FIXED POINT RESULTS FOR NONLINEAR CONTRACTIONS WITH GENERALIZED Ω-DISTANCE MAPPINGS

by Issam Abu-Irwaq¹, Wasfi Shatanawi², Anwar Bataihah³ and Inam Nuseir⁴

Khojasteh et al. [F. Khojasteh, S. Shukla and S. Radenovic, A new approach to the study of fixed point theory for simulation functions, Filomat 29:6 (2015)] defined a new class of mappings namely simulation functions in which they used it to unify several fixed point results in the literature. In this paper we introduce the notion of $(\Omega, \phi, \mathcal{Z})$-contraction with respect to $z$ through generalized $\Omega$-distance mappings which introduced by Abodayeh et al. [K. Abodayeh, A. Bataihah and W. Shatanawi, Generalized $\Omega$-distance mappings and some fixed point theorems, U.P.B. Sci. Bull. Series A, Vol. 79, Iss.2, 2017] and we prove some fixed point results. Also, we give an example to support our main result.

Keywords: fixed point, simulation mappings, G-metric spaces, generalized Omega-distance

1. Introduction

The fixed point theory considered as a main tool in pure and applied mathematics since it gives a solution for the equation $f(x) = x$ for a self mapping $f$ under some considerations. In fact the fixed point theory has been studied in various directions for instance see [12]–[34]. The concept of $b$-metric spaces was introduced by Bakhtin [3] which has became well known by Czerwik [4]. In 2014 Aghanjani et al. [2] introduced the concept of $G_b$-metric spaces (or generalized $b$-metric spaces) using the concepts of $G$-metric spaces and $b$-metric spaces and studied some fixed point results, for more fixed point results on $G_b$-metric spaces we refer the reader to see [5, 6].

2. Preliminaries

The concept of $G_b$-metric spaces is defined as follows:

**Definition 2.1.** [2] Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfies:

$(G_b1)$ $G(x, y, z) = 0$ if $x = y = z$;

$(G_b2)$ $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$;

$(G_b3)$ $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

$(G_b4)$ $G(x, y, z) = G(p[x, y, z])$, where $p$ is a permutation of $x, y, z$ (symmetry);

$(G_b5)$ $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)] \forall x, y, z, a \in X$ (rectangle inequality).

Then the function $G$ is called generalized $b$ metric and the pair $(X, G)$ is called a generalized $b$ metric space or $G_b$-metric space.

¹Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Irbid 22110, Jordan, e-mail: imabuirwaq@just.edu.jo

²Department of General Sciences, Prince Sultan University, Riyadh, Saudi Arabia and Department of Mathematics, Faculty of Science, Hashimite University, P.O. Box 150459, Zarqa, Jordan e-mail: ushatanawi@psu.edu.sa and swasfi@hu.edu.jo

³Department of Mathematics, Faculty of Science, The University of Jordan, Amman, Jordan anwerbataihah@gmail.com

⁴Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Irbid 22110, Jordan, e-mail: immuseir@just.edu.jo
Aghanjani et al. [2] remarked that the class of $G_b$-metric spaces is larger than that of G-metric spaces. The following example shows that $G_b$-metric on X need not be G-metric on X.

**Example 2.1.** [2] Let $(X,G)$ be a G-metric space and $p > 1$. Define $G_\epsilon : X \times X \times X \rightarrow \mathbb{R}^+$ by $G_\epsilon(x,y,z) = G(x,y,z)^p$. Then $G_\epsilon$ is $G_b$-metric on X with $s = 2^{p-1}$.

Now, we present some definitions and propositions in $G_b$-metric space.

**Definition 2.2.** [2] Let X be a $G_b$-metric space. A sequence $(x_n)$ in X is said to be

1. $G_b$-convergent to $x \in X$ if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_m) < \epsilon \ \forall n,m \geq k$.
2. $G_b$-Cauchy if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_{m,i}) < \epsilon \ \forall n,m,l \geq k$.

**Proposition 2.1.** [2] Let X be a $G_b$-metric space. Then the following are equivalent:

1. The sequence $(x_n)$ is $G_b$-convergent to x.
2. $G(x_n,x) \rightarrow 0$ as $n \rightarrow \infty$.
3. $G(x_n,x,x) \rightarrow 0$ as $n \rightarrow \infty$.

**Proposition 2.2.** [2] Let X be a $G_b$-metric space. The sequence $(x_n)$ is $G_b$-Cauchy iff for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_{m,i}) < \epsilon \ \forall n,m \geq k$.

**Definition 2.3.** [2] A $G_b$-metric space X is called $G_b$-complete if every $G_b$-Cauchy sequence is $G_b$-convergent in X.

Very recently, Abodayeh et al. [1] defined the concept of generalized $\Omega_b$-distance mappings (or $\Omega_b$-distance) related to $G_b$-metric spaces and proved some fixed point theorems (see also [19]).

The notion of a generalized $\Omega_b$-distance mapping is given by:

**Definition 2.4.** [1] Let X be a $G_b$-metric space. Then a mapping $\Omega : X \times X \times X \rightarrow [0,\infty)$ is called a generalized $\Omega_b$-distance mapping or an $\Omega_b$-distance mapping on X if the following conditions are satisfied:

1. $\Omega(x,y,z) \leq s \cdot [\Omega(x,a,a) + \Omega(a,y,z)] \forall x,y,z,a \in X$ and $s \geq 1$.
2. For any $x,y \in X$, $\Omega(x,y,.)$ is continuous.
3. For every $\epsilon > 0$, there is a $\delta > 0$ such that $\Omega(x,a,a) \leq \delta$ and $\Omega(a,y,z) \leq \delta$ imply $G_b(x,y,z) \leq \epsilon$.

**Example 2.2.** [1] Let $X = \mathbb{R}$. Consider the $G_b$-metric $G : X \times X \times X \rightarrow [0,\infty)$ defined by $G(x,y,z) = (|x-y| + |y-z| + |x-z|)^2 \forall x,y,z \in \mathbb{R}$. Define $\Omega(x,y,.) : X \times X \times X \rightarrow [0,\infty)$ by $\Omega(x,y,z) = (|x-y| + |y-z|)^2 \forall x,y,z \in \mathbb{R}$. Then $\Omega$ is a generalized $\Omega_b$-distance mapping with $s = 2$.

**Definition 2.5.** [1] Let $(X,G)$ be a $G_b$-metric space and $\Omega_b$ be an $\Omega_b$-distance mapping on X. Then we say that X is $\Omega_b$-bounded if there exists $M > 0$ such that $\Omega_b(x,y,z) \leq M$ for all $x,y,z \in X$.

**Lemma 2.1.** [1] Let X be a $G_b$-metric space and $\Omega_b$ be a generalized $\Omega_b$-distance mapping on X. Let $(x_n)$, $(y_n)$ be sequences in X and let $(\alpha_n)$, $(\beta_n)$ be sequences in $[0,\infty)$ converging to zero and let $x,y,z \in X$. Then we have the following:

1. If $\Omega_b(x_n,x_n,x_0) \leq \alpha_n$ and $\Omega_b(x_n,y_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $G(y_n,z) \rightarrow 0$ and hence $y_n \rightarrow z$.
2. If $\Omega_b(x_n,x_0,x_n) \leq \alpha_n$ and $\Omega_b(x_n,y,z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y,y,z) \leq \epsilon$ and hence $y = z$.
3. If $\Omega_b(x_n,x_0,x_n) \leq \alpha_n$ for any $n,m,l \in \mathbb{N}$ with $n \leq m \leq l$, then $(x_n)$ is a $G_b$-Cauchy sequence.
4. If $\Omega_b(x_n,a,a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $(x_n)$ is a $G_b$-Cauchy sequence.

Khojasteh et al. [8] in 2015 introduced the concept of simulation mappings in which they used it to unify several fixed point results in the literature.

**Definition 2.6.** [8] Let $\zeta : [0,\infty) \times [0,\infty) \rightarrow \mathbb{R}$ be a function. Then $\zeta$ is called a simulation function if it satisfies the following conditions:

1. $\zeta(0,0) = 0$. 

Let \( \zeta(t,s) < s - t \) for all \( s, t > 0 \).

(3) If \( (t_n, s_n) \) are sequences in \([0, \infty)\) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \), then \( \limsup_{n \to \infty} \zeta(t_n, s_n) < 0 \).

The set of all simulation functions are denoted by \( \mathcal{Z} \).

Now, we give some examples of simulation functions. In the following \( \zeta \) is defined from \([0, \infty) \times [0, \infty)\) to \( \mathbb{R} \).

**Example 2.3.** [8] Let \( h_1, h_2 : [0, \infty) \to [0, \infty) \) be two continuous functions such that \( h_1(t) = h_2(t) = 0 \) if and only if \( t = 0 \) and \( h_2(t) < t < h_1(t) \) for all \( t \in [0, \infty) \) and define \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by \( \zeta(t,s) = h_2(s) - h_1(t) \) for all \( t, s \in [0, \infty) \). Then \( \zeta \) is a simulation function.

**Example 2.4.** [8] Let \( g : [0, \infty) \to [0, \infty) \) be a continuous function such that \( g(t) = 0 \) if and only if \( t=0 \) and define \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by \( \zeta(t,s) = s - g(s) - t \) for all \( t, s \in [0, \infty) \). Then \( \zeta \) is a simulation function.

**Example 2.5.** [11] Let \( \eta : [0, \infty) \to [0, \infty) \) be an upper semi continuous function such that \( \eta(t) < t \forall t > 0 \) and \( \eta(0) = 0 \) and define \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by \( \zeta(t,s) = \eta(s) - t \) for all \( t, s \in [0, \infty) \). Then \( \zeta \) is a simulation function.

**Example 2.6.** [11] Let \( \gamma : [0, \infty) \to [0, \infty) \) be a function such that \( \int_0^\infty \gamma(u) du \) exists \( \forall t > 0 \) and define \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by \( \zeta(t,s) = s - \int_0^s \gamma(u) du \) for all \( t, s \in [0, \infty) \). Then \( \zeta \) is a simulation function.

For more work on simulation functions in fixed point theory, we refer the reader to [9]-[11] and references therein.

### 3. Main Result

In our main result, we use a contraction condition equipped with c-comparison functions with base \( s \) which introduced by Shatanawi [7].

**Definition 3.1.** [7] Let \( s \) be a constant with \( s \geq 1 \). A function \( \phi : [0, +\infty) \to [0, +\infty) \) is called a c-comparison function with base \( s \) if \( \phi \) satisfies the following:

(i) \( \phi \) is monotone nondecreasing.

(ii) \( \sum_{n=0}^{\infty} s^n \phi^k(st) \) converges for all \( t \geq 0 \).

**Remark 3.1.** [7] If \( \phi \) is a c-comparison function with base \( s \), then \( \phi(t) < t \) for all \( t > 0 \).

The following example inspired from [7].

**Example 3.1.** Let \( s \geq 1 \). Define \( \phi_1, \phi_2 : [0, \infty) \to [0, \infty) \) by \( \phi_1(t) = kt \) where \( 0 \leq k < \frac{1}{s} \) and \( \phi_2(t) = \frac{1}{a+s} \) where \( a > 0 \). Then \( \phi_1 \) and \( \phi_2 \) are c-comparison functions with base \( s \).

Now, we introduce the following definition

**Definition 3.2.** Let \((X, G)\) be a \( G_0 \)-metric space equipped with a generalized \( \Omega \)-distance mapping \( \Omega \) with base \( s \geq 1 \) and \( \zeta \in \mathcal{Z} \). A self mapping \( T : X \to X \) is said to be \((\Omega, \phi, \mathcal{Z})\)-c-contraction with respect to \( \zeta \) if there is a c-comparison function \( \phi \) with base \( s \) such that \( T \) satisfies the following condition:

\[
\zeta(s \Omega(Tx, T^2x, Ty), \phi s \Omega(x, Tx, y)) \geq 0 \quad \forall x, y \in X. \tag{1}
\]

**Lemma 3.1.** Let \((X, G)\) be a \( G_0 \)-metric space equipped with a generalized \( \Omega \)-distance mapping \( \Omega \) with base \( s \geq 1 \). Let \( \zeta \in \mathcal{Z} \) and \( \phi \) be a c-comparison function with base \( s \). Suppose that \( T : X \to X \) is \((\Omega, \phi, \mathcal{Z})\)-c-contraction with respect to \( \zeta \). If \( T \) has a fixed point (say) \( u \in X \), then it is unique.

**Proof.** First we show that for all \( w \in X \) if \( f w = w \), then \( \Omega(w, w, w) = 0 \). Assume that \( \Omega(w, w, w) > 0 \). From (1) and (\( \zeta_2 \)), we have
0 \leq \zeta(s\Omega(Tw, T^2w, Tw), \phi s\Omega(w, Tw, w)) \\
= \zeta(s\Omega(w, w, w), \phi s\Omega(w, w, w)) \\
< \phi s\Omega(w, w, w) - s\Omega(w, w, w), \\
< s\Omega(w, w, w) - s\Omega(w, w, w), \\
= 0,

a contradiction. Hence \( \Omega(w, w, w) = 0 \).

Now, assume that there is \( v \in X \) such that \( Tv = v \) and \( \Omega(u, v, v) > 0 \). Since \( T \) is \( (\Omega, \phi, \mathcal{Z}) \)-contraction with respect to \( \zeta \), then by substituting \( x = u \) and \( y = v \) in (1) and taking into account (\( \zeta \)), we have

\[
0 \leq \zeta(s\Omega(Tu, T^2u, Tu), \phi s\Omega(u, Tu, v)) \\
= \zeta(s\Omega(u, u, v), \phi s\Omega(u, u, v)) \\
< \phi s\Omega(u, u, v) - s\Omega(u, u, v) \\
< s\Omega(u, u, v) - s\Omega(u, u, v) = 0,
\]

a contradiction. Hence \( \Omega(u, v, v) = 0 \). Thus by the definition of \( \Omega \) we have \( G(u, v, v) = 0 \) and so \( u = v \).

**Theorem 3.1.** \((X, G)\) be a \( G_{\Omega, s}\)-metric space equipped with a generalized \( \Omega \)-distance mapping \( \Omega \) with base \( s \geq 1 \) such that \( X \) is \( \Omega \)-bounded and \( \zeta \in \mathcal{Z} \). Suppose that there is a \( c \)-comparison function \( \phi \) with base \( s \) such that the mapping \( T : X \rightarrow X \) is \( (\Omega, \phi, \mathcal{Z})_s \)-contraction with respect to \( \zeta \) that satisfies the following condition

\[
\forall u \in X \ if \ Tu \neq u, \ then \ \inf\{\Omega(x, Tx, u) : x \in X\} > 0. \quad (2)
\]

Then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be arbitrary and define the sequence \( (x_n) \) in \( X \) inductively by \( x_n = Tx_{n-1} \) \( n \in \mathbb{N} \).

Let \( p \geq 0 \) be a nonnegative integer. Then by (1), we have for all \( n \in \mathbb{N} \)

\[
0 \leq \zeta(s\Omega(x_{n-1}, T^2x_{n-1}, T_{x_{n+p-1}}), \phi s\Omega(x_{n-1}, T_{x_{n-1}}, x_{n+p-1})), \\
= \zeta(s\Omega(x_{n-1}, x_{n+1}, x_{n+p}), \phi s\Omega(x_{n-1}, x_{n+p}), \\
< \phi s\Omega(x_{n-1}, x_{n+1}, x_{n+p}) - s\Omega(x_{n-1}, x_{n+1}, x_{n+p}).
\]

Thus,

\[
s\Omega(x_{n-1}, x_{n+1}, x_{n+p}) < \phi s\Omega(x_{n-1}, x_{n+p}). \quad (3)
\]

Also, by (1) we have

\[
0 \leq \zeta(s\Omega(x_{n-2}, T^2x_{n-2}, T_{x_{n+p-2}}), \phi s\Omega(x_{n-2}, T_{x_{n-2}}, x_{n+p-2})), \\
= \zeta(s\Omega(x_{n-2}, x_{n+1}, x_{n+p-1}), \phi s\Omega(x_{n-2}, x_{n+1}, x_{n+p-1})), \\
< \phi s\Omega(x_{n-2}, x_{n+1}, x_{n+p-1}) - s\Omega(x_{n-2}, x_{n+1}, x_{n+p-1}).
\]

Therefore,

\[
s\Omega(x_{n-2}, x_{n+1}, x_{n+p-1}) < \phi s\Omega(x_{n-2}, x_{n+p-2}). \quad (4)
\]

Since \( \phi \) is nondecreasing, then \( \phi s\Omega(x_{n-1}, x_{n+p-1}) < \phi^2 s\Omega(x_{n-2}, x_{n-1}, x_{n+p-2}) \). Hence, (3) becomes

\[
s\Omega(x_{n-1}, x_{n+p-1}) < \phi^2 s\Omega(x_{n-2}, x_{n-1}, x_{n+p-2}). \quad (5)
\]

If we apply the previous steps repeatedly, we get \( s\Omega(x_{n}, x_{n+p}) \leq \phi^n s\Omega(x_0, x_1, x_p) \). Since \( X \) is \( \Omega \)-bounded, there is \( M \geq 0 \), such that \( \Omega(x, y, z) \leq M, \forall x, y, z \in X \). Thus

\[
s\Omega(x_n, x_{n+p}) \leq \phi^n (sM). \quad (6)
\]

Now, by using the definition of \( \Omega \) and (6), we have for all \( l \geq m \geq n \)
\[
\Omega(x_n, x_m, x_l) \leq s \Omega(x_n, x_{n+1}, x_{n+1}) + s^2 \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\
+ s^{m-n-1} \Omega(x_{m-1}, x_{m-1}, x_{m-1}) + s^{m-n} \Omega(x_{m-1}, x_m, x_l) \\
\leq \phi^n(sM) + s^{\phi_{n+1}}(sM) + \cdots + s^{m-n-2} \phi^{m-1}(sM) + s^{m-n-2} \phi^{m-1}(sM) \\
\leq \phi^n(sM) + s^{\phi_{n+1}}(sM) + \cdots \\
= s^n \sum_{k=n}^{\infty} s^k \phi^k(sM).
\]

Since \( \phi \) is a \( c \)-comparison function with base \( s \), then \( \sum_{k=n}^{\infty} s^k \phi^k(sM) : n \in \mathbb{N} \) converges to 0. Thus for any \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( \sum_{k=n}^{\infty} s^k \phi^k(M) \leq \varepsilon \) \( \forall \ n \geq N. \)

Hence for \( l \geq m \geq n \geq N \), we have

\[
\Omega(x_n, x_m, x_l) \leq s^{-n} \sum_{k=n}^{\infty} s^k \phi^k(M) \leq \varepsilon \ \forall \ n \geq N.
\]

By Lemma 2.1, \((x_n)\) is a \( G_p\)-Cauchy sequence. Therefore there is \( u \in X \) such that \( \lim_{n \to \infty} x_n = u. \)

Consider \( \delta > 0 \). Then there exists \( r_0 \in \mathbb{N} \) such that \( \Omega(x_n, x_m, x_l) \leq \delta \ \forall n, m, l \geq r_0. \)

Therefore, \( \lim_{l \to \infty} \Omega(x_n, x_m, x_l) \leq \lim_{l \to \infty} \delta = \delta. \ \forall n, m \geq r_0. \)

By the lower semi continuity of \( \Omega \), we have \( \Omega(x_{n}, x_{m}, u) \leq \liminf_{p \to \infty} \Omega(x_{n}, x_{m}, x_{p}) \leq \delta \ \forall n, m \geq r_0. \)

Consider \( m = n + 1. \) Then \( \Omega(x_n, x_{n+1}, u) \leq \liminf_{p \to \infty} \Omega(x_{n}, x_{n+1}, x_{p}) \leq \delta \ \forall n \geq r_0. \)

If \( Tu \neq u \), then (2) implies that

\[
0 < \inf\{\Omega(x, Tx, u) : x \in X\} \\
\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \geq r_0\} \\
\leq \delta,
\]

for each \( \delta > 0 \) which is a contradiction. Therefore \( Tu = u. \) The uniqueness follows from Lemma 3.1.  

\[ \Box \]

**Example 3.2.** Let \( X = [0, 1] \) and let \( G : X \times X \times X \to [0, \infty), \) \( \Omega : X \times X \times X \to [0, \infty), \) \( T : X \to X \) and \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) be defined as follow:

\[
G(x, y, z) = (|x - y| + |y - z| + |x - z|)^2, \quad \Omega(x, y, z) = (|x - y| + |y - z|)^2, \quad T = ax, \quad \zeta(u, v) = bv - u \quad \text{and} \quad \phi(t) = ct \text{ where } a, b \in [0, 1), \ c \in [0, \frac{1}{2}) \text{ and } a^2 \leq b c. \quad \text{Then}
\]

(1) \( (X, G) \) is a complete \( G_p\)-metric space and \( \Omega \) is a generalized \( \Omega \)-distance on \( X \) with base \( s = 2, \)

(2) \( \zeta \in \mathscr{L}, \) \( \phi \) is a \( c \)-comparison function with base \( s = 2, \)

(3) \( T \) is \( (\Omega, \phi, \mathscr{L})_c \)-contraction with respect to \( \zeta, \)

(4) for every \( u \in X \) if \( Tu \neq u, \) then \( \inf\{\Omega(x, Tx, u) : x \in X\} > 0. \)

**Proof.** We shall prove (3) and (4).

To prove that \( T \) is \( (\Omega, \phi, \mathscr{L})_c \)-contraction with respect to \( \zeta \), let \( x, y \in X. \) Then

\[
\zeta(s\Omega(Tx, T^2x, Ty), \phi \Omega(x, Tx, y)) \\
= \zeta(2\Omega(Tx, T^2x, Ty), 2\Omega(x, Tx, y)) \\
= 2bc(|x - ax| + |y - y|)^2 - 2(|ax - a^2x| + |ax - ay|)^2 \\
= 2bc((1 - a)|x| + |y - y|)^2 - 2a^2((1 - a)|x| + |y - y|)^2 \\
= 2(bc - a^2)(|x| + |y - y|) \\
\geq 0.
\]

To prove (4), given \( u \in X \) such that \( Tu \neq u. \) Then \( u \neq 0. \) Therefore
Let base s. Assume that there is an upper semi continuous function \( g \) following condition:
\[
\inf\{ \Omega(x, T^2x, Ty) : x, y, z \in X \} = \inf\{ \Omega(x, ax, u) : x \in X \} = \inf\{ |x - ax| + |x - u| : x \in X \} = \inf\{ |(1 - a)x| + |x - u| : x \in X \} = (1 - a)u > 0.
\]

Thus all hypotheses of Theorem 3.1 hold true. Hence \( T \) has a unique fixed point in \( X \). Here the unique fixed point of \( T \) is 0.

Now, we utilized our main result to derive the following results. To facilitate our work, we let \( \mathcal{H} = \{ h : [0, \infty) \rightarrow [0, \infty) : h \text{ is a continuous function} \} \) with \( h^{-1}(\{0\}) = \{0\} \).

**Corollary 3.1.** Let \( (X, G) \) be a complete \( G_s \)-metric space and \( \Omega \) be a generalized \( \Omega \)-distance mapping on \( X \) with base \( s \geq 1 \). Let \( T : X \rightarrow X \) be a self mapping and \( \phi \) be a \( c \)-comparison function with base \( s \). Assume that there are \( h_1, h_2 \in \mathcal{H} \) where \( h_2(t) < t \leq h_1(t) \forall t > 0 \) such that \( T \) satisfies the following condition:
\[
h_1s\Omega(Tx, T^2x, Ty) \leq h_2s\Omega(x, Tx, y) \forall x, y, z \in X.
\]
Also, suppose that for all \( u \in X \) if \( Tu \neq u \), then \( \inf\{ \Omega(x, Tx, u) : x \in X \} > 0 \).
Then \( T \) has a unique fixed point in \( X \).

**Proof.** Define \( \xi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) by \( \xi(u, v) = h_2(v) - h_1(u) \). Clearly \( \xi \in \mathcal{Z} \) and \( T \) is \( (\Omega, \phi, \mathcal{Z}) \) contraction with respect to \( \xi \). Hence the result follows from Theorem 3.1.

By choosing \( h_1(t) = t \) and \( h_2(t) = \lambda t \) where \( 0 \leq \lambda < 1 \) in Corollary 3.1 we have the following:

**Corollary 3.2.** Let \( (X, G) \) be a complete \( G_s \)-metric space and \( \Omega \) be a generalized \( \Omega \)-distance mapping on \( X \) with base \( s \geq 1 \). Let \( T : X \rightarrow X \) be a self mapping and \( \phi \) be a \( c \)-comparison function with base \( s \). Assume that there is \( \lambda \in [0, 1) \) such that \( T \) satisfies the following condition:
\[
\Omega(Tx, T^2x, Ty) \leq \frac{\lambda}{3} \phi s\Omega(x, Tx, y) \forall x, y, z \in X.
\]
Also, suppose that for all \( u \in X \) if \( Tu \neq u \), then \( \inf\{ \Omega(x, Tx, u) : x \in X \} > 0 \).
Then \( T \) has a unique fixed point in \( X \).

**Corollary 3.3.** Let \( (X, G) \) be a complete \( G_s \)-metric space and \( \Omega \) be a generalized \( \Omega \)-distance mapping on \( X \) with base \( s \geq 1 \). Let \( T : X \rightarrow X \) be a self mapping and \( \phi \) be a \( c \)-comparison function with base \( s \). Assume that there is \( g \in \mathcal{H} \) such that \( T \) satisfies the following condition:
\[
s\Omega(Tx, T^2x, Ty) \leq \phi s\Omega(x, Tx, y) - g\phi s\Omega(x, Tx, y) \forall x, y, z \in X.
\]
Also, suppose that for all \( u \in X \) if \( Tu \neq u \), then \( \inf\{ \Omega(x, Tx, u) : x \in X \} > 0 \).
Then \( T \) has a unique fixed point in \( X \).

**Proof.** Define \( \xi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) by \( \xi(u, v) = v - g(v) - u \). Clearly \( \xi \in \mathcal{Z} \) and \( T \) is \( (\Omega, \phi, \mathcal{Z}) \) contraction with respect to \( \xi \). Hence the result follows from Theorem 3.1.

**Corollary 3.4.** Let \( (X, G) \) be a complete \( G_s \)-metric space and \( \Omega \) be a generalized \( \Omega \)-distance mapping on \( X \) with base \( s \geq 1 \). Let \( T : X \rightarrow X \) be a self mapping and \( \phi \) be a \( c \)-comparison function with base \( s \). Assume that there is an upper semi continuous function \( \eta \) such that \( T \) satisfies the following condition:
\[
s\Omega(Tx, T^2x, Ty) \leq \eta \phi s\Omega(x, Tx, y) \forall x, y, z \in X.
\]
Also, suppose that for all \( u \in X \) if \( Tu \neq u \), then \( \inf\{ \Omega(x, Tx, u) : x \in X \} > 0 \).
Then \( T \) has a unique fixed point in \( X \).
**Proof.** Define \( \zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) by \( \zeta(u, v) = \eta(v) - u \). Clearly \( \zeta \in \mathcal{Z} \) and \( T \) is \((\Omega, \phi, \mathcal{Z})\)-contraction with respect to \( \zeta \). Hence the result follows from Theorem 3.1.

**Corollary 3.5.** Let \((X, G)\) be a complete \(G_s\)-metric space and \( \Omega \) be a generalized \(\Omega\)-distance mapping on \(X\) with base \(s \geq 1\). Let \( T : X \rightarrow X \) be a self mapping and \( \phi \) be a \(c\)-comparison function with base \(s\). Assume that there is a function \( \gamma : [0, \infty) \rightarrow [0, \infty) \) where \( \int_0^x \gamma(t)dt \) exists and \( \int_0^x \gamma(t)dt > \varepsilon \) \(\forall \varepsilon > 0\) such that \( T \) satisfies the following condition:

\[
\int_0^{\Omega(Tx, T^2x)} \gamma(u)du \leq \phi s \Omega(x, Tx, y) \forall x, y \in X.
\]

Also, suppose that for all \( u \in X \) if \( Tu \neq u \), then \( \inf\{\Omega(x, Tx, u) : x \in X\} > 0 \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** Define \( \zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \) by \( \zeta(u, v) = v - \int_0^u \gamma(t)dt \). Clearly \( \zeta \in \mathcal{Z} \) (see Example 2.6) and \( T \) is \((\Omega, \phi, \mathcal{Z})\)-contraction with respect to \( \zeta \). Hence the result follows from Theorem 3.1.

## 4. Conclusion

In this paper, we introduced and studied some fixed point theorems in the setting of generalized \(\Omega\)-distance mappings [1] using contraction conditions depend on simulation functions [8] in which our work gives a more general cases in the study of fixed point theory. Also, an example is introduced to support our main result.

**REFERENCES**


13. L. Gholizadeh, A fixed point theorem in generalized ordered metric spaces with application, J. Nonlinear Sci. Appl. 6 (2013), 244-251

