ON Δ-WEAK ϕ-AMENABILITY OF BANACH ALGEBRAS

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Dedicated to Professor Alireza Medghalchi with appreciation and respect

Let $A$ be a Banach algebra and $ϕ ∈ Δ(A) ∪ \{0\}$. We introduce and study the notion of Δ-weak ϕ-amenability of Banach algebra $A$. It is shown that $A$ is Δ-weak ϕ-amenable if and only if $\ker(ϕ)$ has a bounded Δ-weak approximate identity. We prove that every Δ-weak ϕ-amenable Banach algebra has a bounded Δ-weak approximate identity. Finally, we examine this notion for some algebras over locally compact groups and give a characterization of Δ-weak ϕ-amenability of the Figa-Talamanca-Herz algebras.

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1. Introduction

Let $A$ be a Banach algebra, $Δ(A)$ be the character space of $A$, i.e., the space of all non-zero homomorphisms from $A$ into $\mathbb{C}$ and $A^*$ be the dual space of $A$ consisting of all bounded linear functions from $A$ into $\mathbb{C}$.

Throughout this paper, we assume that $A$ is a Banach algebra such that $Δ(A) ≠ \emptyset$.

Let $\{e_α\}$ be a net in a Banach algebra $A$. The net $\{e_α\}$ is called,

(1) an approximate identity if, for each $a ∈ A$, $||ae_α - a|| + ||e_αa - a|| → 0$,
(2) a weak approximate identity if, for each $a ∈ A$, $|f(αe_α) - f(a)| + |f(e_αa) - f(a)| → 0$ for all $f ∈ A^*$,
(3) a Δ-weak approximate identity if, for each $a ∈ A$, $|ϕ(αe_α) - ϕ(a)| → 0$ for all $ϕ ∈ Δ(A)$.

The notion of weak approximate identity was originally introduced for the study of the second dual $A^{**}$ of a Banach algebra $A$. For technical reasons, bounded approximate identities are of interest for mathematicians. It is proved that every Banach algebra $A$ which has a bounded weak approximate identity, also has a bounded approximate identity and conversely [4, Proposition 33.2]. But in [9], Jones and Lahr proved that the approximate identity and Δ-weak approximate identity of a Banach algebra are different. They showed that there exists a Banach algebra $A$ which has a bounded Δ-weak approximate identity, but it does not have any approximate identity. Indeed, if $S = \mathbb{Q}^+$ is the semigroup of positive rationales under addition, ...
they showed that the semigroup algebra $l^1(S)$ has a bounded $\Delta$-weak approximate identity, but it does not have any bounded or unbounded approximate identity.

**Definition 1.1.** Let $A$ be a Banach algebra. A $\Delta$-weak approximate identity for subspace $B \subseteq A$ is a net $\{a_\alpha\}$ in $B$ such that

$$\lim_\alpha |\phi(aa_\alpha) - \phi(a)| = 0 \quad (a \in B, \phi \in \Delta(A)).$$

For $a \in A$ and $f \in A^*$, the linear functional $f.a$ defined as follows

$$<f.a,b> = <f,ab> = f(ab) \quad (b \in A).$$

**Definition 1.2.** Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. The Banach algebra $A$ is said to be $\phi$-amenable, if there exists an $m \in A^{**}$ such that the following relations hold,

1. $m(\phi) = 1$,
2. $m(f.a) = \phi(a)m(f) \quad (a \in A, f \in A^*)$.

The concept of character amenability was first introduced by Monfared in [12]. Also, Kaniuth, Lau and Pym in [10], have investigated the concept of $\phi$-amenability of Banach algebras and gave the following result.

**Theorem 1.1.** Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. Then $\ker(\phi)$ has a bounded right approximate identity if and only if $A$ is $\phi$-amenable and has a bounded right approximate identity.

**Proof.** See [10, Corollary 2.3].

In the next section of this paper, first we give the basic definition of our work, that is $\Delta$-weak $\phi$-amenability of a Banach algebra $A$. Then we characterize it through the existence of a bounded $\Delta$-weak approximate identity for $\ker(\phi)$. Suppose that $A$ is a Banach algebra and $I$ is a closed ideal of $A$ such that $I$ and $A/I$ both have bounded approximate identities. Then $A$ has a bounded approximate identity [4, Proposition 7.1]. We give a variant of this theorem. Using this theorem, we prove that each Banach algebra $A$ that is $\Delta$-weak $\phi$-amenable, has a bounded $\Delta$-weak approximate identity.

In Section 3, we investigate some of the hereditary properties of $\Delta$-weak $\phi$-amenability. In section 4, we study group algebras and Figa-Talamanca Herz algebras of a locally compact group with respect to this notion and prove that when $1 < p < \infty$ and $\phi \in \Delta(A_p(G)) \cup \{0\}$, $A_p(G)$ is $\Delta$-weak $\phi$-amenable if and only if $G$ is an amenable group.

In the final section, we just give some examples which shows the different situations that might occur for definitions and theorems in Section 2 and 3.

## 2. Main definition and its characterization

We begin this section with the following definition that is our main concern.

**Definition 2.1.** Let $A$ be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. We say that $A$ is $\Delta$-weak $\phi$-amenable, if there exists an $m \in A^{**}$ such that $m(\phi) = 0$ and $m(\psi.a) = \psi(a)$ for each $a \in \ker(\phi)$ and $\psi \in \Delta(A)$.
Now, we characterize the concept of Δ-weak φ-amenability as follows. Recall that if A is a Banach algebra, for each \( a \in A \), \( \tilde{a} \in A^{**} \) is defined by \( \tilde{a}(f) = f(a) \) for all \( f \in A^* \).

**Theorem 2.1.** Let \( A \) be a Banach algebra and \( \phi \in \Delta(A) \cup \{0\} \). Then \( A \) is Δ-weak φ-amenable if and only if \( \ker(\phi) \) has a bounded Δ-weak approximate identity.

**Proof.** Let \( \{e_\alpha\} \) be a bounded Δ-weak approximate identity for \( \ker(\phi) \). So, \( \{\tilde{e}_\alpha\} \) is a bounded net in \( A^{**} \). Therefore, the Banach-Alaoglu’s Theorem ([2, Theorem A.3.20]) yields \( \{\tilde{e}_\alpha\} \) has a \( w^* \)-accumulation point \( m \), i.e., there exists a subnet that we denote also by \( \{\tilde{e}_\alpha\} \) such that \( m = w^* - \lim_\alpha (\tilde{e}_\alpha) \).

Now, we have
\[
m(\phi) = \lim_\alpha \tilde{e}_\alpha(\phi) = \lim_\alpha \phi(e_\alpha) = 0,
\]
and for all \( \psi \in \Delta(A) \) and \( a \in \ker(\phi) \),
\[
m(\psi.a) = \lim_\alpha \tilde{e}_\alpha(\psi.a) = \lim_\alpha \psi(ae_\alpha) = \lim_\alpha \psi(a).
\]

Conversely, since \( m \in A^{**} \), Goldstine’s Theorem ([2, Theorem A.3.29(i)]) yields there exists a net \( \{e_\alpha\} \) in \( A \) such that \( m = w^* - \lim_\alpha \tilde{e}_\alpha \) and \( ||e_\alpha|| \leq ||m|| \).

So, \( \{e_\alpha\} \) is a bounded net such that for all \( \psi \in \Delta(A) \) and \( a \in \ker(\phi) \) we have
\[
\psi(a) = m(\psi.a) = \lim_\alpha \tilde{e}_\alpha(\psi.a) = \lim_\alpha \psi.a(e_\alpha) = \lim_\alpha \psi(a).
\]

Hence \( \lim_\alpha |\psi(ae_\alpha) - \psi(a)| = 0 \).

Note that the net \( \{e_\alpha\} \) is not a subset of \( \ker(\phi) \). To construct a net in \( \ker(\phi) \), we have two cases \( \phi = 0 \) or \( \phi \in \Delta(A) \). If \( \phi = 0 \), then the net \( \{e_\alpha\} \) is a bounded Δ-weak approximate identity for \( A = \ker(\phi) \).

If \( \phi \in \Delta(A) \), then there exists \( x_0 \in A \) such that \( \phi(x_0) \neq 0 \). Put \( a_0 = \frac{x_0}{\phi(x_0)} \) and suppose that \( a_\alpha = e_\alpha - \phi(e_\alpha)a_0 \) for all \( \alpha \). Obviously \( \{a_\alpha\} \subseteq \ker(\phi) \). On the other hand, since \( m(\phi) = 0 \) and \( m = w^* - \lim_\alpha \tilde{e}_\alpha \), we conclude that
\[
\lim_\alpha \phi(e_\alpha) = \lim_\alpha \tilde{e}_\alpha(\phi) = m(\phi) = 0.
\]

Hence, for each \( a \in \ker(\phi) \) and \( \psi \in \Delta(A) \) we have
\[
\lim_\alpha |\psi(aa_\alpha) - \psi(a)| = \lim_\alpha |\psi(ae_\alpha) - \phi(e_\alpha)\psi(a_0) - \psi(a)|
\leq \lim_\alpha |\psi(ae_\alpha) - \psi(a)| + \lim_\alpha |\phi(e_\alpha)\psi(a_0)|
= 0.
\]

Therefore, \( \lim_\alpha |\psi(aa_\alpha) - \psi(a)| = 0 \) for each \( a \in \ker(\phi) \) and \( \psi \in \Delta(A) \), which completes the proof.

For simplicity of notation, let bΔ-w.a.i stand for bounded Δ-weak approximate identity and b.a.i stand for bounded approximate identity.

By Theorems 1.1 and 2.1 one can see that every φ-amenable Banach algebra which has a bounded right approximate identity is Δ-weak φ-amenable. But the converse of this assertion is not valid in general.
Remark 2.1. Let $A$ be a $\Delta$-weak $\phi$-amenable Banach algebra and $\{e_\alpha\}$ be a $\Delta$-weak approximate identity of $\ker(\phi)$. If there exists $a_0 \in A$ with $\phi(a_0) = 1$ and $\lim_\alpha |\psi(a_0 e_\alpha) - \psi(a_0)| = 0$ for all $\psi \in \Delta(A) \setminus \{\phi\}$, then there exists $m \in A^{**}$ such that

\begin{enumerate}
  \item $m(\phi) = 0$,
  \item $m(\psi.a) = \psi(a)$ \quad ($a \in A, \psi \in \Delta(A) \setminus \{\phi\}$).
\end{enumerate}

By a similar argument as in the above theorem we can show the existence of $m$. Let $a \in A$. It is clear that $a - \phi(a)a_0 \in \ker(\phi)$. So, for all $\psi \in \Delta(A) \setminus \{\phi\}$ we have $m(\psi.(a - \phi(a)a_0)) = \psi(a - \phi(a)a_0)$. Therefore $m(\psi.a) = \psi(a)$.

Note that in part (2), it is necessary that $\psi \neq \phi$. If $\psi = \phi$ we have $m(\phi.a) = \phi(a)$. On the other hand, there exists an $a \in A$ such that $\phi(a) \neq 0$. So, we have $\phi(a) = m(\phi.a) = m(\phi(a)\phi) = \phi(a)m(\phi)$.

Therefore, $m(\phi) = 1$ and this is a contradiction.

The following lemma is needed in the sequel.

Lemma 2.1. Let $A$ be a Banach algebra such that $\phi, \psi \in \Delta(A)$ and $\phi \neq \psi$. Then there exists $a \in A$ such that $\phi(a) = 0$ and $\psi(a) = 1$.

Proof. See the proof of [11, Theorem 3.3.14]. \hfill $\square$

Remark 2.2. If $A$ is a Banach algebra with $\Delta(A) \setminus \{\phi\} \neq \emptyset$, then $A$ is $\Delta$-weak $\phi$-amenable if and only if there exists a bounded net $\{e_\alpha\}$ in $\ker(\phi)$ such that $\lim_\alpha \psi(e_\alpha) = 1$ for each $\psi \in \Delta(A) \setminus \{\phi\}$ or equivalently there exists an $m \in A^{**}$ with $m(\phi) = 0$ and $m(\psi) = 1$ for all $\psi \in \Delta(A) \setminus \{\phi\}$.

The following proposition allow us to produce Banach algebras which are $\Delta$-weak $\phi$-amenable, but they are not $\phi$-amenable.

Proposition 2.1. Let $A$ be a Banach algebra such that $0 < |\Delta(A)| \leq 2$, i.e., $\Delta(A)$ has 1 or 2 elements. Then $A$ is $\Delta$-weak $\phi$-amenable.

Proof. If $A$ only has one character, the proof is easy. Therefore, we omit it. In the second case, let $\Delta(A) = \{\phi, \psi\}$ and $\phi \neq \psi$. Hence, by Lemma 2.1 there exists an $e \in A$ with $\phi(e) = 0$ and $\psi(e) = 1$. Now, put $m = e$. Clearly, $m(\phi) = 0$ and $m(\psi) = 1$, so by Remark 2.2 $A$ is $\Delta$-weak $\phi$-amenable. \hfill $\square$

The following theorem is a useful tool in the rest of this section.

Theorem 2.2. Let $A$ be a Banach algebra, $I$ be a closed two-sided ideal of $A$ which has a b.$\Delta$-w.a.i and the quotient Banach algebra $A/I$ has a bounded left approximate identity (b.l.a.i). Then $A$ has a b.$\Delta$-w.a.i.

Proof. Let $\{e_\alpha\}$ be a b.$\Delta$-w.a.i for $I$ and $\{f_\delta + I\}$ be a b.l.a.i for $A/I$. Suppose that $F = \{a_1, \ldots, a_m\}$ is a finite subset of $A$ and $n$ is a positive integer. Let $M$ be an upper bound for $\{||e_\alpha||\}$. For $\lambda = (F, n)$, there exists $f_\delta$ such that

$$||f_\delta a_i - a_i + I|| < \frac{1}{2(1 + M)n} \quad (i = 1, 2, 3, \ldots, m).$$

Therefore, there exists $y_i \in I$ such that

$$||f_\delta a_i - a_i + y_i|| < \frac{1}{2(1 + M)n} \quad (i = 1, 2, 3, \ldots, m).$$
Let $\psi \in \Delta(A)$. Since $\{e_\alpha\}$ is a b.∆-w.a.i, for each $y_i$ with $i \in \{1, 2, 3, \ldots, m\}$ which satisfy the above relation, there exists $e_{\alpha_\lambda} \in \{e_\alpha\}$ such that

$$|\psi(e_{\alpha_\lambda} y_i - \psi(y_i))| < \frac{1}{2n} \quad (i = 1, 2, 3, \ldots, m).$$

Now, for each $i \in \{1, 2, 3, \ldots, m\}$ we have

$$|\psi((e_{\alpha_\lambda} + f_{\delta_\lambda} - e_{\alpha_\lambda} f_{\delta_\lambda})a_i) - \psi(a_i)| \leq |\psi(f_{\delta_\lambda} a_i - a_i + y_i)|$$

$$+ |\psi(e_{\alpha_\lambda} y_i) - \psi(y_i)|$$

$$+ |\psi(e_{\alpha_\lambda} a_i - e_{\alpha_\lambda} f_{\delta_\lambda} a_i - e_{\alpha_\lambda} y_i)|$$

$$\leq |||f_{\delta_\lambda} a_i - a_i + y_i||$$

$$+ \frac{1}{2n} + M||a_i - f_{\delta_\lambda} a_i - y_i||$$

$$< \frac{1}{n}.$$

Therefore, $\{e_{\alpha_\lambda} + f_{\delta_\lambda} - e_{\alpha_\lambda} f_{\delta_\lambda}\}_{\lambda \in \Lambda}$ is a $\Delta$-w.a.i for $A$, where $\Lambda = \{(F, n) : F \subseteq A$ is finite, $n \in \mathbb{N}\}$ is a directed set with $(F_1, n_1) \leq (F_2, n_2)$ if $F_1 \subseteq F_2$ and $n_1 \leq n_2$.

Now, we show that there exists a $\Delta$-w.a.i for $A$. Since $\{f_{\delta} + I\}$ is bounded, there exists a positive integer $K$ such that $||f_{\delta} + I|| < K$ for each $\delta$. So, there exists $y_\delta \in I$ such that $||f_{\delta} + I|| < ||f_{\delta} + y_\delta|| < K$. Put $f_{\delta}' = f_{\delta} + y_\delta$. Hence, $\{f_{\delta}' + I\}$ is a bounded approximate identity for $A/I$ which $\{f_{\delta}'\}$ is bounded. Now, we have

$$||e_{\alpha_\lambda} + f_{\delta_\lambda}' - e_{\alpha_\lambda} f_{\delta_\lambda}'|| \leq ||e_{\alpha_\lambda}|| + ||f_{\delta_\lambda}'|| + ||e_{\alpha_\lambda}||||f_{\delta_\lambda}'|| < M + K + KM.$$

Therefore, $A$ has a b.∆-w.a.i. □

It is straightforward to see that for every closed two-sided ideal $I$ with codimension one of a Banach algebra $A$, the quotient Banach algebra $A/I$ has a bounded approximate identity. So, we have the following corollary.

**Corollary 2.1.** Let $A$ be a Banach algebra and $\phi \in \Delta(A) \cup \{0\}$. If $A$ is $\Delta$-weak $\phi$-amenable, then $A$ has a b.∆-w.a.i.

**Proof.** Since $A$ is $\Delta$-weak $\phi$-amenable, by Theorem 2.1, $\ker(\phi)$ has a b.∆-w.a.i. Also, $A/\ker(\phi)$ has a b.a.i, because the codimension of $\ker(\phi)$ is one. Then, by Theorem 2.2, $A$ has a b.∆-w.a.i. □

**Corollary 2.2.** Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. If $A$ is $\Delta$-weak $\phi$-amenable, then $A$ is $\Delta$-weak $\theta$-amenable.

The converse of the above corollary is not valid in general (see Example 5.6). Note that there exist Banach algebras which do not have any b.∆-w.a.i. As an example, let $G$ be a locally compact group and $1 < p < \infty$. Consider $S_p(G) = L^1(G) \cap L^p(G)$ with the norm defined by $||f||_{S_p(G)} = \max\{||f||_1, ||f||_p\}$ which is a Segal algebra (see [18] for a full discussion on Segal algebras). Now, by [8, Remark 2], if $G$ is an infinite abelian compact group, $S_p(G)$ has no b.∆-w.a.i.

There exists a $\Delta$-weak version of an identity in a Banach algebra $A$.

**Definition 2.2.** Let $A$ be a Banach algebra. We say that $e \in A$ is a $\Delta$-weak identity for $A$ if for each $\phi \in \Delta(A)$, $\phi(e) = 1$ or equivalently

$$\phi(ea) = \phi(a) \quad (a \in A, \phi \in \Delta(A)).$$
It is obvious that the identity of a Banach algebra $A$ is a $\Delta$-weak identity of $A$, but the converse is not valid in general.

The following theorem gives a necessary condition for $\Delta$-weak $\phi$-amenability of finite dimensional Banach algebras.

**Theorem 2.3.** Let $A$ be a finite dimensional Banach algebra. If $A$ is $\Delta$-weak $\phi$-amenable, then it has a $\Delta$-weak identity.

**Proof.** In view of Corollary 2.1, $A$ has a $b.\Delta$-w.a.i, say $\{e_n\}$. By the Heine-Borel’s Theorem ([15, Theorem 2.38]), we know that every closed and bounded subset of a finite dimensional normed linear space is compact.

So, $\{e_n\}$ is relatively compact, i.e., its closure, $\overline{\{e_n\}}$ is compact, because it is closed and bounded. Therefore, there exists $e \in A$ and a convergent subnet that we denote also by $\{e_n\}$ such that converges to $e$. Now, for each $\psi \in \Delta(A)$ and $a \in A$, we have

$$\psi(ea) = \lim_{n} \psi(e_n a) = \psi(a).$$

Therefore, $e$ is a $\Delta$-weak identity for $A$. \hfill \square

3. Hereditary properties

In this section, we give some of the hereditary properties of $\Delta$-weak $\phi$-amenability.

**Theorem 3.1.** Let $A$ and $B$ be Banach algebras, $\phi \in \Delta(B)$ and $h : A \to B$ be a dense range continuous homomorphism. If $A$ is $\Delta$-weak $\phi \circ h$-amenable, then $B$ is $\Delta$-weak $\phi$-amenable.

**Proof.** Let $A$ be $\Delta$-weak $\phi \circ h$-amenable. So, there exists $m \in A^{**}$ such that, $m(\phi \circ h) = 0$ and $m(\psi . a) = \psi (a)$ for all $a \in \ker(\phi \circ h)$ and $\psi \in \Delta(A)$.

Define $n \in B^{**}$ as follows

$$n(g) = m(g \circ h) \quad (g \in B^*).$$

So, $n(\phi) = m(\phi \circ h) = 0$. For each $b \in \ker(\phi)$, there exists a sequence $\{e_n\}$ in $A$ such that $\lim_{n} h(e_n) = b$. Put $a_n = e_n - \phi \circ h(e_n) a_0$ where $\phi \circ h(a_0) = 1$. It is obvious that $a_n \in \ker(\phi \circ h)$ for each $n$ and $\lim_{n} h(a_n) = b$. Also, for each $\psi' \in \Delta(B)$, $(\psi' \circ h). a_n \to (\psi' \circ h). b$ in $A^*$, since

$$||(\psi' \circ h). a_n - (\psi' \circ h). b|| = \sup_{||a|| \leq 1} ||\psi' (h(a_n)) - \psi' (h(b))||$$

$$\leq \sup_{||a|| \leq 1} ||h(a_n)h(a) - bh(a)||$$

$$\leq ||h(a_n) - b|| ||h||.$$
So, $B$ is $\Delta$-weak $\phi$-amenable. \hfill \Box

Let $I$ be a closed ideal of a Banach algebra $A$. If $I$ has an approximate identity, then every $\phi \in \Delta(I)$ extends to some $\hat{\phi} \in \Delta(A)$. To see this let $\{e_\alpha\}$ be an approximate identity of $I$ and $u \in I$ be an element with $\phi(u) = 1$. If $a \in A$ and $b \in \ker(\phi)$, then we have

$$\phi(ab) = \lim_\alpha \phi(ae_\alpha b) = 0.$$ 

Therefore, $ab \in \ker(\phi)$ and this shows that $\ker(\phi)$ is a left ideal in $A$. Now, Define $\tilde{\phi} : A \to \mathbb{C}$ by $\tilde{\phi}(a) = \phi(au)$ for all $a \in A$. Since for $a, b \in A$, $bu - ubu \in \ker(\phi)$ therefore, $ab - aubu = a(bu - ubu) \in \ker(\phi)$. So, we conclude that $\phi(abu) = \phi(au)\phi(bu)$. Hence

$$\tilde{\phi}(ab) = \phi(abu) = \phi(au)\phi(bu) = \tilde{\phi}(a)\tilde{\phi}(b) \quad (a, b \in A).$$

**Proposition 3.1.** Let $A$ be a Banach algebra, $I$ a closed ideal of $A$, $\phi$ has a bounded approximate identity and $\phi \in \Delta(A)$ with $I \nsubseteq \ker(\phi)$. If $A$ is $\Delta$-weak $\phi$-amenable, then $I$ is $\Delta$-weak $\phi_I$-amenable.

**Proof.** Let $\{a_\beta\}$ be a bounded approximate identity for $I$ and $\{e_\alpha\}$ be a b.$\Delta$-w.a.i for $\ker(\phi)$. It is clear that $\lim_\beta \psi(a_\beta) = 1$ for all $\psi \in \Delta(I)$.

Put $c_{(\alpha, \beta)} = e_\alpha a_\beta$ for all $\alpha, \beta$. Then $\{c_{(\alpha, \beta)}\}_{(\alpha, \beta)}$ is a bounded net in $I$. Now, for each $a \in \ker(\phi_I)$ and $\psi \in \Delta(I)$ we have

$$\lim_{(\alpha, \beta)} \psi(ac_{(\alpha, \beta)}) = \lim_{(\alpha, \beta)} \psi(ac_\alpha a_\beta)$$

$$= (\lim_{(\alpha, \beta)} \psi(ae_\alpha))(\lim_{(\alpha, \beta)} \psi(a_\beta))$$

$$= (\lim_{(\alpha, \beta)} \tilde{\psi}(ae_\alpha))(\lim_{(\alpha, \beta)} \psi(a_\beta))$$

$$= \tilde{\psi}(a) = \psi(a).$$

Therefore, $I$ is $\Delta$-weak $\phi_I$-amenable by Theorem 2.1. \hfill \Box

For each Banach algebra $A$, we can extend each $\phi \in \Delta(A)$ uniquely to a character $\phi$ of $A^{**}$ defined by $\hat{\phi}(a^{**}) = a^{**}(\phi)$ for all $a^{**} \in A^{**}$. So, we have the following result.

**Proposition 3.2.** Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. If $A^{**}$ is $\Delta$-weak $\hat{\phi}$-amenable, then $A$ is $\Delta$-weak $\phi$-amenable.

**Proof.** Let $A^{**}$ be $\Delta$-weak $\hat{\phi}$-amenable. So, there exists $m^{**} \in A^{****}$ which satisfies the following relations,

1. $m^{**}(\hat{\phi}) = 0$,
2. $m^{**}(\Psi F) = \Psi(F) \quad (\Psi \in \Delta(A^{**}), F \in \ker(\hat{\phi})).$

Put $m(f) = m^{**}(\hat{f})$ for all $f \in A^*$. Therefore, $m(\phi) = m^{**}(\hat{\phi}) = 0$ and for each $a \in \ker(\phi) \subseteq \ker(\hat{\phi})$ we have

$$m(\psi.a) = m^{**}(\hat{\psi}\hat{a}) = m^{**}(\hat{\psi}\hat{a}) = \hat{\psi}(\hat{a}) = a(\hat{\psi}) = \psi(a) \quad (\psi \in \Delta(A)).$$

Therefore, $A$ is $\Delta$-weak $\phi$-amenable. \hfill \Box
4. Some results on algebras over locally compact groups

Let $G$ be a locally compact group. For $1 < p < \infty$ let $A_p(G)$ denotes the subspace of $C_0(G)$ consisting of functions of the form $u = \sum_{i=1}^{\infty} f_i \ast \tilde{g}_i$ where $f_i \in L^p(G)$, $g_i \in L^{q}(G)$, $1/p + 1/q = 1$, $\sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q < \infty$ and $f(x) = f(x^{-1})$ for all $x \in G$. $A_p(G)$ is called the Figa-Talamanca-Herz algebra and with the pointwise operation and the following norm is a Banach algebra,

$$||u||_{A_p(G)} = \inf \{ \sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q : u = \sum_{i=1}^{\infty} f_i \ast \tilde{g}_i \}.$$ 

It is obvious that for each $u \in A_p(G)$, $||u|| \leq ||u||_{A_p(G)}$ where $||u||$ is the norm of $u$ in $C_0(G)$. Also, we know that $\Delta(A_p(G)) = G$, i.e., each character of $A_p(G)$ is an evaluation function at some $x \in G$ [5, Theorem 3].

The dual of the Banach algebra $A_p(G)$ is the Banach space $PM_p(G)$ consisting of all limits of convolution operators associated to bounded measures [3, Chapter 4].

The group $G$ is said to be amenable if, there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_xf) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$ where $L_xf(y) = f(x^{-1}y)$ [17, Definition 4.2].

First we give the following lemma which is a generalization of Leptin-Herz Theorem ([17, Theorem 10.4]).

**Lemma 4.1.** Let $G$ be a locally compact group and $1 < p < \infty$. Then $A_p(G)$ has a b.Δ-w.a.i if and only if $G$ is amenable.

**Proof.** Let $\{e_\alpha\}$ be a b.Δ-w.a.i for $A_p(G)$ and $e \in A_p(G)^{**}$ be a $w^*$-cluster point of $\{e_\alpha\}$.

So, for each $\phi \in \Delta(A_p(G)) = G$, we have

$$< e, \phi > = \lim_\alpha \phi(e_\alpha) = 1.$$ 

Therefore, by [19, Proposition 2.8] $G$ is weakly closed in $PM_p(G) = A_p(G)^*$. Now, by [1, Corollary 2.8] we conclude that $G$ is an amenable group. □

For a locally compact group $G$, let $L^1(G)$ be the group algebra of $G$ endowed with the norm $||.||_1$ and the convolution product as defined in [6]. By [7, Theorem 23.7] we know that

$$\Delta(L^1(G)) = \{ \phi_\rho ; \rho \in \hat{G} \},$$

where $\hat{G}$ is the space of all continuous homomorphisms from $G$ into the circle group $T$ and $\phi_\rho$ defined by

$$\phi_\rho(h) = \int_G \rho(x)h(x)dx \quad (h \in L^1(G)).$$

**Theorem 4.1.** Let $G$ be a locally compact group.

1. If $G$ is an amenable group, then $L^1(G)$ is $\Delta$-weak $\phi$-amenable for each $\phi \in \Delta(L^1(G)) \cup \{0\}$.

2. For $1 < p < \infty$ and $\phi \in \Delta(A_p(G)) \cup \{0\}$, $A_p(G)$ is $\Delta$-weak $\phi$-amenable if and only if $G$ is an amenable group.
Proof. (1): It follows from [12, Corollary 2.4] that $L^1(G)$ is $\phi$-amenable for all $\phi \in \Delta(L^1(G))$. Also, we know that the group algebra has a bounded approximate identity and this completes the proof of (1) by using Theorem 1.1.

(2): If $G$ is an amenable group, then by Leptin-Herz’s Theorem we know that $A_p(G)$ has a bounded approximate identity. On the other hand, by [12, Corollary 2.4], $A_p(G)$ is $\phi$-amenable for each $\phi \in \Delta(A_p(G))$. So, the result follows from Theorem 1.1.

Conversely, let $A_p(G)$ be $\Delta$-weak $\phi$-amenable. If $\phi = 0$ the result follows from Lemma 4.1. If $\phi \in \Delta(A_p(G))$, by Corollary 2.1 we know that $A_p(G)$ has a $b$.$\Delta$-$w.a.i$. So, the result follows from Lemma 4.1.

5. Examples

In this section, we only give some instructive examples.

The following example shows that the $\Delta$-weak 0-amenity and 0-amenity are different.

Example 5.1. Let $S = \mathbb{Q}^+$ be the semigroup of positive rational numbers under addition. So, $A = l^1(\mathbb{Q}^+)$ is $\Delta$-weak 0-amenable, but it is not 0-amenable. The reason is that $A$ has a $b$.\$\Delta$-$w.a.i$, but it does not have any approximate identity [9].

The following three examples give Banach algebras which are $\Delta$-weak $\phi$-amenable, but they are not $\phi$-amenable.

Example 5.2. Let $X$ be a Banach space and take $\phi \in X^* \setminus \{0\}$ with $||\phi|| \leq 1$. Define a product on $X$ by $ab = \phi(a)b$ for all $a,b \in X$. With this product $X$ is a Banach algebra which we denote it by $A_\phi(X)$. It is clear that $\Delta(A_\phi(X)) = \{\phi\}$ and $A_\phi(X)$ is $\phi$-amenable if and only if $\text{dim}(X) = 1$ [16, Example 2.4].

Thus, if we take a Banach space $X$ with $\text{dim}(X) > 1$ and a non-injective $\phi \in X^*$, then $A_\phi(X)$ is not $\phi$-amenable, but it is $\Delta$-weak $\phi$-amenable by Proposition 2.1.

On the other hand, let $x_0 \in X$ be such that $\phi(x_0) = 1$. Then $x_0$ is a $\Delta$-weak identity for $A_\phi(X)$, but it is not an identity. Because, for $0 \neq a \in \ker(\phi)$ we have $ax_0 = \phi(a)x_0 = 0$. Therefore, $ax_0 \neq a$. So, $x_0$ is not an identity. Moreover, it is clear that the $\Delta$-weak identity of the Banach algebra $A_\phi(X)$ is not unique, since each element $a_0 \in X$ such that $\phi(a_0) = 1$, is a $\Delta$-weak identity for $A_\phi(X)$.

Let $A$ and $B$ be Banach algebras with $\Delta(B) \neq \emptyset$ and $\theta \in \Delta(B)$. The $\theta$-Lau product $A \times_\theta B$ is defined as the Cartesian product $A \times B$ with the following multiplication,

$$(a,b)(a_1,b_1) = (aa_1 + \theta(b)a_1 + \theta(b_1)a, bb_1) \quad (a,a_1 \in A, b,b_1 \in B).$$

With the $l^1$-norm and the above multiplication, $A \times_\theta B$ is a Banach algebra; see [13].

Example 5.3. Let $A = B = A_\phi(X)$ be the Banach algebra defined in the above example and $\text{dim}(X) > 1$. Consider the $\phi$-Lau product $A_\phi(X) \times_\phi A_\phi(X)$. By [13, Proposition 2.8], we know that $|\Delta(A_\phi(X) \times_\phi A_\phi(X))| = 2$. Let $\Delta(A_\phi(X) \times_\phi A_\phi(X)) = \{\Theta_1, \Theta_2\}$. So, by Proposition 2.1, $A_\phi(X) \times_\phi A_\phi(X)$ is $\Delta$-weak $\Theta_1$-amenable and $\Delta$-weak $\Theta_2$-amenable. On the other hand, $A_\phi(X)$ is not $\phi$-amenable. Hence, by [14, Lemma 6.8 (iii)] there exists a character $\Theta_i$, $i = 1$ or $2$ of $A_\phi(X) \times_\phi A_\phi(X)$ such that $A_\phi(X) \times_\phi A_\phi(X)$ is not $\Theta_i$-amenable.
Example 5.4. Let \( n \geq 2 \) be an integer number and let \( A \) be the Banach algebra of all upper-triangular \( n \times n \) matrix over \( \mathbb{C} \). We have \( \Delta(A) = \{ \phi_1, \phi_2, \phi_3, \ldots, \phi_n \} \) where \( \phi_k([a_{ij}]) = a_{kk} \quad (k = 1, 2, 3, \ldots, n) \).

Then for each \( \phi_k \), \( A \) is \( \Delta \)-weak \( \phi_k \)-amenable. To see this, let \( e_0 = [a_{ij}] \) be an element of \( A \) such that

\[
a_{ij} = \begin{cases} 
1 & i = j, i \neq k \\
0 & i = j = k \\
0 & i \neq j 
\end{cases}
\]

Obviously, \( e_0 \) is in \( \ker(\phi_k) \). But \( \phi_i(e_0) = 1 \) for all \( i \neq k \). So, the result follows by using Theorem 2.1.

Also, \( A \) is not \( \phi_k \)-amenable for each \( k \geq 2 \). Because, \( \ker(\phi_k) \) does not have a right identity. Therefore, by [10, Proposition 2.2] \( A \) is not \( \phi_k \)-amenable.

Some Banach algebras satisfy both concepts of \( \phi \)-amenability and \( \Delta \)-weak \( \phi \)-amenability.

Example 5.5. If \( A \) is a \( C^* \)-algebra, then \( A \) is \( \phi \)-amenable and \( \Delta \)-weak \( \phi \)-amenable, because each \( C^* \)-algebra and their closed ideals have bounded approximate identity [2, Theorem 3.2.21].

There exists a Banach algebra that is neither \( \phi \)-amenable nor \( \Delta \)-weak \( \phi \)-amenable as the next example shows. Also, the following example shows that the converse of Corollary 2.2 is not valid in general.

Example 5.6. Let \( A = C^1[0, 1] \) be the Banach algebra consisting of all continuous functions on \([0, 1]\) with continuous derivation and norm \(||f||_1 = ||f||_\infty + ||f'||_\infty \). We know that

\[
\Delta(A) = \{ \phi_t : \phi_t(f) = f(t) \text{ for each } t \in [0, 1] \}.
\]

By [10, Example 2.5(1)], \( A \) is not \( \phi_t \)-amenable for any \( t \in [0, 1] \). Moreover, there does not exist \( t_0 \in [0, 1] \) such that \( A \) is \( \Delta \)-weak \( \phi_{t_0} \)-amenable. To see this, let \( \{ f_n \} \) be a b.\( \Delta \)-w.a.i for \( \ker(\phi_{t_0}) \). So, it has the following properties

1. \( \lim_{n} f_n(t) = 1 \quad (t \in [0, 1] \setminus \{ t_0 \}) \),
2. \( \{ ||f_n||_\infty \} \) is bounded.

Hence, there exists a non-negative constant \( M \) with \( ||f'_n||_\infty = \sup_{t \in [0, 1]} |f'_n(t)| < M \) for all \( n \in \mathbb{N} \). Hence, for positive integer \( n_0 \) we have

\[
\lim_{t \to t_0} \left| \frac{f_{n_0}(t) - f_{n_0}(t_0)}{t - t_0} \right| = |f'_{n_0}(t_0)| < M.
\]

Therefore, there exists \( \epsilon > 0 \) such that for each \( t \in N(t_0, \epsilon) = \{ t : 0 < |t - t_0| < \epsilon \} \) we have

\[
|f_{n_0}(t) - f_{n_0}(t_0)| < M|t - t_0|.
\]

But the above relation is not valid in general, because the right hand side of the inequality tends to zero as \( t \to t_0 \), but the left hand side does not.

Hence, \( A \) is not \( \Delta \)-weak \( \phi_{t_0} \)-amenable for each \( t \in [0, 1] \).

Also, this Banach algebra is \( \Delta \)-weak 0-amenable, because the sequence \( \{ \frac{n-t_0}{n} \} \) is a bounded \( \Delta \)-weak approximate identity for \( A \).
The converse of Theorem 3.1 does not hold in general as the following example shows.

**Example 5.7.** Let $A = C^1[0,1]$, $B = C[0,1]$ and $h : A \to B$ be the inclusion homomorphism. It is clear that $A$ is dense in $B$. By Examples 5.5 and 5.6 for each $t \in [0,1]$, $B$ is $\Delta$-weak $\phi_t$-amenable, but $A$ is not $\Delta$-weak $\phi_t$-amenable.

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