A COUPLING METHOD OF HOMOTOPY PERTURBATION AND LAPLACE TRANSFORMATION FOR FRACTIONAL MODELS

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This paper suggests a novel coupling method of homotopy perturbation and Laplace transformation for fractional models. This method is based on He's homotopy perturbation, Laplace transformation and the modified Riemann-Liouville derivative. However, all the previous works avoid the term of fractional order initial conditions and handle them as a restricted variation. In order to overcome this shortcoming, a fractional Laplace homotopy perturbation transform method (FLHPTM) is proposed with modified Riemann-Liouville derivative. The results from introducing a modified Riemann-Liouville derivative, fractional order initial conditions and Laplace transform in the cases studied show the high accuracy, simplicity and efficiency of the approach.

Keywords: Laplace transform; modified Riemann-Liouville derivative; homotopy perturbation method

1. Introduction

There is a long-standing interest in extending the classical calculus to non-integer orders [1-4] because the applications of fractional calculus (integrals and derivatives of any real or complex order) have attracted a great deal of attention in recent years. For example, fractional differential equations are increasingly used to model many problems in biology, chemistry, economic, engineering, physics and other areas of applications. The fractional differential equations have become a useful tool for describing nonlinear phenomena of science and engineering models. Several authors including Beyer and Kempfle [5], Schneider and Wyss [6], Mainardi [7], Huang and Liu [8], He [9, 10], Faraz et al. [11] discussed some methods and solutions of fractional differential equations. However, applications of this linear inhomogeneous Klein-Gordon equation are pointed out in [12, 13].

No analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method was first proposed to solve fractional differential equations with greatest

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success, see Ref. [10]. Following the above idea, Khan et al. [14], Odibat and Momani [15], Das [16] and Faraz et al. [17] applied the variational iteration method to more complex fractional differential equations, showing effectiveness and accuracy of the used method. In previous papers [18–23] many authors have already established as well as successfully exhibited the applicability of Adomian decomposition method to obtain the solutions of different types of fractional differential equations. Momani [24], Ganji [25] and Yildirim [26] applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that HPM is an alternative analytical method for solving fractional differential equations.

The purpose of this paper is to introduce a new method for fractional differential equations. Our aim is to extend the application of the proposed method to obtain the analytical solutions to Klein–Gordon fractional partial differential equation. In this study, He's homotopy perturbation method [27, 28] is implemented. We have introduced fractional order initial conditions. Point to be noted that for fractional differential equations one should use fractional Taylor series. To make the calculation easy and simple, first time we have used Laplace transform to solve the systems of equations after applying the homotopy perturbation instead of applying homotopy inverse operator. Through Laplace transform of fractional order term, it is easy to judge that one must use fractional order initial conditions. By introducing the Laplace transform, calculations become simple and easy to understand as compared to applying the homotopy inverse operator. It is easy to judge, by applying the Laplace transformation that it is essential to use fractional order initial condition to analyze any physical phenomena, which has been expressed in terms of fractional differential equations. The elegance of this article can be attributed to its endeavor in finding the solution in a simple way by considering only FLHPTM. Klein–Gordon fractional partial differential equation is solved which show that only a few iterations are needed to obtain accurate approximate solutions.

2. Fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. Let $f : R \to R,\ \ x \to f(x)$, denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $FW(h)$ (the symbol: = means that the left side by the right one) [29]

$$FW(h).f(x) := f(x+h); \tag{1}$$

Then the fractional difference of order $\alpha, \alpha \in R, 0 < \alpha \le 1,$ of $f(x)$ is defined by the expression [29]

$$\Delta^{\alpha}.f(x) := (FW-1)^{\alpha}.f(x) \\ = \sum_{k=0}^{\infty}(-)^{k} C_{k}^{\alpha} f[x+(\alpha-k)h], \tag{2}$$

and its fractional derivative is the limit [29]

$$f^{(\alpha)}(x) = \lim_{h\downarrow 0} \frac{\Delta^{\alpha}[f(x)-f(0)]}{h^{\alpha}}. \tag{3}$$

This definition is close to the standard definition of derivative (calculus for beginners), and as a direct result, the $\alpha-th$ derivative of a constant is zero.

Proposition. Refer to the function of Definition 2.1. Then its fractional derivative or order α, $\alpha<0$, is defined by the expression [29]

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)}\int_{0}^{x}(x-\xi)^{-\alpha-1} f(\xi)d\xi, \alpha<0. \tag{4}$$

For positive α one will set

$$f^{(\alpha)}(x) = \left(f^{(\alpha-1)}(x)\right)^{/}, 0<\alpha<1, \\ f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}(x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi \tag{5}$$

and

$$f^{(\alpha)}(x) = \left(f^{(\alpha-n)}(x)\right)^{(n)}, n\le\alpha<n+1, n\ge 1. \tag{6}$$

Remark the difference between Eq. (4) and (5). The second one involves the constant $f(0)$ whilst the first one does not. We shall refer to this fractional derivative as to the *modified Riemann Liouville derivative*. With this definition, the Laplace's transform $L\{.\}$ of the fractional derivative is [29]

$$L\left\{f^{(\alpha)}(x)\right\} = s^{\alpha}L\left\{f(x)\right\} - s^{\alpha-1}f(0), 0<\alpha<1. \tag{7}$$

Definition 2.2. (*Principle of Derivative Increasing Orders*). The fractional derivative of fractional order $D^{\alpha+\theta}$ expressed in terms of D^{α} and D^{θ} is defined by the equality [29]

$$D^{\alpha+\theta}f(x) := D^{\max(\alpha,\theta)}\left(D^{\min(\alpha,\theta)}f(x)\right). \tag{8}$$

On doing so, we merely follow the practical rule in accordance of which we increase the derivation order rather than the opposite. Or again, we start from the lower order derivative to define the larger order one.

On the decomposition of fractional derivatives

Let α be such that $0 < 3\alpha < 1$. There are two different ways to obtain $D^{3\alpha} f(x)$. One can calculate $D^{\alpha} D^{\alpha} D^{\alpha} f(x)$ to obtain Laplace's transform [29]

$$L\left\{D^{\alpha} D^{\alpha} D^{\alpha} f(x)\right\} = s^{3\alpha} f(s) - s^{3\alpha-1} f(0) - s^{2\alpha-1} f^{(\alpha)}(0) - s^{\alpha-1} f^{(2\alpha)}(0) \quad (9)$$

Proposition. Assume that the continuous function $f : R \to R,\ x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any $\alpha, 0 < \alpha \le 1;$ then the following equality holds, which is [29]

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\alpha k!} f^{\alpha k}(x), 0 < \alpha \le 1. \quad 0 < \alpha \le 1 \quad (10)$$

On making the substitution $h \to x$ and $x \to 0$ we obtain the fractional Mc-Laurin series [29]

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\alpha k!} f^{\alpha k}(0), \quad 0 < \alpha \le 1 \quad (11)$$

3. Fractional Laplace homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation transform method, we consider the following fractional differential equation:

$$D^{n\alpha} u(x,t) = R[x]u(x,t) + q(x,t), \quad t > 0, x \in \mathbb{R}, \quad 0 < n\alpha \le 1 \quad (12)$$

Where $D^{n\alpha} = \dfrac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R[x]$ is a linear operator in x, $f(x)$ and $q(x,t)$ are continuous functions. Using the HPM [27, 28] as introduced by He, we can construct a homotopy for Eq. (12) as follows:

$$(1-p)D^{n\alpha} u(x,t) + p[D^{n\alpha} u(x,t) - R[x]u(x,t) - q(x,t)] = 0, \quad (13)$$

or

$$D^{n\alpha} u(x,t) = p[R[x]u(x,t) + q(x,t)], \quad (14)$$

where $p \in [0,1]$ is an embedding parameter. If $p = 0$, Eq. (13) and Eq. (14) become

$$D^{n\alpha} u(x,t) = 0, \quad (15)$$

And when $p = 1$, both Eq. (13) and Eq. (14) turn out to be the original fractional differential equation (12).

The homotopy perturbation method [27, 28] admits a solution in the form

$$u = p^0 u_0 + p^1 u_1 + p^2 u_2 + \ldots \quad (16)$$

Setting $p = 1$ results in the solution of Eq. (16); we get

$$u = u_0 + u_1 + u_2 + \ldots \quad (17)$$

From the Eq. (16), one can realize that the problems can be solved by using the homotopy perturbation method (HPM). Thus, He [27] proposed the following conditions for the convergence of the homotopy perturbation method:

The second derivative of nonlinear term with respect to u must be small, because the parameter p may be relatively large, i.e. $p \to 1$.

2) The norm of $\mathrm{L}^{-1}\,\mathrm{dNonlinear\ term}/\mathrm{du}$ must be smaller than one, in order that the series converges.

Invoking Eq. (16) in Eq. (14) and collecting the terms with the same powers of p, we can obtain a series of equations of the following form:

$$
\begin{aligned}
&p^0 : D^{n\alpha} u_0(x,t) = 0,\\
&p^1 : D^{n\alpha} u_1(x,t) = R[x]u_0(x,t) + q(x,t),\\
&p^2 : D^{n\alpha} u_2(x,t) = R[x]u_1(x,t),\\
&p^3 : D^{n\alpha} u_3(x,t) = R[x]u_2(x,t),\\
&\vdots
\end{aligned}
\tag{18}
$$

Taking Laplace transform of both sides of Eq. (18) gives:

$$
\begin{aligned}
&p^0 : s^{n\alpha}u_0(x,s) - s^{n\alpha-1}u_0(x,0) - s^{(n-1)\alpha-1}u_0^{\alpha}(x,0) - s^{(n-2)\alpha-1}u_0^{2\alpha}(x,0) - \ldots - s^{\alpha-1}u_0^{(n-1)\alpha}(x,0) = 0,\\
&p^1 : s^{n\alpha}u_1(x,s) = Ru_0(x,s) + q(x,s),\\
&p^2 : s^{n\alpha}u_2(x,s) = Ru_1(x,s),\\
&p^3 : s^{n\alpha}u_3(x,s) = Ru_2(x,s),\\
&\vdots
\end{aligned}
\tag{19}
$$

On solving Eq. (19) for $u_0, u_1, u_2, u_3 \ldots$ respectively by using the fractional initial conditions, we can get following form:

$$
\begin{aligned}
&p^0 : u_0(x,t) = L^{-1}\left\{ \frac{1}{s^{n\alpha}} \left(s^{n\alpha-1}u_0(x,0) + s^{(n-1)\alpha-1}u_0^{\alpha}(x,0) + s^{(n-2)\alpha-1}u_0^{2\alpha}(x,0) + \ldots + s^{\alpha-1}u_0^{n\alpha}(x,0) \right) \right\},\\
&p^1 : u_1(x,t) = L^{-1}\left\{ \frac{1}{s^{n\alpha}} \left(Ru_0(x,s) + q(x,s) \right) \right\}\\
&p^2 : u_2(x,t) = L^{-1}\left\{ \frac{1}{s^{n\alpha}} \left(Ru_1(x,s) \right) \right\},\\
&p^3 : u_3(x,t) = L^{-1}\left\{ \frac{1}{s^{n\alpha}} \left(Ru_2(x,s) \right) \right\},\\
&\vdots
\end{aligned}
\tag{20}
$$

Substituting successive iterations in Eq. (17) will give required result.

4. Application

We consider the one-dimensional linear inhomogeneous fractional Klein-Gordon equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^{2}u}{\partial x^{2}}-u+6x^{3}t+(x^{3}-6x)t^{3},0<\alpha\leq 2,$$

subject to the initial conditions (21)

$$u(x,0)=0,\qquad u_{t}(x,0)=0,$$

In order to illustrate the efficiency of our method, we replace the fractional order α ($0<\alpha\leq 2$) by the order 2α ($0<\alpha\leq 1$) in Eq. (21).

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}=\frac{\partial^{2}u}{\partial x^{2}}-u+6x^{3}t+(x^{3}-6x)t^{3},0<\alpha\leq 1,$$

subject to the initial conditions (22)

$$u(x,0)=0,\quad u^{\alpha}(x,0)=0,$$

By applying the aforesaid method, we have

$$\begin{aligned}
&p^{0}:D^{2\alpha}u_{0}(x,t)=0,u_{0}(0,t)=0,u_{0}^{\alpha}(0,t)=0\\
&p^{1}:D^{2\alpha}u_{1}(x,t)=u_{xx0}(x,t)-u_{0}(x,t)+6x^{3}t+(x^{3}-6x)t^{3},u_{1}(0,t)=u_{1}^{\alpha}(0,t)=0\\
&p^{2}:D^{2\alpha}u_{2}(x,t)=u_{xx1}(x,t)-u_{1}(x,t),u_{2}(0,t)=u_{2}^{\alpha}(0,t)=0\\
&p^{3}:D^{2\alpha}u_{3}(x,t)=u_{xx2}(x,t)-u_{2}(x,t),u_{3}(0,t)=u_{3}^{\alpha}(0,t)=0\\
&\vdots
\end{aligned}\tag{23}$$

In view of Eq. (9), Eq. (23) takes the form as follows

$$\begin{aligned}
&p^{0}:s^{2\alpha}u_{0}(x,s)=0,\\
&p^{1}:s^{2\alpha}u_{1}(x,s)=\frac{6x^{3}}{s^{2}}+\frac{(x^{3}-6x)6}{s^{4}},u_{1}(0,t)=u_{1}^{\alpha}(0,t)=0\\
&p^{2}:s^{2\alpha}u_{2}(x,s)=\frac{36x}{s^{2+2\alpha}}+\frac{36x}{s^{4+2\alpha}}-\frac{6x^{3}}{s^{2+2\alpha}}-\frac{(x^{3}-6x)6}{s^{2\alpha+4}},u_{2}(0,t)=u_{2}^{\alpha}(0,t)=0\\
&\vdots
\end{aligned}\tag{24}$$

The inverse Laplace transform applied to Eq. (24) results in:

$$
\begin{aligned}
p^0 &: u_0(x,t) = 0,\\
p^1 &: u_1(x,t) = L^{-1}\left\{\frac{1}{s^{2\alpha}}\left(\frac{6x^3}{s^2}+\frac{(x^3-6x)6}{s^4}\right)\right\}\\
p^2 &: u_2(x,t) = L^{-1}\left\{\frac{1}{s^{2\alpha}}\left(\frac{36x}{s^{2\alpha+2}}+\frac{36x}{s^{2\alpha+4}}-\frac{6x^3}{s^{2\alpha+2}}-\frac{(x^3-6x)6}{s^{2\alpha+4}}\right)\right\},\\
&\vdots
\end{aligned}
\tag{25}
$$

The most refined form of Eq. (25) is

$$
\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= \left(\frac{6x^3t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{6(x^3-6x)t^{2\alpha+3}}{\Gamma(2\alpha+4)}\right)\\
u_2(x,t) &= \left(\frac{36xt^{4\alpha+1}}{\Gamma(4\alpha+2)}+\frac{36xt^{4\alpha+3}}{\Gamma(4\alpha+4)}-\frac{6x^3t^{4\alpha+1}}{\Gamma(4\alpha+2)}-\frac{6(x^3-6x)t^{4\alpha+3}}{\Gamma(4\alpha+4)}\right)\\
u_3(x,t) &= \left(\frac{6x^3t^{6\alpha+1}}{\Gamma(6\alpha+2)}-\frac{72xt^{6\alpha+1}}{\Gamma(6\alpha+2)}+\frac{6(x^3-6x)t^{6\alpha+3}}{\Gamma(6\alpha+4)}-\frac{72xt^{6\alpha+3}}{\Gamma(6\alpha+4)}\right),\\
&\vdots
\end{aligned}
\tag{26}
$$

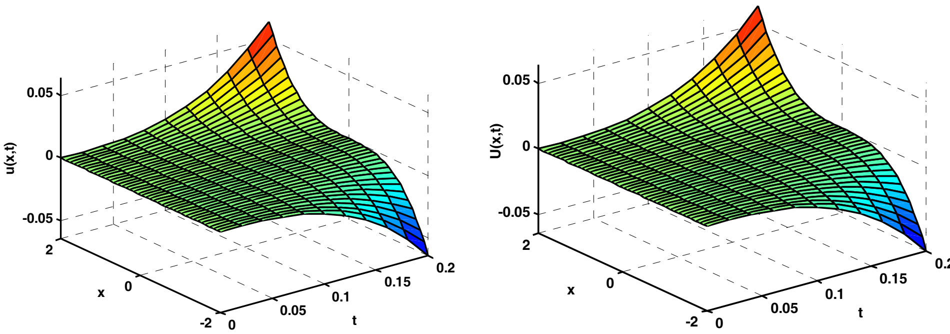
In the same manner, the rest of the components can be obtained. Consequently, we obtain the following expansion:

$$
u(x,t)=\frac{6x^3t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{6(x^3-6x)t^{2\alpha+3}}{\Gamma(2\alpha+4)}+\frac{36xt^{4\alpha+1}}{\Gamma(4\alpha+2)}+\frac{36xt^{4\alpha+3}}{\Gamma(4\alpha+4)}-\frac{6x^3t^{4\alpha+1}}{\Gamma(4\alpha+2)}-\frac{6(x^3-6x)t^{4\alpha+3}}{\Gamma(4\alpha+4)}+\dots \tag{27}
$$

when $\alpha=1$, it is obvious that the “noise terms” appearing in $u(x,t)$ are cancelled and finally we obtain the solution for the classical Klein-Gordon equation as

$$
u(x,t)=x^3t^3 \tag{28}
$$

In order to corroborate the efficiency and accuracy of the method, we compare our approximate solutions of fractional differential equation (22) with the exact solutions. Fig. 1. (A) and Fig. 1. (B) show the numerical results of $u(x,t)$ at $\alpha=1$ and exact solution, respectively. It can be seen from the figures that a very good approximation is achieved with the exact solution by using only two terms of the FLHPTM series.

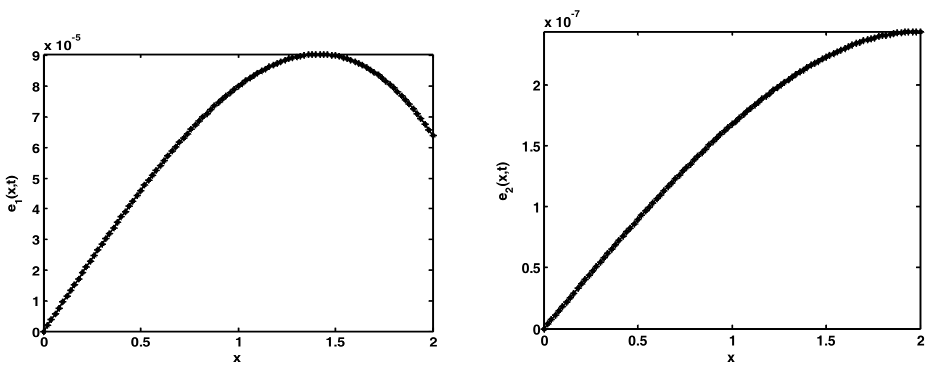


A. B.

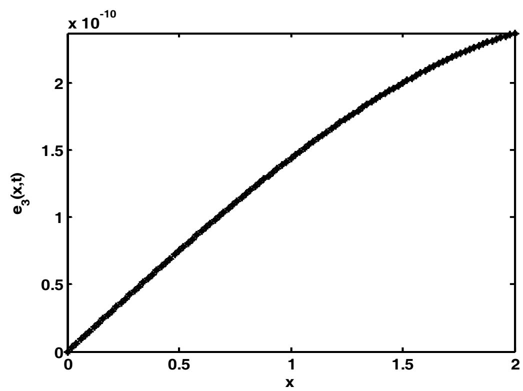
B.

Fig.1: Comparison between approximate and exact solutions: (A) Approximate solution at $\alpha = 1$, (B) Exact solution

We determine the absolute error function $e_n(x,t)$ between exact and approximate solutions with considering different number of components or terms of FLHPTM series. Figs. 2 (A)-(C) show the results of the absolute errors for the first-order, second-order and third-order terms of the FLHPTM series solution, respectively. The figures demonstrate the speed of convergence of the proposed method is very fast, and the accuracy of the method increases with increasing number of components of approximated solutions.



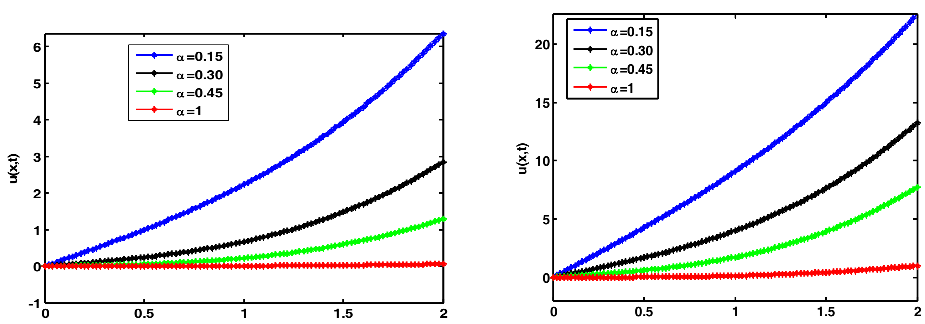
A. B.



C.

Fig.2: Absolute error functions at $t = 0.2$: (A) $e_1(x,t)$ (B) $e_2(x,t)$ and (C) $e_3(x,t)$.

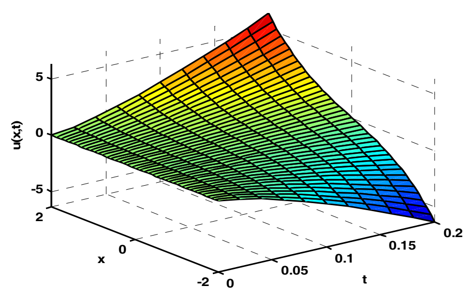
To demonstrate the influence of varying the order of the fractional derivative on the behaviour of solution, we take four different values of α as $\alpha = 0.15$, $\alpha = 0.30$, $\alpha = 0.45$, and $\alpha = 1$. The numerical solutions of $u(x,t)$ are presented in Fig. 3 (A) and Fig. 3 (B) for $t = 0.2$ and $t = 0.5$, respectively.



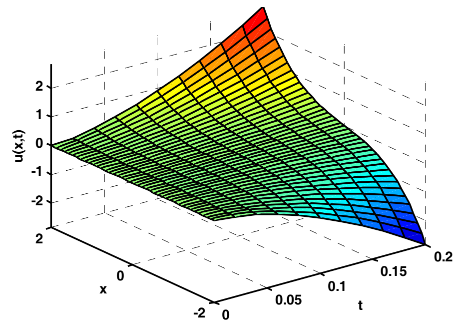
A. B.

Fig.3: Plot of numerical solution of Eq. (4.9) for different values α: (A) at $t = 0.2$; (B) at $t = 0.5$.

It is evident that the solution continuously depends on the time fractional derivative. Moreover, the numerical results of $u(x,t)$ for various values of α are depicted through Figs. 4 (A)-(C). As seen from the figures, as the value of fractional order α decreases, the approximate solution dramatically increases.



A.



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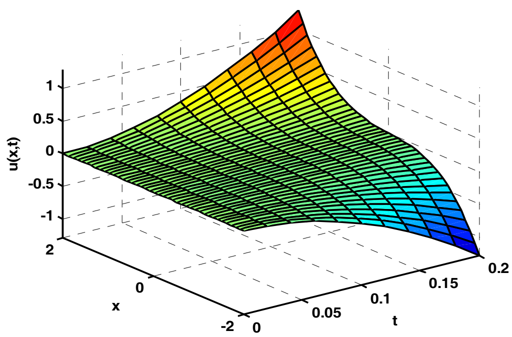

C.

Fig.4: Numerical solutions of Eq. (4.9) for different values α: (A) $\alpha = 0.15$, (B) $\alpha = 0.3$, (C) $\alpha = 0.45$.

6. Conclusions

In this article, we proposed a fractional Laplace homotopy perturbation transform method (FLHPTM) for finding the solution of partial differential equations with fractional time derivative. The method is applied in a direct way without using linearization, discretization or restrictive assumptions. It may be concluded that the FLHPTM is very powerful and efficient in finding the analytical solutions for a wide class of initial value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of partial differential equations.

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