MINIMIZING BANKRUPTCY PROBABILITY OF A LIFE INSURER - SOME ANALYTICAL CONSIDERATIONS

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This paper studies the portfolio problem faced by a life insurance company which sells an annuity, collects fees/premiums for it as a lump-sum, and for solvency considerations invests in a financial market with several investment opportunities. The company has to choose the investment strategy (a portfolio) which minimizes the probability of being unable to pay the annuity before it stops being in force, and this occurs when portfolio value adjusted for annuity present value becomes negative. We manage to solve this stochastic control problem in closed form for constant mortality intensity, and we found out that the optimal investment in stock is decreasing in initial wealth. Moreover if the initial wealth exceeds a threshold the optimal investment in stock is decreasing in mortality intensity. The stochastic mortality intensity case is more involved and we perform duality techniques and asymptotic expansions to tackle it, and established the following qualitative result: in a model with stochastic intensity the probability of company default is higher than in a model with constant intensity fact explained by the extra source of risk (longevity risk) faced by the insurance company.

1. The Introduction

The class of continuous time cohort mortality models has emerged in the recent stream of actuarial/insurance literature, see for instance Biffis (2005), and Luciano and Vigna (2008). Here, the individual death is modelled as the first hitting time of a doubly stochastic Cox process, where mortality intensity follows a stochastic process. This work considers a portfolio problem faced by a life insurance company in the following setting: suppose that the insurance company issues an annuity whose payment rate is K, at time t, i.e., the insurance company has to pay out continuously $KI(t)dt$, where $I(t)$ denotes the proportion of clients who are alive at time t. The fees/premiums are collected by the insurance company as a lump-sum at time $t = 0$, and the payment phase occurs continuously over time. Assume without loss of generality that initially there is a unit mass of clients with the same characteristics (one cohort with specific stochastic mortality intensity) that buys one unit of annuity. The life insurance company can invest in a financial market with several investment opportunities. The problem is to find the investment strategy (portfolio) which minimizes the probability of being unable to pay the annuity before it stops being in force. The insurance company is unable to pay the annuity instalments if the value of its portfolio less annuity payment becomes negative. The time when the company stops operating is modelled as a stopping time, exponentially distributed.

There have been many studies on dynamic portfolio choice problems since the seminal work of Merton (1969) and (1971). We point the interested reader in portfolio optimization

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problems within continuous trading to Karatzas (1989). The study of investment strategies that minimize the probability of bankruptcy goes back to Ferguson (1965) which considers a discrete time setting. The portfolio selection problem in a continuous time Brownian motion paradigm with the objective of minimizing the probability of bankruptcy was studied in Browne (1995). That work shows that the strategy which minimizes portfolio probability of bankruptcy is the strategy which maximizes portfolio's exponential utility at a given time benchmark. Our work is in the paradigm of optimal investment in a framework similar to Browne (1995) but with the insurance aspect added to the dynamic portfolio choice problem.

The optimization problem faced by the insurance company is tackled via dynamic programming and as such the value function is characterized through Hamilton Jacobi Bellman (HJB) equation. We consider first the case of constant mortality intensity and we are able to solve in closed form the HJB equation which in turn yields the value function and the optimal investment strategy. The explicit formulas show that the optimal investment in stock is decreasing in the initial wealth as the company becomes more conservative, i.e., when it has more capital. It is interesting to point out that our plots reveal a linear, decreasing, dependence of the optimal investment in mortality intensity. In fact we proved that when wealth exceeds a threshold the optimal investment in stock is decreasing in mortality intensity. The stochastic mortality intensity is considered and we use duality theory to simplify the HJB equation and work with the dual value function. We performed asymptotic analysis on the the dual value function and found out that the dual value function is higher in the case of stochastic mortality intensity; this in turn says that the value function is higher in the case of stochastic mortality intensity or in other words the probability of default is higher if the longevity uncertainty is present.

The reminder of the paper is organized as follows. Section **2** describes the model and formulates the stochastic control problem. Section **4** presents the case of constant mortality intensity and Section **5** addresses the case of stochastic mortality intensity. The paper ends with an appendix.

2. The Model

Let $\{(W(t)\}_{t\in[0,\infty)}$ be a 1−dimension Brownian motion on a probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathcal{F}, \mathbb{P})$. We consider a simple financial market consisting of a riskless bond with interest rate equal to zero, i.e., $r = 0$, [1] and a stock whose dynamics is driven by a geometric Brownian motion (GBM) with drift μ and volatility σ

$$dS(t) = S(t)(\mu dt + \sigma dW(t)).$$

Suppose that a life insurance company issues annuity whose payment rate is K, i.e. at time t, the insurance company has to pay out $KI(t)dt$, where $I(t)$ denotes the proportion of clients who are alive at time t. The fees are collected as a lump-sum at a time prior to $t = 0$, from this time onward, we consider only payment phase, there is no restriction of time horizon for sake of simplicity. Without loss of generality, we assume that initially there is an unit mass of clients with same characteristics (a single cohort) that buy one unit of annuity.

Following [4] and considering a special case we model the stochastic intensity as follows

$$d\lambda(t) = \hat{\sigma} dB(t),$$

for some positive constants $\hat{\sigma}$, and $\{(B(t)\}_{t\in[0,\infty)}$ a 1−dimension Brownian motion independent of $\{(W(t)\}_{t\in[0,\infty)}$. Notice that in the absence of the volatility term the stochastic intensity follows the model of constant intensity.

The proportion of clients who are alive at time t, $I(t)$ it is given by

[1]This can be achieved by taking the bond as numeraire; our setup can be extended to allow for stochastic interest rates.

$$I(t) = e^{-\int_0^t \lambda(u)du}.$$

Let x be the initial wealth that the company holds to hedge away the future payment. The payment fund is maintained by a portfolio consisting of risky and risk-free assets. Its dynamics is given by

$$dX(t) = \pi(t)X(t)(\mu dt + \sigma dW(t)) - Ke^{-\int_0^t \lambda(u)du}dt, \quad X(0) = x,$$

where $\pi(t)$ denotes the proportion of wealth invested in the stock at time t. Let

$$Y(t) = X(t)e^{\int_0^t \lambda(u)du},$$

then by the stochastic product rule we have that

$$dY(t) = [(\lambda(t) + \mu\pi(t))Y(t) - K]dt + \sigma\pi(t)Y(t)dW(t), \quad Y(0) = X(0) = x.$$

Suppose that τ_d is the random time when the company stop operating (switching of managers, random default time, etc). Assume that τ_d is exponentially distributed with intensity γ.

Next, let us denote τ_0 the first time when the company cannot meet its obligation with clients, i.e.

$$\tau_0 := \inf\{s \geq 0 : X(s) = 0\} = \inf\{s \geq 0 : Y(s) = 0\}.$$

3. Objective

We are ready at this point to formulate our objective. The company tries to chose a portfolio to minimize the probability of being unable to pay debt before it stops operating, i.e.,

$$\inf_{\pi} P[\tau_0 \leq \tau_d | \lambda(0) = \lambda, Y(0) = y].$$

Let us denote the value function

$$V(\lambda, y) = \inf_{\pi} P[\tau_0 \leq \tau_d | \lambda(0) = \lambda, Y(0) = y]. \quad (\mathbf{3}.1)$$

Theorem 3.1. *Solution of the following Hamilton Jacobi Bellman (HJB) equation*

$$\gamma V(\lambda, y) = (\lambda y - K)V_y + \frac{\hat{\sigma}^2}{2}V_{\lambda\lambda} + \min_{\pi}\{\mu y \pi V_y(\lambda, y) + \frac{1}{2}\sigma^2 y^2 \pi^2 V_{yy}(\lambda, y)\}, \quad (\mathbf{3}.2)$$

with boundary condition

$$V(\lambda, 0) = 1,$$

yields the value function in (**3**.1).

Proof. See the appendix.

This HJB equation can not be explicitly solved for stochastic intensity. We treat first the case of constant intensity since it leads to closed form solutions to the problem (objective) at stake.

4. Constant mortality intensity

In this case, we assume perfect diversification and no longevity risk, i.e. $I(t)$ decreases exponentially $I(t) = e^{-\lambda t}$, where λ denotes the average mortality intensity of clients. We split the analysis into two sections.

4.1. Non zero mortality intensity. In this case we denote the value function

$$V(y) = \inf P[\tau_0 \le \tau_d | Y(0) = y].$$

The HJB equation now reads

$$\gamma V = (\lambda y - K)V_y + \min_{\pi}\{\mu y \pi V_y + \frac{1}{2}\sigma^2 y^2 \pi^2 y^2 V_{yy}\}.$$

The boundary condition is $V(0) = 1$. The optimizer is

$$\pi^* = -\frac{\mu V_y}{\sigma^2 y V_{yy}}. \qquad (\mathbf{4}.1)$$

The equation is rewritten as

$$\gamma V = (\lambda y - K)V_y - \frac{\mu^2 V_y^2}{2\sigma^2 V_{yy}},$$

assuming that V is convex in y. From the dynamics of Y, we can deduce that if $\lambda y - K \ge 0$ then $V(y) = 0$ (by letting $\pi = 0$). In the following we assume that that $\lambda y - K < 0$. Next, we look for V of the form

$$V(y) = l(K - \lambda y)^n \qquad (\mathbf{4}.2)$$

where $l = K^{-n}$. By plugging this ansatz back into the HJB equation we get

$$\gamma = \lambda n - \frac{\mu^2 n}{2\sigma^2(n-1)}.$$

or

$$\lambda n^2 - (\gamma + \lambda + \frac{\mu^2}{2\sigma^2})n + \gamma = 0.$$

This equation in n has two solutions since the discriminant is positive, i.e.,

$$[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma \ge 0.$$

Notice that this inequality is always satisfied, hence there are two separate roots, and it turns out that $n_1 < 1 < n_2$. Since the function V is convex in y it follows that

$$V(y) = l(K - \lambda y)^{n_2}, \qquad (\mathbf{4}.3)$$

where n_2 is

$$n_2 = \frac{(\gamma + \lambda + \frac{\mu^2}{2\sigma^2}) + \sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}}{2\lambda}.$$

4.1.1. *Optimal Investment Strategy.* By plugging the above V into the optimizer formula we get the optimal investment strategy

$$\pi^* = \frac{\mu}{\sigma^2(n_2 - 1)}\left[\frac{K}{\lambda y} - 1\right] > 0.$$

Thus, the optimal strategy is to go long the stock because the stock return exceeds the riskless return. Let us summarize the results of this section in the following Theorem.

Theorem 4.1. *The value function is*

$$V(y) = l(K - \lambda y)^{n_2}, \quad \text{if } \lambda y - K < 0 \text{ and } 0 \text{ otherwise}. \qquad (\mathbf{4}.4)$$

The optimal investment strategy is

$$\pi^* = \frac{\mu}{\sigma^2(n_2 - 1)}\left[\frac{K}{\lambda y} - 1\right] > 0. \qquad (\mathbf{4}.5)$$

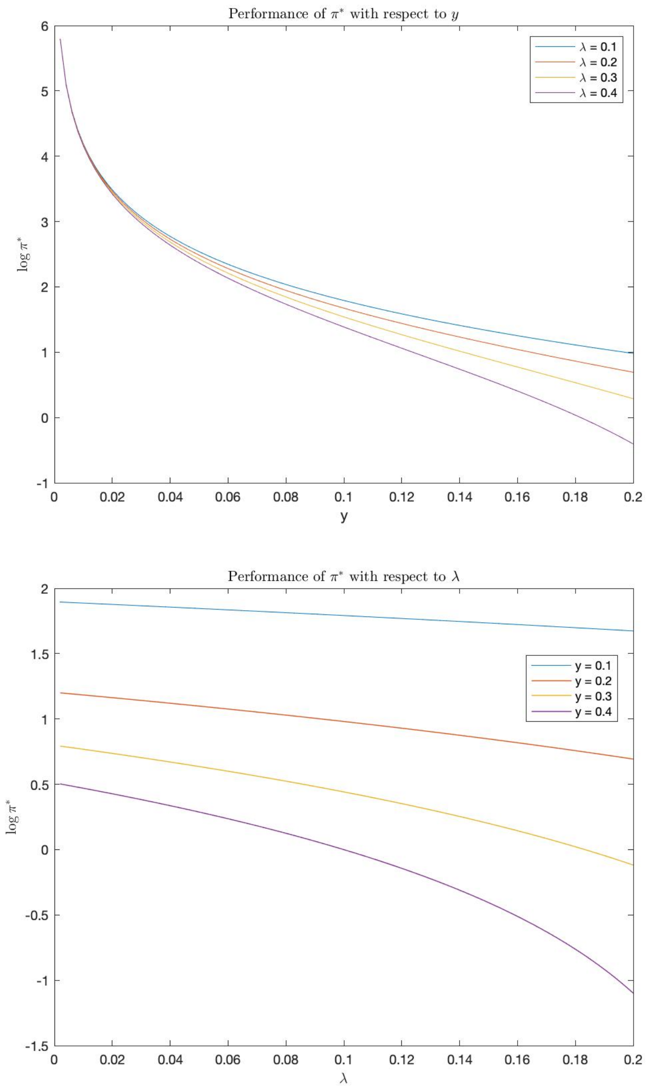

Figure 1. Optimal investment and mortality intensity

It is obvious from (**4**.7) that the optimal investment strategy is decreasing in the current wealth y. Moreover, the dependence of the later on λ is not straightforward. We used following parameters to simulate the model: $\gamma = 0.001$, $\sigma = 0.05$, $\mu = 0.15$, $K = 0.1$. By using these parameters we plotted the optimal investment, π^* and compared the performance of π^* by varying initial wealth, y, and mortality intensity, λ. In Fig. 1 (top), as expected, we found that the optimal investment decreases in y. We also plotted the optimal investment as function of λ as shown in Fig 1 (bottom). The optimal investment decreases in λ. In conclusion the optimal investment in the stock decreases when we increase the initial wealth or mortality intensity.

Lemma 4.1. *The optimal investment strategy is decreasing in λ if and only if the wealth y exceeds $\frac{Kn_1}{\lambda}$.*

Proof. See the appendix.

4.2. **The special case of zero mortality intensity.** In this case the HJB equation reads

$$\gamma V = KV_y + \min_{\pi}\{\mu y \pi V_y + \frac{1}{2}\sigma^2 y^2 \pi^2 V_{yy}\},$$

with the boundary condition $V(0) = 1$. The optimizer is

$$\pi^* = -\frac{\mu V_y}{\sigma^2 y V_{yy}},$$

given the convexity of V in y. The HJB equation is rewritten as

$$\gamma V = -KV_y - \frac{\mu^2 V_y^2}{2\sigma^2 V_{yy}}.$$

The solution is

$$V(y) = \exp \beta y, \qquad \beta = -\left(\gamma + \frac{\mu^2}{2\sigma^2}\right)\frac{1}{K},$$

and is convex in y.

4.2.1. *Optimal Investment Strategy.* By plugging the above V into the optimizer formula we get the optimal investment strategy to be

$$\pi^* = -\frac{\mu}{\sigma^2 \beta}\frac{\exp \beta y}{y} > 0.$$

Thus, the optimal strategy is to go long the stock because the stock return exceeds the riskless return. Let us summarize the results of this section in the following Theorem.

Theorem 4.2. *The value function is*

$$V(y) = \exp \beta y, \qquad \beta = -\left(\gamma + \frac{\mu^2}{2\sigma^2}\right)\frac{1}{K}. \tag{4.6}$$

The optimal investment strategy is

$$\pi^* = -\frac{\mu}{\sigma^2 \beta}\frac{\exp \beta y}{y} \tag{4.7}$$

Next we move to stochastic mortality intensity.

5. Stochastic mortality intensity

In this section $I(t)$ decreases exponentially $I(t) = e^{-\int_0^t \lambda(u)du}$, where $\lambda(t)$ denotes the average mortal intensity of clients, and its dynamics is given by

$$d\lambda(t) = \hat{\sigma} dB(t),$$

for some positive volatility $\hat{\sigma}$, and $\{(B(t)\}_{t\in[0,\infty)}$ a 1−dimension Brownian motion independent of $\{(W(t)\}_{t\in[0,\infty)}$.

The value function is denoted $V(x, \lambda)$ (for notational convenience we use x instead of y). The HJB equation in this case reads

$$\gamma V = (\lambda x - K)V_x + \frac{\hat{\sigma}^2}{2}V_{\lambda\lambda} + \min_{\pi}\{\mu x V_x \pi + \frac{1}{2}\sigma^2 x^2 \pi^2 V_{xx}\}.$$

The boundary condition is $V(\lambda, 0) = 1$. The optimizer, assuming that V is convex in x, is

$$\pi^* = -\frac{\mu V_x}{\sigma^2 x V_{xx}}.$$

The HJB equation is rewritten as

$$\gamma V = (\lambda x - K)V_x + \frac{\hat{\sigma}^2}{2}V_{\lambda\lambda} - \frac{\mu^2 V_x^2}{2\sigma^2 V_{xx}}.$$

Let us introduce the dual function w, by

$$w(\lambda, y) = \inf_{x>0}[V(\lambda, x) + xy], \tag{5.1}$$

so that

$$V(\lambda, x) = \sup_{y>0}[w(\lambda, y) - xy]. \tag{5.2}$$

The following *dual* relations hold

$$w_y(\lambda, y) = x, \quad \text{iff} \quad V_x(\lambda, x) = -y, \tag{5.3}$$

and

$$w_{yy}(\lambda, y) = -\frac{1}{V_{xx}(\lambda, x)}.$$

By the Envelope Theorem

$$V_\lambda = w_\lambda, \quad V_{\lambda\lambda} = w_{\lambda\lambda}.$$

Thus, we arrive at the following PDE for w

$$\frac{\mu^2 y^2}{2\sigma^2} w_{yy} + \frac{\hat{\sigma}^2}{2} w_{\lambda\lambda} + (\gamma - \lambda) y w_y - \gamma w + Ky = 0, \tag{5.4}$$

with the boundary conditions $w(\lambda, 0) = 0, w(\lambda, \infty) = 1, w(-\infty, y) = 1, w(\infty, y) = 0$.

It is clear that $w(\lambda, 0) = 0$. Let us argue that $w(\lambda, \infty) = 1$. It is obvious that $w(\lambda, y) \leq 1$ by sending x to 0 in (**5**.1). For every $\epsilon > 0$ there is a big $N > 0$ such that $\inf_{\frac{1}{N}>x>0} V(\lambda, x) \geq 1 - \epsilon$, so $w(\lambda, N) > 1 - \epsilon$. Thus $w(\lambda, \infty) = 1$.

We further assume that $\hat{\sigma} = \sqrt{\epsilon}$, for some small ϵ. By using a first order expansion we can approximate the solution of PDE (**5**.4) as follows

$$w \approx \hat{w} = w^0 + \epsilon w^1,$$

where w^0 solves

$$\frac{\mu^2 y^2}{2\sigma^2} w^0_{yy} + (\gamma - \lambda) y w^0_y - \gamma w^0 + Ky = 0, \qquad w^0(\lambda, 0) = 0, w^0(\lambda, \infty) = 1,$$

and w^1 solves

$$\frac{\mu^2 y^2}{2\sigma^2} w^1_{yy} + (\gamma - \lambda) y w^1_y - \gamma w^1 + Ky = -\frac{1}{2} w^0_{\lambda\lambda}, \qquad w^1(\lambda, 0) = 0, w^1(\lambda, \infty) = 0.$$

It turns out that

$$w^0(\lambda, y) = \inf_{x>0}[V^0(\lambda, x) + xy], \tag{5.5}$$

where

$$V^0(\lambda, x) = l(K - \lambda x)^{n_2}, \quad \text{if} \quad K - \lambda x > 0, \quad 0 \text{ otherwise},$$

Direct computations lead to

$$w^0(\lambda, y) = \frac{1}{(\lambda)^{\frac{n_2}{n_2-1}}} \left(\frac{1}{n_2} - 1\right) \left(\frac{1}{n_2 l}\right)^{\frac{1}{n_2-1}} y^{\frac{n_2}{n_2-1}} + \frac{Ky}{\lambda} \quad \text{if} \quad y < \lambda n_2 l K^{n_2-1},$$

$$1 \text{ otherwise.}$$

If

$$y > \lambda n_2 l K^{n_2-1}$$

then

$$w^1(\lambda, y) = 0.$$

On the interval $[0, \lambda n_2 l K^{n_2-1}]$ the function w^1 solves

$$\frac{\mu^2 y^2}{2\sigma^2} w^1_{yy} + (\gamma - \lambda) y w^1_y - \gamma w^1 = -\frac{1}{2} w^0_{\lambda\lambda}, \qquad w^1(\lambda, 0) = 0, w^1(\lambda, \lambda n_2 l K^{n_2-1}) = 0.$$

Since

$$w^0_{\lambda\lambda} = V^0_{\lambda\lambda},$$

and V^0 is convex in λ it follows that

$$w^0_{\lambda\lambda} \geq 0,$$

thus

$$\frac{\mu^2 y^2}{2\sigma^2} w^1_{yy} + (\gamma - \lambda) y w^1_y - \gamma w^1 \leq 0.$$

By a standard Maximum Principle one gets

$$w^1 \geq u$$

where u solves

$$\frac{\mu^2 y^2}{2\sigma^2} u_{yy} + (\gamma - \lambda) y u_y - \gamma u = 0, \qquad u(\lambda, 0) = 0, u(\lambda, \lambda n_2 l K^{n_2-1}) = 0.$$

However $u = 0$, hence $w^1 \geq 0$, so

$$\hat{w} \geq 0.$$

Let us define the first order approximation value function by

$$\hat{V}(\lambda, x) = \sup_{y>0} [\hat{w}(\lambda, y) - xy],$$

so

$$\hat{V}(\lambda, x) \geq V^0(\lambda, x).$$

Let us summarize the results of this section in the following Theorem.

Theorem 5.1. *The value function approximation $\hat{V}(\lambda, x)$ in the model with stochastic intensity exceeds $V^0(\lambda, x)$, the value function of the model with constant intensity.*

In conclusion, in a model with stochastic intensity the probability of default is higher than in a model with constant intensity, fact explained by the extra source of risk (longevity risk) faced by the insurance company.

6. Appendix

6.1. Proof of Theorem 3.1. Since τ_d is exponentially distributed with intensity γ, it follows that

$$P[\tau_0 \leq \tau_d] = E[\exp -(\gamma \tau_0)].$$

We employ the martingale optimality principled, see page 102 in [9], to characterize the value function. The HJB equation (**3**.2) says that

$$(\exp -(\gamma t)) V(\lambda(t), Y(t))$$

is supermartingale, and martingale for the optimal (π^*, Y^*). Thus,

$$V(\lambda(0), Y(0)) \leq E[(\exp -(\gamma \tau_0)) V(\lambda(\tau_0), Y(\tau_0))] =$$
$$= E[(\exp -(\gamma \tau_0)) V(\lambda(\tau_0), 0)] = E[\exp -(\gamma \tau_0)] = P[\tau_0 \leq \tau_d]$$

$$V(\lambda(0), Y^*(0)) = E[(\exp -(\gamma\tau_0^*))V(\lambda(\tau_0^*), Y(\tau_0^*)] =$$

$$= E[(\exp -(\gamma\tau_0^*))V(\lambda(\tau_0^*), 0)] = E[\exp -(\gamma\tau_0^*)] = P[\tau_0^* \leq \tau_d],$$

whence the optimality.

6.2. **Proof of Lemma 4.1.** By differentiating

$$\lambda n^2 - (\gamma + \lambda + \frac{\mu^2}{2\sigma^2})n + \gamma = 0$$

with respect to λ one gets

$$2\lambda n n' + n^2 - (\gamma + \lambda + \frac{\mu^2}{2\sigma^2})n' - n = 0.$$

Thus

$$n' = \frac{n - n^2}{2\lambda n - (\gamma + \lambda + \frac{\mu^2}{2\sigma^2})}.$$

In particular

$$n_2' = \frac{n_2 - n_2^2}{2\lambda n_2 - (\gamma + \lambda + \frac{\mu^2}{2\sigma^2})} = \frac{n_2 - n_2^2}{\sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}}.$$

Let

$$\pi^* = \frac{\mu}{\sigma^2(n_2 - 1)}\left[\frac{K}{\lambda y} - 1\right] := F(\lambda).$$

Then, the claim follows since

$$F' = -\frac{\mu}{\sigma^2(n_2-1)^2}\frac{n_2 - n_2^2}{\sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}}\left[\frac{K}{\lambda y} - 1\right] - \frac{\mu}{\sigma^2(n_2-1)}\frac{K}{\lambda^2 y} =$$

$$= \frac{K}{\lambda y}\frac{\mu}{\sigma^2(n_2-1)}\left[\frac{n_2}{\sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}} - \frac{1}{\lambda}\right] -$$

$$\frac{\mu}{\sigma^2(n_2-1)}\frac{1}{\sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}} =$$

$$= \frac{\mu}{\sigma^2(n_2-1)}\frac{1}{\sqrt{[\gamma + \lambda + \frac{\mu^2}{2\sigma^2}]^2 - 4\lambda\gamma}}\left[n_1\frac{K}{\lambda y} - 1\right].$$

REFERENCES

[1] Biffis, E., (2005) *Affine processes for dynamic mortality and actuarial valuations*, Insurance: Mathematics and Economics Vol 37, No 3, 443-468.

[2] Browne, S., (1995) *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin*, Mathematics of Operations Research Vol 20, No 4, 937-958.

[3] Ferguson, T., (1965) *Betting systems which minimize the probability of ruin*, J. SIAM 13, 795-818.

[4] Luciano E., and Vigna E., *Non mean reverting affine processes for stochastic mortality* Working Paper no. 4/2005 March 2005.

[5] Luciano E., and Vigna E.,*Mortality risk via affine stochastic intensities: calibration and empirical relevance*, Belgian Actuarial Bulletin 8, 5-16.

[6] Karatzas, I. (1989) *Optimization problems in the theory of continuous trading*, SIAM Journal of Control and Optimization 27, 1221-1259.

[7] Merton, R. C. (1969) *Lifetime portfolio selection under uncertainty: the continuous-time case*, Rev. Econom. Statist 51, 247-257.

[8] Merton, R. C. (1971) *Optimum consumption and portfolio rules in a continuous-time model*, J. Economic Theory, 3, 373-413.

[9] Huyen P. (2009), *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer, DOI 10.1007/978-3-540-89500-8.