \).

Thus, we get
\[
\limsup_{n \to \infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \limsup_{n \to \infty} \frac{\psi(s_n) + t_n - \varphi(t_n)}{s_n} \leq \limsup_{n \to \infty} \frac{\psi(s_n)}{s_n} < 1.
\]

It follows that \((\eta_2)\) holds.

In this paper, we present variant fixed point results for multivalued mappings involving manageable contractions via \( \alpha \)-admissible mappings in the class of metric-like spaces. Some examples and an application on a boundary value problem are given illustrating the presented concepts and obtained results.

2. Fixed points via manageable functions

Now, we state and prove our first main result.

**Theorem 2.1.** Let \((X, \sigma)\) be a complete metric-like space and \( T : X \to CB^\sigma(X) \) be a given multi-valued mapping. Suppose that there exist a manageable function \( \eta \in \widehat{\text{Man}}(\mathbb{R}) \) and \( \alpha : X \times X \to [0, \infty) \) such that
\[
\eta(H_\sigma(Tx, Ty), M_\sigma(x, y)) \geq 0
\]
for all \( x, y \in X \) satisfying \( \alpha(x, y) \geq 1 \), where
\[
M_\sigma(x, y) = \max \{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4}[\sigma(x, Ty) + \sigma(Tx, y)]\}.
\]

Assume that
(i) \( T \) is \( \alpha \)-admissible mapping;
(ii) there exist elements \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \);
if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \to x \) in \((X, \sigma)\) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \).

Then \( T \) has a fixed point.

**Proof.** By assumption (ii), there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \). Clearly, if \( x_1 = x_0 \) or \( x_1 \in Tx_1 \), we conclude that \( x_1 \) is a fixed point of \( T \) and so the proof is finished. Now, we assume that \( x_1 \neq x_0 \) and \( x_1 \notin Tx_1 \). So, \( \sigma(x_0, x_1) > 0 \) and \( \sigma(x_1, Tx_1) > 0 \).

Since \( \alpha(x_0, x_1) \geq 1 \), by (2), we have

\[
\eta(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) \geq 0,
\]

where

\[
M_\sigma(x_0, x_1) = \max\{\sigma(x_0, x_1), \sigma(x_0, Tx_0), \sigma(x_1, Tx_1), \frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, Tx_0)]\}
\]

\[
= \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1), \frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, x_1)]\}.
\]

Note that

\[
\frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, x_1)] \leq \frac{1}{4}[\sigma(x_1, Tx_1) + 3\sigma(x_0, x_1)] \leq \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\}.
\]

Therefore

\[
M_\sigma(x_0, x_1) = \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\}.
\]

Define the function \( \lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
\lambda(t, s) = \begin{cases} \frac{t + \eta(t, s)}{s} & \text{if } t, s > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

By \( \eta_1 \), we have

\[
0 < \lambda(t, s) < 1 \quad \text{for all } t, s > 0.
\]

Also, if \( \eta(t, s) \geq 0 \), then

\[
0 < t \leq s\lambda(t, s) \quad \text{for all } t, s > 0.
\]

From (3) and (4), we get

\[
0 < \lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) < 1.
\]

Since \( \sigma(x_1, Tx_1) > 0 \), by using (6), we have

\[
\sigma(x_1, Tx_1) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1))}} \sigma(x_1, Tx_1).
\]

Lemma 1.2 implies the existence of a point \( x_2 \in Tx_1 \) such that

\[
\sigma(x_1, x_2) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1))}} \sigma(x_1, Tx_1).
\]

From (5), we have

\[
H_\sigma(Tx_0, Tx_1) \leq M_\sigma(x_0, x_1)\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) < M_\sigma(x_0, x_1).
\]

Then

\[
\sigma(x_1, Tx_1) \leq H_\sigma(Tx_0, Tx_1) < M_\sigma(x_0, x_1),
\]

which implies that \( M_\sigma(x_0, x_1) = \sigma(x_0, x_1) \). It follows that

\[
\sigma(x_1, Tx_1) \leq \sigma(x_0, x_1)\lambda(H_\sigma(Tx_0, Tx_1), \sigma(x_0, x_1)).
\]

(8)

Combining (7) and (8), we get

\[
\sigma(x_1, x_2) \leq \sqrt{M_\sigma(Tx_0, Tx_1)} \sigma(x_0, x_1) \sigma(x_0, x_1).
\]
Note that $x_2 \neq x_1$ because $x_1 \notin Tx_1$. If $x_2 \notin Tx_2$, we conclude that $x_2$ is a fixed point of $T$ and so the proof is finished. Now, we assume that $x_2 \notin Tx_2$. Since $T$ is $\alpha$-admissible and $x_2 \in Tx_1$, we have
\[
\alpha(x_1, x_2) \geq 1.
\]
Hence by (2)
\[
\eta(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2)) \geq 0,
\]
where
\[
M_\sigma(x_1, x_2) = \max\{\sigma(x_1, x_2), \sigma(x_1, Tx_1), \sigma(x_2, Tx_2), \frac{1}{4}[\sigma(x_1, Tx_2) + \sigma(x_2, Tx_1)]\}
\]
From (9) and (4), we get 0
\[
\sigma(x_2, Tx_2) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2))}}\sigma(x_1, Tx_2).
\]
Lemma 1.2 implies the existence of a point $x_3 \in Tx_2$ such that
\[
\sigma(x_2, x_3) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2))}}\sigma(x_1, x_2).
\]
Similarly, we get $\alpha(x_2, x_3) \geq 1$ and
\[
\sigma(x_2, x_3) \leq \sqrt{\lambda(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2))}\sigma(x_1, x_2).
\]
Continuing in this fashion, we construct a sequence $\{x_n\}$ in $X$ such that for all $n \geq 1$
\[
\text{(i) } \alpha(x_n, x_{n+1}) \geq 1, \; x_n \notin Tx_n, \; x_n \neq x_{n+1}, \; x_{n+1} \in Tx_n;
\]
\[
\text{(ii) } \sigma(x_n, x_{n+1}) \leq \sqrt{\lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n))}\sigma(x_{n-1}, x_n).
\]
From (9) and (4), we get $0 < \sigma(x_n, x_{n+1}) < \sigma(x_{n-1}, x_n)$ for all $n$, which implies that $\{\sigma(x_{n-1}, x_n)\}$ is a non-increasing sequence of positive reals, then it is convergent. Also, we have
\[
0 < H_\sigma(Tx_{n-1}, Tx_n) < \sigma(x_{n-1}, x_n),
\]
for all $n$, which implies that $\{H_\sigma(Tx_{n-1}, Tx_n)\}$ is a bounded sequence. From (\eta_2), we have
\[
\limsup_{n \to \infty} \lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n)) < 1.
\]
Let
\[
\lambda_n = \sqrt{\lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n))}, \quad \forall n \geq 1.
\]
From (9), we get
\[
\sigma(x_n, x_{n+1}) \leq \lambda_n \sigma(x_{n-1}, x_n), \quad \forall n \geq 1.
\]
By (10), there exist $\gamma \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that
\[
\lambda_n \leq \gamma, \quad \forall n \geq n_0.
\]
Hence, by (11), we get
\[
\sigma(x_n, x_{n+1}) \leq \gamma \sigma(x_{n-1}, x_n), \quad \forall n \geq n_0.
\]
Thus
\[
\sigma(x_n, x_{n+1}) \leq \gamma^{n-n_0+1} \sigma(x_{n_0-1}, x_{n_0}), \quad \forall n \geq n_0.
\]
Now, for $m > n \geq n_0$, we have
\[
\sigma(x_n, x_m) \leq \sum_{i=n}^{m-1} \sigma(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \gamma^{i-n_0+1} \sigma(x_{n_0-1}, x_{n_0}) \leq \sigma(x_{n_0-1}, x_{n_0}) \sum_{i=n}^{\infty} k^i \to 0m as n \to \infty.
\]
Thus, 
\[ \lim_{n,m \to \infty} \sigma(x_n, x_m) = 0. \]
So \( \{x_n\} \) is \( \sigma \)-Cauchy in the complete metric-like space \((X, \sigma)\). Then there exists \( u \in X \) such that
\[ \lim_{n \to \infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m \to \infty} \sigma(x_n, x_m) = 0. \]
We will show that \( u \) is a fixed point of \( T \). If there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} = u \) or \( Tx_{n_k} = Tu \) for all \( k \), then \( Tx_{n_k} = Tu \) for all \( k \). Since \( x_{n_k+1} \in Tx_{n_k} \) for all \( k \), then \( x_{n_k+1} \in Tu \) for all \( k \). Hence \( \sigma(u, Tu) \leq \sigma(u, x_{n_k+1}) \) for all \( k \). Letting \( k \to \infty \), we get \( \sigma(u, Tu) \leq 0 \) and so by Lemma 1.1, we have \( u \in Tu = Tu \).
So, without loss of generality, we may suppose that \( x_n \neq u \) and \( x_n \neq Tu \) for all nonnegative integer \( n \). By assumption \((ii)\), there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, u) \geq 1 \) for all \( k \). Hence by \((2)\), we have
\[ \eta(H_\sigma(Tx_{n(k)}, Tu), M_\sigma(x_{n(k)}, u)) \geq 0, \quad \forall k, \]
where
\[ M_\sigma(x_{n(k)}, u) = \max\{\sigma(x_{n(k)}, u), \sigma(u, Tu), \sigma(x_{n(k)}, Tx_{n(k)}), \frac{1}{4}[\sigma(x_{n(k)}, Tu) + \sigma(u, Tx_{n(k)})]\}. \]
From \((5)\), we have
\[ H_\sigma(Tx_{n(k)}, Tu) \leq \lambda(H_\sigma(Tx_{n(k)}, Tu), M_\sigma(x_{n(k)}, u))M_\sigma(x_{n(k)}, u) < M_\sigma(x_{n(k)}, u), \quad \forall k. \]
Since
\[ \sigma(u, Tu) \leq \sigma(u, x_{n(k)+1}) + \sigma(x_{n(k)+1}, Tu) \leq \sigma(u, x_{n(k)+1}) + H_\sigma(Tx_{n(k)}, Tu), \]
then
\[ \sigma(u, Tu) \leq \sigma(u, x_{n(k)+1}) + H_\sigma(Tx_{n(k)}, Tu) \]
\[ \leq \sigma(u, x_{n(k)+1}) + \lambda(H_\sigma(Tx_{n(k)}, Tu), M_\sigma(x_{n(k)}, u))M_\sigma(x_{n(k)), u), \quad \forall k. \]
Suppose that \( \sigma(u, Tu) > 0 \). Then, there exists \( N \in \mathbb{N} \) such that
\[ M_\sigma(x_{n(k)}, u) = \sigma(u, Tu), \quad \forall k \geq N. \]
It follows that
\[ \sigma(u, Tu) \leq \sigma(u, x_{n(k)+1}) + \lambda(H_\sigma(Tx_{n(k)}, Tu), \sigma(u, Tu))\sigma(u, Tu), \quad \forall k \geq N. \]
Passing to \( \limsup \) as \( k \to \infty \), we get
\[ \sigma(u, Tu) \leq \limsup_{k \to \infty} \sigma(u, x_{n(k)+1}) + \sigma(u, Tu) \limsup_{k \to \infty} \lambda(H_\sigma(Tx_{n(k)}, Tu), \sigma(u, Tu)) \]
\[ < \sigma(u, Tu), \]
which is a contradiction. Hence \( \sigma(u, Tu) = 0 \) and so \( u \in Tu \), that is, \( u \) is a fixed point of \( T \).

By using the same techniques, we may state the following result in the setting of partial metric and metric-like spaces. Mention that the partial Hausdorff metric \( H_p \) written in Theorem 2.2 has been already introduced by Aydi et al. [3].

**Theorem 2.2.** Let \((X, p)\) be a complete partial metric space and \( T : X \to CB^p(X) \) be a given multi-valued mapping. Suppose that there exist a manageable function \( \eta \in \text{Man}(\mathbb{R}) \) and \( \alpha : X \times X \to [0, \infty) \) such that
\[ \eta(H_p(Tx, Ty), N_p(x, y)) \geq 0 \quad (12) \]
for all \(x, y \in X\) satisfying \(\alpha(x, y) \geq 1\), where
\[
N_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.
\]
Assume that

(i) \(T\) is \(\alpha\)-admissible mapping;
(ii) there exist elements \(x_0 \in X\) and \(x_1 \in Tx_0\) such that \(\alpha(x_0, x_1) \geq 1\);
(iii) if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\) in \((X, \sigma)\) as \(n \to \infty\), then there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that \(\alpha(x_{n(k)}, x) \geq 1\) for all \(k\).

Then \(T\) has a fixed point.

**Theorem 2.3.** Let \((X, \sigma)\) be a complete metric-like space and \(T : X \to CB^\alpha(X)\) be a given multi-valued mapping. Suppose that there exist a manageable function \(\eta \in \text{Man}(\mathbb{R})\) and \(\alpha : X \times X \to [0, \infty)\) such that
\[
\eta(H_\sigma(Tx, Ty), \sigma(x, y)) \geq 0
\]
for all \(x, y \in X\) satisfying \(\alpha(x, y) \geq 1\). Assume that

(i) \(T\) is \(\alpha\)-admissible mapping;
(ii) there exist elements \(x_0 \in X\) and \(x_1 \in Tx_0\) such that \(\alpha(x_0, x_1) \geq 1\);
(iii) if \(\{x_n\}\) is a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n\) and \(x_n \to x\) in \((X, \sigma)\) as \(n \to \infty\), then there exists a subsequence \(\{x_{n(k)}\}\) of \(\{x_n\}\) such that \(\alpha(x_{n(k)}, x) \geq 1\) for all \(k\).

Then \(T\) has a fixed point.

**Remark 2.1.** Theorem 2.1 is a generalization of Theorem 2 in [5]. Theorem 2.2 is a generalization of Theorem 2.2 in [4] (when considering one mapping).

We give an example to illustrate the utility of Theorem 2.1.

**Example 2.1.** Let \(X = [0, \infty)\) and \(\sigma : X \times X \to [0, \infty)\) defined by
\[
\sigma(x, y) = x + y, \quad \forall x, y \in X
\]
Then \((X, \sigma)\) is a complete metric-like space. Define the map \(T : X \to CB^\alpha(X)\) by
\[
Tx = \begin{cases} [2, \infty) & \text{if } x > 1 \\ \left[0, \frac{x^2}{1+x^2}\right] & \text{if } x \in [0, 1] \end{cases}
\]
Note that \(Tx\) is bounded and closed for all \(x \in X\) in metric-like space \((X, \sigma)\). Take the applications \(\alpha : X \times X \to [0, \infty)\) and \(\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) defined as follow
\[
\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1] \\ 0 & \text{if not} \end{cases}
\]
\[
\eta(t, s) = rs - t \text{ for all } s, t \in \mathbb{R} \text{ with } r \in \left[\frac{1}{2}, 1\right].
\]
It is easy tho show that \(\eta\) is a manageable function and \(T\) is an \(\alpha\)-admissible mapping.
Let \(x, y \in X\) such that \(\alpha(x, y) \geq 1\). This implies that \(x, y \in [0, 1]\). We shall show that
\[
H_\sigma(Tx, Ty) \leq \frac{1}{2} M_\sigma(x, y), \quad \forall x, y \in [0, 1].
\]
For this, we consider the following cases:

Case 1: \( x = y \). We have

\[
H_\sigma(Tx,Ty) = \max\{\sigma(0,Tx), \sigma(\frac{x^2}{1+x},Tx)\}
\]

\[
= \max\{\min\{\sigma(0,0),\sigma(\frac{x^2}{1+x})\}, \min\{\sigma(0,\frac{y^2}{1+y}),\sigma(\frac{x^2}{1+x},\frac{y^2}{1+y})\}\}\]

\[
= \max\{0, \frac{x^2}{1+x}\} = \frac{x^2}{1+x} \leq x = \frac{1}{2}\sigma(x,x) \leq \frac{1}{2}M_\sigma(x,y).
\]

Case 2: \( x \neq y \). Since \( \sigma \) is symmetric, it suffices to consider the case where \( x > y \). We have

\[
H_\sigma(Tx,Ty) = H_\sigma(\{0, \frac{x^2}{1+x}\}, \{0, \frac{y^2}{1+y}\})
\]

\[
= \max\{\max\{\sigma(0,\{0,\frac{y^2}{1+y}\}),\sigma(\frac{x^2}{1+x},\{0,\frac{y^2}{1+y}\})\},\max\{\sigma(\{0,\frac{x^2}{1+x}\},\sigma(\frac{y^2}{1+y},\{0,\frac{x^2}{1+x}\})\)}\}
\]

\[
= \max\{\sigma(\frac{x^2}{1+x},\{0, \frac{y^2}{1+y}\}),\sigma(\frac{y^2}{1+y},\{0, \frac{x^2}{1+x}\})\}\}
\]

\[
= \max\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\} = \frac{x^2}{1+x} \leq \frac{1}{2} \leq \frac{1}{2}(x+y) = \frac{1}{2}\sigma(x,y) \leq \frac{1}{2}M_\sigma(x,y).
\]

Thus

\[
\eta(H_\sigma(Tx,Ty), M_\sigma(x,y)) = rM_\sigma(x,y) - H_\sigma(Tx,Ty) \geq (r - \frac{1}{2})M_\sigma(x,y) \geq 0.
\]

Moreover, the conditions (ii) and (iii) of Theorem 2.1 are verified. Indeed, for \( x_0 = 0 \) and

\[
x_1 = 0, \text{we have } \alpha(x_0, x_1) = 2 > 1.
\]

Also, if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \)

for all \( n \) and \( x_n \rightarrow x \) in \( (X, \sigma) \) as \( n \rightarrow \infty \), we get \( \{x_n\} \subseteq [0,1] \) and \( |x_n - x| \rightarrow 0 \) as \( n \rightarrow \infty \).

So, \( x \in [0,1] \). Hence \( \alpha(x_n, x) = 2 \geq 1 \) for all \( n \). Then all required hypotheses of Theorem 2.1 are satisfied. Here \( u = 0 \) is a fixed point of \( T \).

3. Fixed point theory in ordered metric-like spaces

The study of fixed points in partially ordered sets was developed in [10,20–22,24]. In this section, we give some fixed point results for multi-valued mappings in the concept of metric-like spaces endowed with a partial order. Finally, we say that \( X \) is said to be regular if the following condition holds: for any sequence \( \{x_n\} \) in \( X \) with \( Tx_n \preceq Tx_{n+1} \), for all \( n \in \mathbb{N} \) and \( x_n \rightarrow x \) in \( (X, \sigma) \), there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( Tx_{n(k)} \preceq Tx \), for all \( k \in \mathbb{N} \).

We also have the following concept.

**Definition 3.1.** Let \((X, \sigma, \preceq)\) be a complete partially ordered metric-like space. Suppose that \( T : X \rightarrow CB^\sigma(X) \) is a multi-valued mapping. Suppose that there exists a manageable function \( \eta \in \operatorname{Man}(\mathbb{R}) \) such that

\[
\eta(H_\sigma(Tx,Ty), M_\sigma(x,y)) \geq 0
\]  

(14)
for all \( x, y \in X \), with \( Tx \preceq Ty \), where

\[
M_\alpha(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{1}{4}[\sigma(x, Ty) + \sigma(Tx, y)]\}.
\]

Assume that

(i) for each \( x \in X \) and \( y \in Tx \) with \( Tx \preceq Ty \), we have \( Ty \preceq Tz \) for all \( z \in Ty \);
(ii) there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( Tx_0 \preceq Tx_1 \);
(iii) \( (X, \preceq) \) is regular.

Then \( T \) has a fixed point.

Proof. Take \( \alpha : X \times X \to [0, \infty) \) such that

\[
\alpha(x, y) = \begin{cases} 
1 & \text{if } Tx \preceq Ty \\
0 & \text{otherwise}. 
\end{cases}
\]

The multi-valued mapping \( T \) is \( \alpha \)-admissible. In fact, if \( x \in X \) and \( y \in Tx \) with \( \alpha(x, y) \geq 1 \), then \( Tx \preceq Ty \). By condition (i), we have \( Ty \preceq Tz \) for all \( z \in Ty \), then \( \alpha(y, z) = 1 \). Also, by (16), \( T \) verifies (2) of Theorem 2.1. Proceeding as in proof of Theorem 2.1, we may construct a sequence \( \{x_n\} \) which converges to \( x \in (X, \sigma) \) and \( x_{n+1} \in Tx_n \) for all \( n \in \mathbb{N} \). Finally, by condition (iii) and Lemma 1.1, we conclude that \( x \) is a fixed point of \( T \).

**Theorem 3.2.** Let \( (X, p, \preceq) \) be a complete partially ordered partial metric space. Suppose that \( T : X \to CB^p(X) \) is a multi-valued mapping. Suppose that there exists a manageable function \( \eta \in \text{Man}(\bar{\mathbb{R}}) \) such that

\[
\eta(H_p(Tx, Ty), N_p(x, y)) \geq 0
\]

for all \( x, y \in X \), with \( Tx \preceq Ty \), where

\[
N_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.
\]

Assume that

(i) for each \( x \in X \) and \( y \in Tx \) with \( Tx \preceq Ty \), we have \( Ty \preceq Tz \) for all \( z \in Ty \);
(ii) there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( Tx_0 \preceq Tx_1 \);
(iii) \( (X, \preceq) \) is regular.

Then \( T \) has a fixed point.

**Theorem 3.3.** Let \( (X, \sigma, \preceq) \) be a complete partially ordered metric-like space. Suppose that \( T : X \to CB^\sigma(X) \) is a multi-valued mapping. Suppose that there exists a manageable function \( \eta \in \text{Man}(\bar{\mathbb{R}}) \) such that

\[
\eta(H_\sigma(Tx, Ty), \sigma(x, y)) \geq 0
\]

for all \( x, y \in X \), with \( Tx \preceq Ty \). Assume that

(i) for each \( x \in X \) and \( y \in Tx \) with \( Tx \preceq Ty \), we have \( Ty \preceq Tz \) for all \( z \in Ty \);
(ii) there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( Tx_0 \preceq Tx_1 \);
(iii) \( (X, \preceq) \) is regular.

Then \( T \) has a fixed point.
4. Application

In this section, we consider the following two-point boundary value problem for second order differential equation:

\[
\begin{aligned}
-\frac{d^2 x}{dt^2} &= f(t, x(t)), \quad t \in [0, 1] \\
x(0) &= x(1) = 0,
\end{aligned}
\] (17)

where \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function. The Green’s function associated to (17) is

\[
\begin{aligned}
G(t, s) &= t(1 - s) \quad \text{if } 0 \leq t \leq s \leq 1 \\
G(s, t) &= s(1 - t) \quad \text{if } 0 \leq s \leq t \leq 1.
\end{aligned}
\] (18)

Let us take \( X = C^2(I) \) where \( I = [0, 1] \) the space of all continuous functions defined on \( I \). Consider the metric-like \( \sigma \) given by

\[
\sigma(x, y) = \|x\|_1 + \|y\|_1
\]

for all \( x, y \in X \), where \( \|u\|_\infty = \max_{t \in [0, 1]} |u(t)| \) for each \( u \in X \). Clearly, \((X, \sigma)\) is complete. Note that \( \sigma \) is not a partial metric.

It is well known that \( x \in C^2(I) \) is a solution of (17) if and only if \( x \in X = C^2(I) \) is a solution of the integral equation

\[
x(t) = \int_0^1 G(t, s)f(s, x(s))ds, \quad t \in I.
\] (19)

Inspired from [6], we state the following result.

**Theorem 4.1.** Suppose the following conditions hold:

- there exists a continuous function \( \beta : I \rightarrow [0, \infty) \) such that
  \[
  |f(s, a)| \leq 8 \beta(s) |a|,
  \]
  for each \( s \in I \) and \( a \in \mathbb{R} \);
- \( \sup_{s \in I} \beta(s) = k \in (0, 1) \).

Then the problem (17) has a solution \( u \in X \).

**Proof.** Consider the mapping \( T : X \rightarrow X \) defined by

\[
Tx(t) = \int_0^1 G(t, s)f(s, x(s))ds,
\]

for all \( x \in X \) and \( t \in I \). Note that problem (17) is equivalent to finding \( u \in X \) that is a fixed point of \( T \). For \( x, y \in X \), we have

\[
|Tx(t)| = \left| \int_0^1 G(t, s)f(s, x(s))ds \right|
\]

\[
\leq \int_0^1 G(t, s)|f(s, x(s))|ds
\]

\[
\leq 8 \int_0^1 G(t, s)\beta(s)|x(s)|ds
\]

\[
\leq 8k \|x\|_\infty \sup_{t \in I} \int_0^1 G(t, s)ds
\]

\[
= k \|x\|_\infty.
\]
New fixed point results for multi-valued maps via manageable functions and an application on a boundary value problem

We have used the fact that for each \( t \in I \), we have \( \int_0^1 G(t, s) ds = -\frac{t^2}{2} + \frac{t}{2} \), and so
\[
\sup_{t \in I} \int_0^1 G(t, s) ds = \frac{1}{8}.
\]
Thus
\[
\|Tx\|_{\infty} \leq k \|x\|_{\infty}.
\]
Equation (20)

Proceeding similarly, one can get
\[
\|Ty\|_{\infty} \leq k \|y\|_{\infty}.
\]
Equation (21)

Summing (20) to (21), we find
\[
\sigma(Tx, Ty) = \|Tx\|_{\infty} + \|Ty\|_{\infty}
\leq k (\|x\|_{\infty} + \|y\|_{\infty})
= k \sigma(x, y) \leq k M(x, y).
\]
Thus
\[
\eta(H_{\sigma}(Tx, Ty), M_{\sigma}(x, y)) = k M(x, y) - H_{\sigma}(Tx, Ty) \geq 0.
\]
So all hypotheses of Theorem 2.1 are satisfied (with \( \alpha(x, y) = 1 \)), and so \( T \) has a fixed point
\[
u \in X,
\]
that is, the problem (17) has a solution \( \nu \in C^2(I) \).

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REFERENCES


