SRIGHT WEAKLY REGULAR SEMIGROUPS CHARACTERIZED BY THEIR GENERALIZED FUZZY IDEALS

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We first give some useful characterizations of regular semigroups and right weakly regular semigroups by the properties of their bi-ideals, interior ideals, left ideals and right ideals. Based on these characterizations, we also characterize regular semigroups and right weakly regular semigroups by the properties of their $(\in, \in \vee q_k)$*-fuzzy (generalized) bi-ideals,* $(\in, \in \vee q_k)$*-fuzzy interior ideals,* $(\in, \in \vee q_k)$*-fuzzy left ideals and* $(\in, \in \vee q_k)$*-fuzzy right ideals.*

Keywords: Regular semigroups, right weakly regular semigroups, $(\in, \in \vee q_k)$-fuzzy ideals, fuzzy sets.

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1. Introduction

The real world is full of uncertainty, imprecision and vagueness. As one can see, many concepts used in common sense reasoning of human are vague rather than precise. In recent years, scientists and engineers have shown increasing interests in vague concepts. This might be due to the fact that many practical problems emerging in almost all disciplines including economics, engineering, environmental science, management science, social sciences, and medical science are full of complexities and various types of uncertainties while dealing with them in most occasions. To solve these problems, many theories had been developed such as probability theory, fuzzy set theory, rough set theory and the theory of soft sets [18, 8, 9].

By allowing partial membership, the theory of fuzzy sets provides an appropriate framework for representing and manipulating vague concepts. Since the introduction of fuzzy sets, researchers have accumulated a vast literature on its theory and applications. Particularly, many authors have applied fuzzy sets to generalized the basic theories of various algebraic structures. In 1971, Rosenfeld [23] first applied fuzzy sets to group structures, and he initiated a novel notion called fuzzy subgroups. The theory of fuzzy semigroups and fuzzy ideals in semigroups was introduced by

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Kuroki in [16, 17]. The theoretical exposition of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines and fuzzy languages was considered by Mordeson [19, 20] in a systematic way. Murali [21] proposed the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on fuzzy subsets. By using these ideas, Bhakat and Das [1, 2], gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. These fuzzy subgroups are further studied in [3, 4]. The concept of $(\in, \in \vee q_k)$-fuzzy subgroups is a viable generalization of Rosenfeld's fuzzy subgroups. Davvaz defined $(\in, \in \vee q_k)$-fuzzy subnearrings and ideals of a near ring in [7]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [11] which is the generalization of fuzzy interior ideals [12]. In [15], Kazanci and Yamak studied $(\in, \in \vee q_k)$-fuzzy bi-ideals of a semigroup.

In this paper, we consider characterizations of regular semigroups and right weakly regular semigroups by the properties of their right ideal, left ideal, bi-ideal, generalized bi-ideal and interior ideal. Moreover, we also characterize regular and right weakly regular semigroups in terms of their $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal and $(\in, \in \vee q_k)$-fuzzy interior ideals.

2. $(\in, \in \vee q_k)$-fuzzy Ideals in Semigroups

Throughout this paper S will denote a semigroup and k be an arbitrary element of $[0, 1)$ unless otherwise specified. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. A non-empty subset J of S is called a left (right) ideal of S if $SJ \subseteq I$ $(JS \subseteq I)$. J is called a two-sided ideal or simply an ideal of S if it is both left and right ideal of S. A non-empty subset B of S is called a generalized bi-ideal of S if $BSB \subseteq B$. A non-empty subset B of S is called a bi-ideal of S if it is both a subsemigroup and a generalized bi-ideal of S. A subsemigroup I of S is called an interior ideal of S if $SIS \subseteq I$.

Recall that a fuzzy set μ in a universe U is defined by its *membership function* $\mu : U \to [0, 1]$. For $x \in U$, the membership value $\mu(x)$ essentially specifies the degree to which $x \in U$ belongs to the fuzzy set μ. By $\mu \subseteq \nu$, we mean that $\mu(x) \leq \nu(x)$ for all $x \in U$. Clearly $\mu = \nu$ if $\mu \subseteq \nu$ and $\nu \subseteq \mu$. That is, $\mu(x) = \nu(x)$ for all $x \in U$. In what follows, we denote by $F(U)$ the set of all fuzzy sets in U.

There are a number of different definitions for fuzzy set operations. With the min-max system proposed by Zadeh, fuzzy set *intersection, union,* and *complement* are defined as follows:

- $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$,
- $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$,
- $\mu^c(x) = 1 - \mu(x)$,

where $\mu, \nu \in F(U)$ and $x \in U$.

Definition 2.1. For a fuzzy set f of a semigroup S and $t \in [0,1]$, the crisp set $U(f;t) = \{x \in S \mid f(x) \geq t\}$ is called a level subset of the fuzzy set f.

Definition 2.2. Let f and g be any two fuzzy subsets of S. Then the product $f \circ g$ is a fuzzy subset of S defined by

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} (f(b) \wedge g(c)), & \text{if there exists } b, c \in S \text{ such that } a = bc, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. For any $t \in (0,1]$, a fuzzy subset f of S defined as

$$f(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{otherwise,} \end{cases}$$

is called a *fuzzy point* with support x and value t, which is denoted by x_t.

A fuzzy point x_t is said to *belong* to (*resp. quasi-coincident with*) a fuzzy set f, written as $x_t \in f$ (*resp.* $x_t q f$) if $f(x) \geq t$ (*resp.* $f(x) + t > 1$). If $x_t \in f$ or $x_t q f$, then we write $x_t \in \vee q f$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. For any two fuzzy subsets f and g of S, $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in S$.

Generalizing the concept of $x_t q f$, Jun [12, 13], defined $x_t q_k f$, where $k \in [0,1)$, as $f(x) + t + k > 1$. By $x_t \in \vee q_k f$ we shall mean that $x_t \in f$ or $x_t q_k f$.

Definition 2.4. [24] A fuzzy subset of S is called an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if for all $x, y \in S$ and $t, r \in (0,1]$ the following condition holds:

$$x_t \in f, y_r \in f \Rightarrow (xy)_{\min\{t,r\}} \in \vee q_k f.$$

Lemma 2.1. [24] Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if and only if $f(xy) \geq \min\left\{f(x), f(y), \frac{1-k}{2}\right\}$.

Definition 2.5. [24] A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if for all $x, y \in S$ and $t \in (0,1]$ the following condition holds:

$$y_t \in f \Rightarrow (xy)_t \in \vee q_k f \text{ (resp. } x_t \in f \Rightarrow (xy)_t \in \vee q_k f).$$

Lemma 2.2. [24] Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S if and only if $f(xy) \geq \min\left\{f(y), \frac{1-k}{2}\right\}$.

Lemma 2.3. [24] Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy right ideal of S if and only if $f(xy) \geq \min\left\{f(x), \frac{1-k}{2}\right\}$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy ideal of S if it is both an $(\in, \in \vee q_k)$-fuzzy left ideal and an $(\in, \in \vee q_k)$-fuzzy right ideal of S.

Definition 2.6. [24] A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, if for all x, y,$z \in S$ and t, $r \in (0,1]$, we have

$$x_t \in f, z_r \in f \Rightarrow (xyz)_{\min\{t,r\}} \in \vee q_k f.$$

An $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if it is also an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S.

Lemma 2.4. [24] A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if it satisfies the following conditions.

(i) $f(xy) \geq \min\left\{f(x), f(y), \frac{1-k}{2}\right\}$ for all $x, y \in S$ and $k \in [0, 1)$.

(ii) $f(xyz) \geq \min\left\{f(x), f(z), \frac{1-k}{2}\right\}$for all $x, y, z \in S$ and $k \in [0, 1)$.

Lemma 2.5. [24] A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if $f(xyz) \geq \min\left\{f(x), f(z), \frac{1-k}{2}\right\}$for all $x, y, z \in S$ and $k \in [0, 1)$.

Definition 2.7. A fuzzy subsemigroup f of S is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if

$$y_t \in f \Rightarrow (xyz)_t \in \vee q_k f$$

for all $x, y, z \in S$ and $t \in (0, 1]$.

Lemma 2.6. [24] A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if and only if it satisfies the following conditions.

(i) $f(xy) \geq \min\left\{f(x), f(y), \frac{1-k}{2}\right\}$ for all $x, y \in S$ and $k \in [0, 1)$.

(ii) $f(xyz) \geq \min\left\{f(y), \frac{1-k}{2}\right\}$for all $x, y, z \in S$ and $k \in [0, 1)$.

Definition 2.8. Let A be any subset of S. Then the characteristic function $(C_A)_k$ is defined as

$$(C_A)_k(a) = \begin{cases} \geq \frac{1-k}{2}, & \text{if } a \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1. Let $S = \{1, 2, 3\}$ be a semigroup with binary operation " $\cdot$ ", as defined by the following Cayley table:

$\cdot$	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

Clearly, $(S, \cdot)$ is semigroup and $\{1\}$, $\{2\}$ and $\{3\}$ are left ideals of S. Let δ be a fuzzy subset of S such that

$$\delta(1) = 0.9,\ \delta(2) = 0.6,\ \delta(3) = 0.5.$$

Then it is easy to verify that δ is an $(\in, \in \vee q_k)$-fuzzy ideal of S.

Example 2.2. Let $S = \{1, 2, 3, 4\}$ be a semigroup with binary operation " $\cdot$ ", as defined by the following Cayley table:

$\cdot$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	3	3	3
4	1	2	2	2

Let δ be a fuzzy subset of S such that

$$\delta(1) = 0.8,\ \delta(2) = 0.7,\ \delta(3) = 0.5,\ \delta(4) = 0.6.$$

Then it is easy to verify that δ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Example 2.3. Let $S = \{a, b, c, d\}$ be a semigroup with binary operation " $\cdot$ ", as defined by the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	c
c	a	c	c	c
d	a	c	c	c

Let δ be a fuzzy subset of S such that

$$\delta(a) = 0.9,\ \delta(b) = 0.7,\ \delta(c) = 0.5, \delta(d) = 0.6.$$

Then it is easy to verify that δ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.

Lemma 2.7. [24] *A non-empty subset R of a semigroup S is right (left) ideal if and only if $(C_R)_k$ is an $(\in, \in \vee q_k)$-fuzzy right (left) ideal of S.*

Lemma 2.8. [24] *A non-empty subset I of a semigroup S is an interior ideal if and only if $(C_I)_k$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.*

Lemma 2.9. [24] *A non-empty subset B of a semigroup S is bi-ideal if and only if $(C_B)_k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Definition 2.9. [24] Let $\mu, \nu \in F(S)$. Then we define the fuzzy subsets μ_k, $\mu \wedge_k \nu$ and $\mu \circ_k \nu$ of S as follows:

- $\mu_k(x) = \mu(x) \wedge \frac{1-k}{2}$;
- $(\mu \wedge_k \nu)(x) = \mu(x) \wedge \nu(x) \wedge \frac{1-k}{2}$;
- $(\mu \circ_k \nu)(x) = (\mu \circ \nu)(x) \wedge \frac{1-k}{2}$, for all $x \in S$.

Lemma 2.10. [24]*Let f and g be any fuzzy subsets of semigroup S. Then following properties hold.*

(i) $(f \wedge_k g) = (f_k \wedge g_k)$.

(ii) $(f \circ_k g) = (f_k \circ g_k)$.

Lemma 2.11. [24] *Let A and B be any non-empty subsets of a semigroup S. Then the following properties hold.*

(i) $(C_A \wedge_k C_B) = (C_{A\cap B})_k$.

(ii) $(C_A \circ_k C_B) = (C_{AB})_k$.

3. Characterizations of Regular Semigroups in Terms of $(\in, \in \vee q_k)$-fuzzy Ideals

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $a = axa$ and S is called a *regular semigroup*, if every element of S is regular. In this section, we first give some characterizations of regular semigroups by the properties of their bi-ideals, interior ideals, left ideals and right ideals.

Theorem 3.1. *For a semigroup S the following are equivalent.*

(i) *S is regular.*

(ii) *$B \cap I \cap L \subseteq BIL$ for every bi-ideal B, interior ideal I and left ideal L of a semigroup S.*

(iii) *$B[a] \cap I[a] \cap L[a] \subseteq B[a]I[a]L[a]$, for some a in S.*

Proof. $(i) \Rightarrow (ii)$: Let S be regular semigroup, B a bi-ideal, I an interior ideal and L a left ideal of S. Let $a \in B \cap I \cap L$ then $a \in B$, $a \in I$ and $a \in L$. Since S is regular semigroup so for a there exists $x \in S$ such that

$$a = axa = axaxa \in BSISL \subseteq BIL.$$

Therefore $a \in BIL$. So $B \cap I \cap L \subseteq BIL$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: As $a \cup a^2 \cup aSa$, $a \cup a^2 \cup SaS$ and $a \cup Sa$ are bi-ideal, interior ideal and left ideal of S generated by an element a of S respectively. Thus by assumption we have

$$\begin{aligned} & (a \cup a^2 \cup aSa) \cap (a \cup a^2 \cup SaS) \cap (a \cup Sa) \\ \subseteq \ & (a \cup a^2 \cup aSa)(a \cup a^2 \cup SaS)(a \cup Sa) \\ \subseteq \ & (a \cup a^2 \cup aSa)S(a \cup Sa) \subseteq (a \cup a^2 \cup aSa)(a \cup Sa) \\ = \ & a^2 \cup aSa \cup a^3 \cup a^2Sa \cup aSa^2 \cup aSaSa \\ \subseteq \ & a^2 \cup a^3 \cup aSa. \end{aligned}$$

Therefore $a = a^2 = aa = a^2a = aaa$ or $a = a^3 = aaa$ or $a = axa$ for some x in S. Hence S is regular. □

Theorem 3.2. *For a semigroup S the following are equivalent.*

(i) *S is regular.*

(ii) *$R_1 \cap R_2 \cap B \subseteq R_1R_2B$ for right ideals R_1, R_2 and bi-ideal B of semigroup S.*

(iii) *$R_1[a] \cap R_2[a] \cap B[a] \subseteq R_1[a]R_2[a]B[a]$, for some a in S.*

Proof. $(i) \Rightarrow (ii)$: Let S be regular semigroup and R_1, R_2 be two right ideals and B a bi-ideal of S. Let $a \in R_1 \cap R_2 \cap B$ then $a \in R_1$, $a \in R_2$ and $a \in B$. Since S is regular semigroup so for $a \in S$ there exists $x \in S$ such that

$$a = axa = axaxaxa \in R_1SR_2SBSB \subseteq R_1R_2B.$$

Therefore $a \in R_1R_2B$.So $R_1 \cap R_2 \cap B \subseteq R_1R_2B$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: As $a \cup aS$, $a \cup aS$ are right ideals and $a \cup a^2 \cup aSa$ is a bi-ideal of S generated by an element a of S respectively. Then by hypothesis,

$$\begin{aligned} & (a \cup aS) \cap (a \cup aS) \cap (a \cup a^2 \cup aSa) \\ \subseteq\ & (a \cup aS)(a \cup aS)(a \cup a^2 \cup aSa) \\ \subseteq\ & (a \cup aS)S(a \cup a^2 \cup aSa) \\ \subseteq\ & (a \cup aS)(a \cup a^2 \cup aSa) \\ \subseteq\ & a^2 \cup a^3 \cup a^2Sa \cup aSa \cup aSa^2 \cup aSaSa \\ \subseteq\ & a^2 \cup a^3 \cup aSa. \end{aligned}$$

Therefore $a = a^2 = aa = a^2a = aaa = aua$ or $a = a^3 = aaa = ava$ or $a = axa$, for some x, u, v in S. Hence S is regular. □

Next, we characterize regular semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, $(\in, \in \vee q_k)$-fuzzy interior ideals, $(\in, \in \vee q_k)$-fuzzy left ideals and $(\in, \in \vee q_k)$-fuzzy right ideals.

Theorem 3.3. *For a semigroup S, the following conditions are equivalent.*

(i) *S is regular.*

(ii) *$f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy interior ideal g and $(\in, \in \vee q_k)$-fuzzy left ideal h of a semigroup S.*

(iii) *$f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy interior g and $(\in, \in \vee q_k)$-fuzzy left ideal h of a semigroup S.*

Proof. $(i) \Rightarrow (iii)$: Let f, g and h be any $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy interior ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S. Since S is regular therefore for each $a \in S$ there exists $x \in S$ such that

$$a = axa = axaxa = axaxaxaxa = (axa)(xax)(axa).$$

Thus

$$\begin{aligned}
(f \circ_k g \circ_k h)(a) &= (f \circ g \circ h)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a=pq} \{ f(p) \wedge (g \circ h)(q) \} \right) \wedge \frac{1-k}{2} \\
&\geq f(axa) \wedge (g \circ h)((xax)(axa)) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \left(\bigvee_{(xax)(axa)=bc} \{ g(b) \wedge h(c) \} \right) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(xax) \wedge h(axa) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}.
\end{aligned}$$

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$: Let $B[a]$, $I[a]$ and $L[a]$ be bi-ideal, interior ideal and left ideal of S generated by a respectively.

Then $(C_{B[a]})_k$, $(C_{I[a]})_k$ and $(C_{L[a]})_k$ be $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy interior ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of semigroup S respectively. Let $a \in S$ and $b \in B[a] \cap I[a] \cap L[a]$. Then $b \in B[a]$, $b \in I[a]$ and $b \in L[a]$. Now

$$\begin{aligned}
\frac{1-k}{2} &\leq (C_{B[a] \cap I[a] \cap L[a]})_k(b) = ((C_{B[a]})_k \wedge_k (C_{I[a]})_k \wedge_k (C_{L[a]})_k))(b) \\
&\leq ((C_{B[a]})_k \circ_k (C_{I[a]})_k \circ_k (C_{I[a]})_k)(b) = (C_{B[a]I[a]L[a]})_k(b).
\end{aligned}$$

Thus $b \in B[a]I[a]L[a]$. Therefore $B[a] \cap I[a] \cap L[a] \subseteq B[a]I[a]L[a]$.

So by theorem 3.1, S is regular. □

Theorem 3.4. *For a semigroup S, the following conditions are equivalent.*

(i) S is regular.

(ii) $f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideals f, g and $(\in, \in \vee q_k)$-fuzzy bi-ideal h of a semigroup S.

(iii) $f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideals f, g and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal h of a semigroup S.

Proof. $(i) \Rightarrow (iii)$: Let f, g be $(\in, \in \vee q_k)$-fuzzy right ideals and h be $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Since S is regular therefore for each $a \in S$ there exists $x \in S$ such that

$$a = axa = axaxaxa.$$

Thus

$$\begin{aligned}
(f \circ_k g \circ_k h)(a) &= (f \circ g \circ h)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a=pq} \{f(p) \wedge (g \circ h)(q)\} \right) \wedge \frac{1-k}{2} \\
&\geq f(ax) \wedge (g \circ h)((ax)(axa)) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \left(\bigvee_{(ax)(axa)=bc} \{g(b) \wedge h(c)\} \right) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(ax) \wedge h(axa) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge h(a) \wedge \frac{1-k}{2}.
\end{aligned}$$

Thus $f \circ_k g \circ_k h \geq f \wedge_k g \wedge_k h$

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$: Let $R_1[a]$, $R_2[a]$ be two right ideals and $B[a]$ a bi-ideal of S generated by an element a of S.

Let $b \in R_1[a] \cap R_2[a] \cap B[a]$, so $b \in R_1[a]$, $b \in R_2[a]$ and $b \in B[a]$.

Then $(C_{R_1[a]})_k$, $(C_{R_2[a]})_k$ be $(\in, \in \vee q_k)$-fuzzy right ideals and $(C_{B[a]})_k$ be $(\in, \in \vee q_k)$-fuzzy bi-ideal of semigroup S. Now

$$\begin{aligned}
\frac{1-k}{2} &\leq (C_{R_1[a] \cap R_2[a] \cap B[a]})_k(b) = ((C_{R_1[a]})_k \wedge_k (C_{R_2[a]})_k \wedge_k (C_{B[a]})_k))(b) \\
&\leq ((C_{R_1[a]})_k \circ_k (C_{R_2[a]})_k \circ_k (C_{B[a]})_k)(b) = (C_{R_1[a]R_2[a]B[a]})_k(b)
\end{aligned}$$

Thus $b \in R_1[a]R_2[a]B[a]$. Therefore $R_1[a] \cap R_2[a] \cap B[a] \subseteq R_1[a]R_2[a]B[a]$.

So by theorem 3.2, S is regular. □

4. Characterizations of Right Weakly Regular Semigroups in Terms of $(\in, \in \vee q_k)$-fuzzy Ideals

A semigroup S is called a *right weakly regular* if for every $a \in S$ there exist $x, y \in S$ such that $a = axay$. To begin with this section, we first give the following characterization of right weakly regular semigroups by the properties of their right ideals.

Theorem 4.1. *For a semigroup S the following are equivalent*

(i) S is right weakly regular.

(ii) $R_1 \cap R_2 \subseteq R_1 R_2$, where R_1 and R_2 are right ideals of S.

(iii) $R_1[a] \cap R_2[a] \subseteq R_1[a]R_2[a]$, for some a in S.

Proof. $(i) \Rightarrow (ii)$: Let S be a right weakly regular semigroup and R_1, R_2 be two right ideals of S. Let $a \in R_1 \cap R_2$ which implies that $a \in R_1$ and $a \in R_2$. Since S is right weakly regular semigroup so for $a \in S$ there exist $x, y \in S$ such that $a = axay$ so we have

$$a = axay \in R_1 S R_2 S \subseteq R_1 R_2.$$

Therefore $R_1 \cap R_2 \subseteq R_1 R_2$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: Let $a \in S$, then $R[a] = a \cup aS$ is a right ideal generated by a and so by hypothesis, we have

$$\begin{aligned}(a \cup aS) \cap (a \cup aS) &\subseteq (a \cup aS)(a \cup aS) \\ &= a^2 \cup a^2 S \cup aSa \cup aSaS.\end{aligned}$$

Thus $a = a^2$ or $a = a^2 x$ or $a = aua$ or $a = asat$ for some x, u, s and t in S. If $a = a^2$ then $a = aaaa$. If $a = a^2 x$ then $a = aax = aa^2 xx = aaaxx = aaav$, where $v = xx$. If $a = aua = auaua = auay$, where $y = ua$. Therefore S is right weakly regular. □

Now, we give some characterizations of right weakly regular semigroups by the properties of their $(\in, \in \vee q_k)$-fuzzy right ideals.

Theorem 4.2. *For a semigroup S the following conditions are equivalent.*

(i) S is right weakly regular.

(ii) $f \wedge_k g \leq f \circ_k g$ for all $(\in, \in \vee q_k)$-fuzzy right ideals f and g of S.

Proof. $(i) \Rightarrow (ii)$: Let f and g be $(\in, \in \vee q_k)$-fuzzy right ideals of S. Since S is right weakly regular, then for each $a \in S$ there exist $x, y \in S$ such that $a = axay$, so we have

$$\begin{aligned}(f \circ_k g)(a) &= (f \circ g)(a) \wedge \frac{1-k}{2} = \left(\bigvee_{a=pq} \{f(p) \wedge g(q)\} \right) \wedge \frac{1-k}{2} \\ &\geq f(ax) \wedge g(ay) \wedge \frac{1-k}{2} \geq f(a) \wedge g(a) \wedge \frac{1-k}{2} \\ &= (f \wedge_k g)(a).\end{aligned}$$

Therefore $f \wedge_k g \leq f \circ_k g$.

$(ii) \Rightarrow (i)$: Let $a \in S$ and $b \in R_1[a] \cap R_2[a]$. Since $R_1[a]$ and $R_2[a]$ are right ideals of S generated by a. Then by lemma 2.7, $\left(C_{R_1[a]}\right)_k$ and $\left(C_{R_2[a]}\right)_k$are $(\in, \in \vee q_k)$-fuzzy right ideals of S. Then by hypothesis,

$$\begin{aligned}\frac{1-k}{2} &\leq (C_{R_1[a] \cap R_2[a]})_k(b) = (C_{R_1[a]})_k \wedge_k (C_{R_2[a]})_k)(b) \\ &\leq (C_{R_1[a]})_k \circ_k (C_{R_2[a]})_k)(b) = (C_{R_1[a] R_2[a]})_k(b).\end{aligned}$$

Therefore $b \in R_1[a] R_2[a]$. Thus $R_1[a] \cap R_2[a] \subseteq R_1[a] R_2[a]$. Hence it follows from theorem 4.1, that S is right weakly regular. □

Theorem 4.3. *For a semigroup S, the following conditions are equivalent.*

(i) S is right weakly regular.

(ii) $(f \circ_k g) \wedge (g \circ_k f) \geq f \wedge_k g$ for every $(\in, \in \vee q_k)$-fuzzy right ideals f and g of S.

Proof. $(i) \Rightarrow (ii)$: Let f and g be any $(\in, \in \vee q_k)$-fuzzy right ideals of S, respectively. Since S is right weakly regular then for each $a \in S$ there exist $x,\ y \in S$ such that $a = axay$, so we have

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=pq} \left\{ f(p) \wedge g(q) \wedge \frac{1-k}{2} \right\} \\ &\geq f(ax) \wedge g(ay) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Also

$$\begin{aligned}(g \circ_k f)(a) &= \bigvee_{a=pq} \left\{ g(p) \wedge f(q) \wedge \frac{1-k}{2} \right\} \\ &\geq g(ax) \wedge f(ay) \wedge \frac{1-k}{2} \\ &\geq g(a) \wedge f(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Therefore $(f \circ_k g) \wedge (g \circ_k f) \geq f \wedge_k g$.

$(ii) \Rightarrow (i)$: Let f and g be any $(\in, \in \vee q_k)$-fuzzy right ideals of S. Then by assumption $(f \circ_k g) \geq f \wedge_k g$, so by using theorem 4.2, S is right weakly regular. □

Theorem 4.4. *For a semigroup S, the following conditions are equivalent.*

(i) *S is right weakly regular.*

(ii) *$f \wedge_k g \wedge_k h \leq f \circ_k g \circ_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideals f, g and h of S.*

Proof. $(i) \Rightarrow (ii)$: Let f, g and h be any $(\in, \in \vee q_k)$-fuzzy right ideals of S. Since S is right weakly regular, so for each $a \in S$ there exist $x,\ y \in S$ such that $a = axay$. Then

$$a = axay = (axay)xay = (ax)(ayx)(ay).$$

Thus

$$\begin{aligned}
(f\circ_k g\circ_k h)(a) &= (f\circ g\circ h)(a)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\{f(p)\wedge(g\circ h)(q)\}\right)\wedge\frac{1-k}{2}\\
&\geq f(ax)\wedge(g\circ h)((ayx)(ay))\wedge\frac{1-k}{2}\\
&\geq f(a)\wedge\left(\bigvee_{(ayx)(ay)=bc}\{g(b)\wedge h(c)\}\right)\wedge\frac{1-k}{2}\\
&\geq f(a)\wedge g(ayx)\wedge h(ay)\wedge\frac{1-k}{2}\\
&\geq f(a)\wedge g(a)\wedge h(a)\wedge\frac{1-k}{2}.
\end{aligned}$$

Therefore $f\wedge_k g\wedge_k h\leq f\circ_k g\circ_k h$.

$(ii)\Rightarrow(i)$: Let f and h be any $(\in,\in\vee q_k)$-fuzzy right ideals of S. Since S itself is right ideal so by theorem $(C_S)_k$ is an $(\in,\in\vee q_k)$-fuzzy right ideal of S, so by (ii), we have

$$\begin{aligned}
(f\wedge_k h)(a) &= (f\wedge h)(a)\wedge\frac{1-k}{2} = (f\wedge C_S\wedge h)(a)\wedge\frac{1-k}{2}\\
&= (f\wedge_k C_S\wedge_k h)(a)\leq(f\circ_k C_S\circ_k h)(a)\\
&= (f\circ C_S\circ h)(a)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\{f(p)\wedge(C_S\circ h)(q)\}\right)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\left\{f(p)\wedge(\bigvee_{q=bc}C_S(b)\wedge h(c)\right\}\right)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\left\{f(p)\wedge(\bigvee_{q=bc}1\wedge h(c)\right\}\right)\wedge\frac{1-k}{2}\\
&\leq \left(\bigvee_{a=pq}\left\{f(p)\wedge(\bigvee_{q=bc}h(bc))\right\}\right)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\{f(p)\wedge h(q))\}\right)\wedge\frac{1-k}{2}\\
&= \left(\bigvee_{a=pq}\left\{f(p)\wedge h(q)\wedge\frac{1-k}{2}\right\}\right)\\
&= (f\circ_k h)(a).
\end{aligned}$$

Therefore $f \wedge_k h \leq f \circ_k h$ for every $(\in, \in \vee q_k)$-fuzzy right ideals f and h of S. Hence by theorem 4.2, S is right weakly regular. □

Theorem 4.5. *For a semigroup S, the following are equivalent,*

(i) S is right weakly regular.

(ii) $f_k = f \circ_k f$ for every $(\in, \in \vee q_k)$-fuzzy right ideal f of S.

Proof. $(i) \Rightarrow (ii)$: Let S be a right weakly regular semigroup and f be any $(\in, \in \vee q_k)$-fuzzy right ideal of S. Then for each $a \in S$ there exist $x, y \in S$ such that $a = axay$, we have

$$(f \circ_k f)(a) = \bigvee_{a=pq} \left\{ f(p) \wedge f(q) \wedge \frac{1-k}{2} \right\} \geq f(ax) \wedge f(ay) \wedge \frac{1-k}{2}$$
$$\geq f(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} = f_k(a).$$

Therefore $f \circ_k f \geq f_k$ but $f \circ_k f \leq f_k$. Hence $f_k = f \circ_k f$.

$(ii) \Rightarrow (i)$: Let $a \in S$ and $R[a]$ is a right ideal of S generated by a. Then by lemma 2.7, $\left(C_{R[a]}\right)_k$ is an $(\in, \in \vee q_k)$-fuzzy right ideal of S, now

$$\left(C_{R[a]R[a]}\right)_k (a) = \left(C_{R[a]} \circ_k C_{R[a]}\right)(a) = \left(C_{R[a]}\right)_k (a) \geq \frac{1-k}{2}.$$

This implies that,

$$a \in R[a]\, R[a] = (\{a\} \cup aS)(\{a\} \cup aS)$$
$$= \left\{a^2\right\} \cup aaS \cup aSa \cup aSaS.$$

Thus $a = a^2$ or $a = aax$ where $x = a$ or $a = aya$ where $y = a$ or $a = auav$ where $u = v = a$.

when $a = a^2$ then $a = a^2a^2 = aaaa = apaq$ where $p = q = a$.When $a = aax = (aax)ax = alax$ where $l = ax$. When $a = aya = (aya)ya = aya(ya) = ayam$ where $m = ya$. Therefore S is a right weakly regular semigroup. □

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