ON THE OPTIMIZATION OF AN ACKERMANN STEERING LINKAGE

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In the paper the optimization problem of the Ackermann linkage is examined taking into account in turn two criteria: the maximum of the absolute value of the steering error and the root-mean-square of the steering error. For this purpose the transmission function of the mechanism is directly used without resorting to the method of increasing the degree of freedom.

Keywords: Ackermann linkage, optimization, steering, steering angle, vehicle

1. Introduction

The steering system of an automobile with a rigid steering axle (beam and steering knuckles hinge-connected to the beam with king pins) comprises the steering mechanism which is represented by the steering trapezium otherwise known as Ackermann steering linkage. The analysis and synthesis of the Ackermann steering linkage has been tackled in numerous papers [1]. Although in these works the problem of solution finding which should lead to enough good approximation of the correct steering condition has been set, a rigorous approach to optimization has been undertaken only relatively recently. So, in [2] the optimization problem of a planar Ackermann linkage has been investigated by the method of increasing the degree of freedom of mechanism. It is supposed that the function which is generated by the mechanism should ensure positions which are as near as possible to certain chosen positions (design points). For the optimization of a planar Ackermann linkage in [3] a method is applied which minimizes the strength energy of the bars by considering this elasticity. There are some difficulties when performing the numerical optimization.

In the present paper the synthesis of the Ackermann linkage is tackled using systematically normalized lengths and paying more attention to the constraints imposed by the peculiarities of the mechanism [4]. The optimization problem of the Ackermann linkage is examined taking into account two criteria: the criterion of the absolute value of the steering error and the criterion of the root-mean-square of the steering error. For this purpose the transmission function of the mechanism is directly used, without resorting to the method of increasing the degree of freedom.

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In the paper we consider the planar Ackermann linkage and the classical condition of correct steering of a vehicle given by the Ackermann’s relation [1, 5]. Also, in the following we use the notations and the relations derived in [4].

2. The steering error

Schematically, the disposal of an Ackerman steering linkage is shown in Fig. 1. The shown position corresponds to the straight-line motion of automobile. The following notations are used: \(l_j\)-length of the element \(j\) \((j=1,2,3,4)\); \(\phi_1=\angle(BAD)\); \(\phi_2=\angle\) between BC and AD (the positive direction is counter clockwise rotation of BC); \(\phi_3=\angle(CDA)\); \(E_p=l_4\)-king pin track; \(\beta_e\) and \(\beta_i\)-turning angle of outer wheel and inner wheel, respectively; \(\phi_{10}=\phi_{30}\)-the values of \(\phi_1\) and \(\phi_3\) when the wheel are not turned. Also we introduce the normalized lengths: 
\[
\lambda_1 = \frac{l_1}{l_4}, \quad \lambda_2 = \frac{l_2}{l_4}, \quad \lambda_3 = \frac{l_3}{l_4}, \quad l_1 = l_3.
\]

In [4] it has been proved that

\[
\phi_2(\phi_1) = \arccos\left(\frac{1 + \lambda_2^2 - 2\lambda_1 \cos \phi_1}{2\lambda_1 \sqrt{1 + \lambda_2^2 - 2\lambda_1 \cos \phi_1}} - \arctg\left(\frac{\lambda_1 - 1}{\lambda_1 + 1} \cdot \cot g \frac{\phi_1}{2}\right)\right) - \frac{\pi - \phi_1}{2}, \quad (1)
\]

\[
\phi_3(\phi_1) = \arccos\left(\frac{1 + 2\lambda_1^2 - \lambda_2^2 - 2\lambda_1 \cos \phi_1}{2\lambda_1 \sqrt{1 + \lambda_2^2 - 2\lambda_1 \cos \phi_1}} + \arctg\left(\frac{\lambda_1 - 1}{\lambda_1 + 1} \cdot \cot g \frac{\phi_1}{2}\right)\right) + \frac{\pi - \phi_1}{2}, \quad (2)
\]

\[
\tan^{-1}\left(\frac{\lambda_1 - 1}{\lambda_1 + 1} \cdot \cot g \frac{\phi_1}{2}\right) = \frac{1}{\sin(\phi_2 - \phi_1)} - \frac{1}{\sin(\phi_2 + \phi_3)}.
\]

Inspecting Fig 1 one can write the relationship

\[
\beta_e = \pm(\phi_1 - \phi_{10}), \quad \beta_i = \pm(\phi_{10} - \phi_3),
\]

where the sign + is adopted for the trailing Ackerman steering linkage and the sign – corresponds to the leading Ackerman steering linkage. Also the angles \(\phi_{10}\) and \(\phi_{30}\) are given by the relation.
\[ \varphi_{10} = \varphi_{30} = \frac{1 - \lambda_2}{2 \lambda_1}. \] (5)

The transmission angle between element 2 and element 3 for the trailing Ackermann linkage and the leading Ackermann linkage, respectively, is given by the relations [4]

\[ \gamma_{23} = \varphi_2 + \varphi_3, \quad \gamma_{23} = \pi - (\varphi_2 + \varphi_3). \] (6)

The condition of correct steering is written as [1, 5]:

\[ \cot g \beta_e - \cot g \beta_i = E_p / L, \] (7)

where \( L \) is the wheel base of the automobile. If it is supposed that \( \beta_e \) is the input magnitude and \( \beta_i \) is output magnitude then the relation (7) leads to the relation

\[ \beta_i = \arctg \left( \frac{\tan \beta_e}{1 - (E_p / L) \cdot \tan \beta_e} \right). \] (8)

The steering error is defined by the difference between the turning angle of inner wheel which is carried out by the Ackermann steering linkage and the turning angle of the same wheel corresponding to the condition of correct steering for a given turning angle of outer wheel. Therefore, the expression of this error can be written as

\[ e_s(\beta_e, \lambda_1, \lambda_2) = \pm[\varphi_{10} - \varphi_3(\pm \beta_e + \varphi_{10})] - \arctg \left[ \frac{\tan \beta_e}{1 - (E_p / L) \cdot \tan \beta_e} \right] \] (9)

where sign + is for the trailing Ackerman steering linkage and the sign – is for the leading Ackermann steering linkage (the function \( \varphi_3 \) has been previously defined by the relation (1)).

The function (9) corresponding to the trailing Ackermann linkage is defined on the domain \( D_2 \) [4]. The function (9) is defined in the point \((\lambda_1, e_1, 1)\) with \( \lambda_1 > 0 \), but it is not defined in the point \( O_1(0, 1) \). It is readily verified that point \( O_1 \) is an accumulation point of the set defined by \( D_2 \) with \( \lambda_1 > 0 \). Also, it can be readily verified that there is no limit at the point \( O_1 \) (on variable ensemble). Indeed, putting \( \lambda_2 = q, \lambda_1 + 1 \) it is found that the limit of this function is dependent on \( q \). This finding is very important, having outcomes which will be discussed later. The function (9) corresponding to the leading Ackermann linkage is defined on the domain \( D_2 \) (see [4]) but it is not defined in the point \( O_1 \) (the domain \( D_3 \) is similar to the domain \( D_2 \)). Also, this function doesn’t have a limit at the point \( O_1 \).

In Fig.2 the curves \( e_s(\beta_{e0}, \lambda_1, \lambda_2) = 0 \) for different given values of \( \beta_{e0} \) are shown. Also, in the same figure are shown the curves \( b_{k3} \) and \( b_{l3} \) corresponding to the admissible transmission angle that settle the limits of the admissible domains \( D_3 \) and \( D_3 \) [4] (\( b_{k1} \) and \( b_{l1} \) are the boundaries of the existence domains of the Ackermann steering linkage).
It is found that in the case of trailing Ackermann steering linkage some curves are situated out of the admissible domain, at least on certain length. Also, it is noticed that for $\lambda_1\in[0.05, 0.20]$ the curves for $\beta_e>20^\circ$ are very near one with another and they are included in the admissible domain. In exchange, in the case of the leading Ackermann linkage the curves are included in the admissible domain.

3. Criterion of the maximum of absolute steering error value

In comparison with the above discussion it is necessary to use a more adequate criterion for the determination of the parameters $\lambda_1$ and $\lambda_2$. This criterion has been used in [2]. More precisely, one uses an objective function as a maximum of the absolute value of the steering error on the interval $[0, \beta_{emax}]$: 

$$f_0(\lambda_1, \lambda_2) = \max_{0 \leq \beta_e \leq \beta_{emax}} |e_s(\beta_e, \lambda_1, \lambda_2)|.$$  \hspace{1cm} (10)
The optimal parameters are those which achieve minimization of $f_0$ with respect to $\lambda_1$ and $\lambda_2$, being satisfied the above discussed constraints. Although the function $e_\varepsilon(\beta_e,\lambda_1,\lambda_2)$ is differentiable with respect to $\beta_e$, generally, the function $|e_\varepsilon(\beta_e,\lambda_1,\lambda_2)|$ is not differentiable. This fact brings forth difficulties to solve the optimization problem. Also, an analytic approach of the optimization problem is in practice impossible.

Starting from (2) and (3) we have defined the function (10) by the function NMaximize of the programming environment Mathematica®.

For the above reasons one used the optimization methods without derivatives: Simulated Annealing and Differential Method. For a given value of the ratio $E_\varepsilon/L$ the graph of the function (10) is shown in Fig. 3. It is found that $f_0$ has very large values (but finite) on the boundary of the definition domain of function $e_\varepsilon$. This fact can be explained because the critical position of the...
mechanism the transmission ratio $i_{33}=0$ [4]. Also, it has been ascertained that the function $f_0$ decreases fast in proportion as the point $(\lambda_1, \lambda_2)$ moves away from the boundary. After that the function $f_0$ increases. But, what really matters is the domain corresponding to the motion transmitting. In this case the graph of the function $f_0$ is shown in Fig.4. It is found that the minimum values of $f_0$ are obtained on the boundary of the admissible domain. In a direct way this fact is highlighted by contours lines of the function $f_0$ as shown in Fig.5. In this figure are also shown the boundaries of the admissible domains defined by the admissible value $\gamma_2$ of transmission angle. In the case of the trailing Ackermann linkage it is found that as $\lambda_1$ decreases on the domain boundary the value of the criterion decreases also in proportion. The ensemble of the contour lines is focused on the point (0, 1) without it being attained. Of course, for small values of $\lambda_1$, large variations of the criterion are produced when the variations of $\lambda_2$ are small. For optimization it is sufficiently to choose a value of $\lambda_1$ and after that to solve the equation obtained by equating to zero the function of the transmission angle with respect to $\lambda_2$. For example, for $\lambda_1=0.16$, $E_p/L=1/1.9$, $\beta_{emax}=35^\circ$ and $\gamma_2=25^\circ$ the results are: $\lambda_2=0.91911$ and $f_0=3.026^\circ$. In the case of the leading Ackermann linkage as $\lambda_1$ increases on the domain boundary the optimization criterion decreases in proportion. For example, for $\lambda_1=0.16$ the obtained results are: $\lambda_2=1.1184$ and $f_0=1.55^\circ$. For $\lambda_1=0.30$ we have: $\lambda_2=1.260$ and $f_0=1.1528^\circ$. It is clear that the diminution of min $f_0$ is not very marked when $\lambda_1$ increases.

The variation of the steering error with respect to the outer turning angle is shown in Fig. 6. It is found that in the case of the leading Ackermann steering linkage the absolute value of the steering error is smaller than that for the trailing Ackermann steering linkage.

![Fig.6](image_url)
4. Criterion of root-mean-square steering error

Since the steering error should be reduced on the whole, we can use an integral criterion under the form of

\[
I(\lambda_1, \lambda_2, E_p / L) = \int_0^{\beta_{e_{\text{max}}}} e_s^2(\beta_e, \lambda_1, \lambda_2, E_p / L) f_w(\beta_e) d\beta_e, \tag{11}
\]

where \( f_w(\beta_e) > 0 \) is a weight function with the property

\[
\int_0^{\beta_{e_{\text{max}}}} f_w(\beta_e) d\beta_e = 1. \tag{12}
\]

Obviously, the parameters \( \lambda_1 \) and \( \lambda_2 \) are chosen so that \( I(\lambda_1, \lambda_2) \) to have the minimum value. A similar criterion has been used in [6] for the case of a six-bar steering linkage. The weight function is chosen taking into account the concrete operation conditions of the vehicle. In [6] it is assumed that \( f_w = 1 \), in the way and in the present paper we shall act. To make comparisons more easily we use the root mean square steering error defined as

\[
e_{\text{rms}}(\lambda_1, \lambda_2, E_p / L) = \frac{180}{\pi} \sqrt{\int_0^{\beta_{e_{\text{max}}}} e_s^2(\beta_e, \lambda_1, \lambda_2, E_p / L) d\beta_e} / \beta_{e_{\text{max}}} \tag{13}
\]

The optimization consists in the minimization of the root-mean-square steering error with respect to \( \lambda_1 \) and \( \lambda_2 \). So we have a constrained optimization problem. The optimum point may be or the critical point of the function \( e_{\text{rms}} \), either a point situated on the boundary of the admissible domain defined by the constraints. The critical point is defined by the relations

\[
\frac{\partial e_{\text{rms}}}{\partial \lambda_1} = 0, \quad \frac{\partial e_{\text{rms}}}{\partial \lambda_2} = 0, \tag{14}
\]

which lead to

\[
\frac{\partial}{\partial \lambda_j} \int_0^{\beta_{e_{\text{max}}}} e_s^2(\beta_e, \lambda_1, \lambda_2, E_p / L) d\beta_e = 0, \quad (j = 1, 2). \tag{15}
\]

The condition (15) becomes

\[
\int_0^{\beta_{e_{\text{max}}}} e_s(\beta_e, \lambda_1, \lambda_2, E_p / L) \frac{\partial e_s}{\partial \lambda_j} d\beta_e = 0, \quad (j = 1, 2). \tag{16}
\]

For the trailing Ackermann linkage the preceding condition is written as

\[
\frac{\partial \phi_{10}}{\partial \lambda_j} \int_0^{\beta_{e_{\text{max}}}} (1 - \frac{d\phi_3}{d\phi_1}(\beta_e + \varphi_{10})) e_s(\beta_e, \lambda_1, \lambda_2, E_p / L) d\beta_e = 0, \quad (j = 1, 2) \tag{17}
\]

with \( \partial \phi_{10} / \partial \lambda_j \neq 0 \). Further we arrive at

\[
I_t(\lambda_1, \lambda_2, E_p / L) = \int_0^{\beta_{e_{\text{max}}}} \left(1 - \frac{d\phi_3}{d\phi_1}(\beta_e + \varphi_{10})\right) e_s(\beta_e, \lambda_1, \lambda_2, E_p / L) d\beta_e = 0. \tag{18}
\]
Therefore, the conditions (14) are satisfied if a single condition (18) takes place. As such, the critical points of the mentioned function are not isolated points. They are situated on a curve of which projection on the plane \((\lambda_1, \lambda_2)\) is defined by the equation (18). In this case the critical points are of type trough as we can see on the graphical representation in Fig. 7. To establish the expression of the integrand function in (18) one uses the relations (1),(2),(3),(4) and (5). The integral of (18) cannot be expressed by elementary functions and its evaluation is numerically carried out. Using an adequate program in Mathematica® the curve \(I(\lambda_1, \lambda_2, E_p/L)=0\) has been drawn for given values of \(E_p/L\). In the case of the leading Ackermann linkage a similar method has been used. The results are shown in Fig. 8.

In the case of the trailing trapezium the curves are situated higher in proportion as \(E_p/L\) decreases (one can verify that for \(E_p/L=0.26\) the curve is completely situated within the admissible domain, which is not case for \(E_p/L=0.46\)). In the case of the leading trapezium in proportion as the mentioned ratio decreases the curves are situated more down.

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**Fig. 7.** Root mean square steering error as a function of \(\lambda_1\) and \(\lambda_2\) \((E_p/L=1/1.9, \beta_{max}=35^\circ)\) : a) trailing Ackermann linkage ; b) leading Ackermann linkage

**Fig. 8.** Projections of critical lines on the plane \((\lambda_1, \lambda_2)\) : a) trailing Ackermann linkage ; b) leading Ackermann linkage
The contour lines of the function $e_{rms}$, the curve $I_t=0$, the curve corresponding to the critical position of the mechanism for $\beta_{\text{max}}$, and the curve corresponding to the admissible transmission angle $\gamma_a=25^\circ$ are plotted in Fig. 9 (for $E_p/L=1/1.9$). The last two curves are depicted by $B_{t1}$, $B_{t3}$, $B_{t2}$ and $B_{t3}$ (see [4]). For a given value of $\lambda_1$ there is a certain value of $\lambda_2$ corresponding to the curve $I_t=0$ for which the root mean square steering error has the minimum value. The minimum value is dependent on the position of the figurative point on the mentioned curve. This fact is made evident on Fig. 9. At the same time it is necessary to observe that there are situations when curve $I_t=0$ is not in part or totally included in the admissible domain. In the case of Fig. 9a curve $I_t=0$ is situated out of admissible domain defined by $\gamma_a$. Therefore, the constrained optimum is given by the curve corresponding to $\gamma_a$. For a given $\lambda_1$ one can determine $\lambda_2$ from equation derived from (6) putting $\gamma_a=25^\circ$ (obviously, in turn, for each type of Ackermann linkage). After that one can determine the value of $e_{rms}$, which is dependent on the position of the figurative point on the mentioned curve. This fact is highlighted in Fig. 10a. It is found that the constraint related to the transmission angle leads to a large value of $e_{rms}$ comparatively with that of $\min e_{rms}$. In the case of Fig. 9b the curve $I_t=0$ is near the curve corresponding to $\gamma_a$. It is situated on the admissible domain, as $\lambda_1$ increases the optimum value of $e_{rms}$ decreases in proportion (see Fig. 10b). Although the curve corresponding to $\gamma_a$ is very close to the curve corresponding to $I_t=0$ the values of $e_{rms}$ are large enough. This fact is in concordance with the variation of $e_{rms}$ shown in Fig. 10b.
Generally, it is found that the errors are reduced by the leading Ackermann linkage. In the case of the trailing Ackermann linkage the errors decrease when $\lambda_1$ decreases, which is also contrary to the case of leading trapezium. From analysis of the surface shown in Fig. 7 it is noticed that the variations of the $rms$ steering error are very important in the side near to the boundary of the admissible domain. Therefore, for a given $\lambda_1$, the small variations of $\lambda_2$ around the optimum value leads to the important increases of the $rms$ steering error. This result is in accordance with the results of Fig. 9, finding that for the small values of $\lambda_1$ the curves are much nearer together. In consequence, the small variations of $\lambda_2$ lead to the significant variations of the angles involved by these curves. The general recommendations relating to choosing of $\lambda_1 \in [0.12, 0.20]$ [1] are in accordance with this above mentioned especially for the trailing trapezium (it is necessary to have a reduced size). But, at the same time, it is necessary to notice that the manufacture accuracy of the bars for the lengths should be high, otherwise the steering errors will become large. For the leading Ackermann linkage it is desired that $\lambda_1$ to have large values, but there is a size limitation because the knuckle arm nears to the wheel.

Obviously, it is important how the steering error changes as a function of the outer steering angle. To illustrate this dependence also the consequences of choosing of the parameters in concordance with the mentioned criterion one considered certain optimal values of $\lambda_1$ and $\lambda_2$ and plotted the curves shown in Fig. 11 for different values of $E_p/L$. 
It is found that the largest errors are obtained by large turning angles. Generally, for “long” vehicles the steering error is smaller. In any way, the steering error is enough reduced in comparison with that indicated in literature [1]. If, for instance, for $\frac{E_p}{L}=0.46$ one chooses $\lambda_1=0.30$ and $\lambda_2=0.80$, as in [7], then the steering error becomes very large, especially for $\beta_\text{e max}=30^\circ$, namely $15^\circ$. In [2], for $\frac{E_p}{L}=1/1.9$, $\beta_\text{max}=40^\circ$ and $\lambda_1=0.16$ the optimum design of a trailing Ackermann steering linkage is obtained by $\phi_{01}=180^\circ-109.15921^\circ$ (see table1 [2]). The mentioned values lead to $\beta_\text{max}=30.2015^\circ$ and $\lambda_2=0.894978$. For $\frac{E_p}{L}=1/1.9$, $\frac{E_p}{L}=1/1.9$ and $\lambda_1=0.16$ the method used in the present paper results in $\lambda_2=0.895693$. Therefore, the results by the two methods are very close together (percentage difference is 0.08%).

5. Conclusions

The maximum absolute value of the steering error and the root-square-mean steering error as functions of $\lambda_1$ and $\lambda_2$ (normalized lengths of the knuckle arm and of the tie road, respectively) has not a proper minimum point.

In the paper it is shown how to determine the normalized lengths of the bar mechanism so that the mentioned functions get values as reduced as possible.
taking into account different constraints. Generally, these quantities correspond to the minimum admissible value of transmission angle.

The obtained results are in accordance with those which have been obtained in other works, but the method used in the paper is more simple and direct.

In the paper it was demonstrated that the Ackermann steering linkage has intrinsic limitations in term of achieving the correct steering condition.

REFERENCES


