NOISE–INDICATOR ARMA MODEL WITH APPLICATION IN FITTING PHYSICALLY–BASED TIME SERIES

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In this paper we propose modification of a linear autoregressive moving-average (ARMA) model by using the so-called Noise-Indicator time series. The obtained model, named NIN–ARMA model, is nonlinear threshold autoregressive one. The basic stochastic properties of the NIN–ARMA model have been analyzed and the Empirical Characteristic Function (ECF) method has been used for parameters estimation. Finally, the NIN–ARMA model has been applied in fitting of two actual, real-based physical time series.

Keywords: Noise–Indicator, ARMA models, ECF method, parameters estimation, application

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1. Introduction

Time series modeling is a dynamic research area with aim to carefully collect and study past observations of a time series to develop an appropriate model which describes the inherent structure of the series [1]. One of the frequently used stochastic time series models is *the Autoregressive Moving Average (ARMA) model*. The basic assumption made to implement this model is that the considered time series is linear and follows a particular known statistical distribution, such as the normal distribution [2]. Despite the relative simplicity for understanding and implementation of the ARMA models, they exhibit the insufficiency in cases where time series show non-linear patterns. To overcome that drawback of the ARMA model here is proposed its modification by introducing the so-called Noise-Indicator, similar as in [3]-[5]. The modified model obtained in this way is nonlinear threshold time series model, and is called *Noise-Indicator ARMA model* (or, shortly *NIN–ARMA model*). Its definition and basic stochastic properties are explained in Section 2. The estimation procedure of the model's parameters based on the so-called *Empirical Characteristic Function (ECF) method* is considered in Section 3. The numerical simulations of described estimation procedure, as well as the practical application of the developed model in fitting of two actual time series are given in Section 4. Finally, last Section 5 is devoted to some concluding remarks.

2. Definition and the main properties of NIN–ARMA model

Definition and explanation of the main stochastic properties of the NIN–ARMA model primarily require introduction of the Noise-Indicator time series, as follows.

Definition 2.1. *Let $(\varepsilon_t) \sim (0,\sigma^2)$ be a time series of the independent identical distributed random variables (RVs), defined on some probability space $(\Omega, P, \mathscr{F})$. Then*

$$\eta_t(c) := I(\varepsilon_{t-1}^2 \geq c) = \begin{cases} 1, & \varepsilon_{t-1}^2 \geq c \\ 0, & \varepsilon_{t-1}^2 < c. \end{cases} \qquad (\mathbf{2}.1)$$

is the Noise–Indicator of the series (ε_t), with the parameter $c > 0$.

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Notice that (η_t) indicates those innovations (ε_t) which are, in comparison with the given value $c > 0$, statistically significant. In that sense, the parameter c is usually called *critical value of reaction*. Namely, if the value ε_{t-1}^2 is smaller than c, then is $\eta_t = 0$, while in the case of pronounced fluctuation of ε_{t-1}^2, the indicator is $\eta_t = 1$. Furthermore, we denote the Noise-Indicator's mean

$$m_c := E[\eta_t] = P\{\varepsilon_t^2 \geq c\} = 1 - F(c),$$

where $F(\cdot)$ is the cumulative distribution function (CDF) of (ε_t^2). Thus, for a given value $c > 0$ can be determined m_c, and vice versa. This will be especially significant at parameters estimation of developed model, which we are formally introducing in the following way.

Definition 2.2. *Let (η_t) be the Noise-Indicator defined by Eq. (2.1). We say that series (X_t) obeys the Noise-INdicator ARMA model of the order (p,q), or simply the NIN–ARMA(p,q) model, if is valid*

$$X_t = \sum_{i=1}^{p} a_i X_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \, \eta_{t-j} \, \varepsilon_{t-j}, \quad t \in \mathbf{Z}, \tag{1}$$

where $a_i \geq 0$, $i = 1,\dots,p$ and $b_j \geq 0$, $j = 1,\dots,q$.

Let us emphasize that the NIN–ARMA model can be interpreted as the nonlinear modification of the ARMA(p,q) model, with the mean $E[X_t] = 0$ and "temporary" innovations $(\eta_{t-j}\varepsilon_{t-j})$. Moreover, Eq. (1) gives the standard ARMA model, when $c \to 0$, hence the NIN–ARMA model can be referred as the generalization of the ARMA models. Conversely, under the nontrivial condition $0 < c < +\infty$, or, equivalently $m_c \in (0,1)$, our model shows a nonlinear, specific structure, which will be analyzed in following. In this aim, we are starting from the matrix representation of the NIN–ARMA(p,q) model, by the stochastic difference equation of order one:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{U}_t, \tag{2}$$

where, for an arbitrary $t \in \mathbb{Z}$, we denoted:

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \mathbf{X}_t = \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{pmatrix}, \quad \mathbf{U}_t = \begin{pmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and $u_t := \varepsilon_t + \sum_{j=1}^{q} b_j \eta_{t-j} \, \varepsilon_{t-j}$. Some necessary and sufficient stationary conditions of the vector series $(\mathbf{X}_t)$ are specified in following.

Theorem 2.1. *Let the vector series $(\mathbf{X}_t)$ be defined by the recurrence relation* (2)*, where the RVs (ε_t) have absolutely continuous distribution. Then, the following conditions are equivalent:*

(i) *The polynomial $P(\lambda) = \lambda^p - \sum_{j=1}^{p} a_j \, \lambda^{p-j}$ has the roots $\lambda_1, ..., \lambda_p$ which satisfy the condition $|\lambda_j| < 1$, $\forall j = 1,\dots,p$.*

(ii) *Eq.* (2) *has the unique, strong stationary and ergodic solution*

$$\mathbf{X}_t = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{U}_{t-k}, \tag{3}$$

where the sum above converges almost surely and in the mean square sense.

(iii) *The inequality $\sum_{j=1}^{p} a_j < 1$ holds.*

Proof. $(i) \Rightarrow (ii)$: After some computation it can be seen that $\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^p P(\lambda)$, i.e., the eigenvalues of matrix $\mathbf{A}$ are the roots of characteristic polynomial $P(\lambda)$. Then, according to the assumption (i), we have $\mathbf{A}^k \longrightarrow \mathbf{O}_{p\times p}$, $k \to \infty$. According to [6], the existence of almost sure unique, ergodic and stationary solution (3) of the Eq. (2) is equivalent to the above convergence.

$(ii) \Rightarrow (iii)$: If assume that condition (ii) is true, then vector series $\mathbf{Y}_t = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{V}_{t-k}$, $t \in \mathbb{Z}$, where $\mathbf{V}_t = \begin{pmatrix} \eta_t \varepsilon_t^2 & 0 & \cdots & 0 \end{pmatrix}'$, is also strong stationary, with the mean

$$E(\mathbf{Y}_t) = (\mathbf{I} - \mathbf{A})^{-1} E(\mathbf{V}_t) = m_c \sigma^2 \cdot \Big(1 - \sum_{j=1}^{p} a_j\Big)^{-1} \cdot \mathbf{1}_{p\times 1}. \tag{4}$$

Note that components $y_t^{(j)}$, $j = 1, \ldots, p$ of the series $(\mathbf{Y}_t)$ satisfy recurrence relations:

$$\begin{aligned} y_t^{(1)} &= \sum_{j=1}^{p} a_j y_{t-1}^{(j)} + \eta_t \varepsilon_t^2 \\ y_t^{(j)} &= y_{t-1}^{(j-1)}, \quad j = 2, \ldots, p. \end{aligned}$$

According to the assumptions of absolutely continuous distribution of (ε_t) and $m_c > 0$, follows that RVs $y_t^{(j)}$ are (almost surely) strictly positive, for any $j = 1, \ldots, p$ and $t \in \mathbf{Z}$. Finally, Eq. (4) gives $E\big(y_t^{(j)}\big) > 0$ or, equivalently, $1 - \sum_{i=1}^{p} a_j > 0$, and (iii) follows.

$(iii) \Rightarrow (i)$: Let $\mathscr{S}_p(\mathbf{A}) = \max_j \{\lambda_j\}$ be the spectral radius of the matrix $\mathbf{A}$. Then, the inequality $\mathscr{S}_p(\mathbf{A}) \le \|\mathbf{A}\|_\infty$ holds, where $\|\mathbf{A}\|_\infty = \max\Big\{\sum_{j=1}^{p} a_j,\, 1\Big\} = 1$ is the sup-norm. If we suppose that $\mathscr{S}_p(\mathbf{A}) = 1$, then for some $\alpha \in [0, 2\pi)$ there exists an eigenvalue $\lambda' = \mathrm{e}^{\mathrm{i}\alpha}$ which satisfies equality

$$P(\lambda') = \mathrm{e}^{\mathrm{i}\alpha} - \sum_{j=1}^{p} a_j \mathrm{e}^{\mathrm{i}(p-j)\alpha} = 0.$$

After that, the inequality $\left|\mathrm{e}^{\mathrm{i}p\alpha}\right| \le \sum_{j=1}^{p} a_j \left|\mathrm{e}^{\mathrm{i}(p-j)\alpha}\right|$ implies $\sum_{j=1}^{p} a_j \ge 1$, which contradicts (iii). Thus, $\mathscr{S}_p(\mathbf{A}) < 1$ what is equivalent to (i). □

From the developing of the vector series (X_t) at their first components, follows:

Corollary 2.1. *The NIN–ARMA(p,q) series (X_t) is stationary if and only if the conditions of the Theorem 2.1 hold. Then, this series has an unique representation:*

$$X_t = \varepsilon_t + \sum_{k=1}^{\infty} v_k \eta_{t-k} \varepsilon_{t-k}, \qquad t \in \mathbf{Z}, \tag{5}$$

where series $\{v_k\}$ satisfies recurrence relation $v_k - \sum_{j=1}^{\min\{k,p\}} a_j v_{k-j} = \begin{cases} b_k, & k \le q \\ 0, & k > q \end{cases}$, $k = 1, 2, \ldots$ Furthermore, the sum in Eq. (5) converges almost surely and in the mean square sense.

Further, we have considered in a more detail the simplest case of NIN–ARMA$(1,1)$ model, defined by equality:

$$X_t = aX_{t-1} + \varepsilon_t + b\eta_{t-1}\varepsilon_{t-1}, \quad t \in \mathbb{Z}. \tag{6}$$

where $0 < a, b < 1$. If denote $u_t := \varepsilon_t + b\eta_{t-1}\varepsilon_{t-1}$, then series $X_t = aX_{t-1} + u_t$ is the AR(1) model on (u_t), which can be expressed as an infinite MA-model:

$$X_t = \sum_{j=0}^{\infty} a^j u_{t-j} = \varepsilon_t + \sum_{j=1}^{\infty} a^{j-1}(a + b\eta_{t-j})\varepsilon_{t-j}. \tag{7}$$

Finally, according to Eq. (7) and after some computation, the covariance of (X_t) is:

$$\gamma_X(h) := E(X_t X_{t+h}) = \begin{cases} \sigma^2 \dfrac{1 + bm_c(2a+b)}{1-a^2}, & h = 0; \\[2ex] \sigma^2 a^{|h|-1} \dfrac{a + bm_c(1 + a(a+b))}{1-a^2}, & h = \pm 1, \pm 2, \ldots \end{cases} \tag{8}$$

These facts will be used below, in order to estimate the parameters of the NIN–ARMA model.

3. **Parameters estimation**

Since the NIN–ARMA(1,1) model represents the specific modification of the appropriate linear ARMA one, the more complex procedure for estimation of its parameters $\theta = (a, b, m_c, \sigma^2)'$ is need. For instance, due to Eqs. (8), the moment-based estimation methods, known as the Yule-Walker estimators, cannot be used here. Also, since the likelihood function of the series (X_t) is unbounded at the origin, the maximum likelihood and other closely related methods not applicable. For these reasons here is used *the empirical characteristic function (ECF)* estimation method [8], based on matching the ECF with theoretical characteristic function (CF) of NIN–ARMA model. The main advantage of this method is based on the fact that the CF is uniformly bounded implying numerical stability of the ECF method. Moreover, according to bijective correspondence between CFs and corresponding CDFs, the ECF retains all the informations present in the sample. First of all, we are giving a general definition of the CF of order $\ell \geq 1$:

Definition 3.1. *Let* $\mathbf{r} = (r_1, \ldots, r_\ell)' \in \mathbb{R}^\ell$ *and* $\mathbf{X}_t^{(l)} := (X_t, \ldots, X_{t+\ell-1})'$, $t \in \mathbb{Z}$ *be the overlapping blocks of the process* (X_t)*. The* ℓ*–dimensional CF of vector* $(\mathbf{X}_t^{(\ell)})$ *is*

$$\varphi_X^{(\ell)}(\mathbf{r};\theta) := E\left[\exp\left(\mathrm{i}\mathbf{r}'\mathbf{X}_t^{(\ell)}\right)\right] = E\left[\exp\left(i\sum_{j=1}^{\ell} r_j X_{t+j-1}\right)\right]. \tag{9}$$

Following statement gives an explicit expression of the CF of NIN–ARMA$(1,1)$ model, in the case of Gaussian distributed innovations.

Theorem 3.1. *Let* (X_t) *be the NIN–ARMA*$(1,1)$ *model defined by Eqs. (6)-(7), where RVs* (ε_t) *have a Gaussian distribution* $\mathscr{N}(0,\sigma^2)$*. Then, the CF of the first order of RVs* (X_t) *is*

$$\varphi_X^{(1)}(r;\theta) = \mathrm{e}^{-\sigma^2 r^2/2} \prod_{j=1}^{\infty} \left[(1-m_c)\mathrm{e}^{-\sigma^2 r^2 a^{2j}/2} + m_c \mathrm{e}^{-\sigma^2 r^2 a^{2(j-1)}(a+b)^2/2}\right], \tag{10}$$

and CFs of the random vectors $\mathbf{X}_t^{(\ell)} = (X_t, \ldots, X_{t+\ell-1})$, $\ell \geq 2$ *are*

$$\begin{aligned}\varphi_X^{(\ell)}(\mathbf{r};\theta) = \mathrm{e}^{-\frac{\sigma^2}{2}\left(\sum\limits_{j=2}^{\ell} r_j^2\right)} &\left[(1-m_c)\mathrm{e}^{-\frac{\sigma^2}{2}\left(r_1+\sum\limits_{j=2}^{\ell} r_j a^{j-1}\right)^2} + m_c \mathrm{e}^{-\frac{\sigma^2}{2}\left(r_1+(a+b)\sum\limits_{j=2}^{\ell} r_j a^{j-2}\right)^2}\right] \\ \times \prod_{j=1}^{\infty} &\left[(1-m_c)\mathrm{e}^{-\frac{\sigma^2}{2}\left(\sum\limits_{k=1}^{\ell} r_k a^{j+k-1}\right)^2} + m_c \mathrm{e}^{-\frac{\sigma^2}{2}\left((a+b)\sum\limits_{k=1}^{\ell} r_k a^{j+k-2}\right)^2}\right].\end{aligned} \tag{11}$$

Proof. If we denote $\xi_t := (a + b\eta_t)\varepsilon_t$, $t \in \mathbb{Z}$, then Eq. (7) can be rewritten as

$$X_t = \varepsilon_t + \sum_{j=1}^{\infty} a^{j-1} \xi_t t - j. \tag{12}$$

As the RVs (η_t) and (ε_t) are mutually independent, (ξ_t) is series of uncorrelated RVs, with CDF:

$$\begin{aligned} F_\xi(x) &:= P\{\xi_t < x\} = P\{a\,\varepsilon_t < x\}P\{\eta_t = 0\} + P\{(a+b)\,\varepsilon_t < x\}P\{\eta_t = 1\} \\ &= (1-m_c)F_\varepsilon(x/a) + m_c F_\varepsilon\left(x/(a+b)\right), \end{aligned} \tag{13}$$

where $F_\varepsilon(x) := P\{\varepsilon_t < x\}$ is the CDF of RVs (ε_t) with Gaussian distribution $\mathscr{N}(0,\sigma^2)$. According to Eq. (12), the first order CF of RVs (X_t) is

$$\varphi_X(r;\theta) = E\left(\mathrm{e}^{\mathrm{i}rX_t}\right) = \varphi_\varepsilon(r) \prod_{j=1}^{\infty} \varphi_\xi\left(a^{j-1}r;\theta\right), \tag{14}$$

where $\varphi_\varepsilon(r) = \mathrm{e}^{-\sigma^2 r^2/2}$ is the CF of Gaussian RVs (ε_t) and, according to Eq. (13),

$$\varphi_\xi(r;\theta) = (1-m_c)\varphi_\varepsilon(ar) + m_c\,\varphi_\varepsilon((a+b)r) \tag{15}$$

is the CF of RVs (ξ_t). Finally, substitution Eq. (15) in Eq. (14) gives Eq. (10).

On the other hand, according to Eq. (12) it follows $X_{t+k} = \varepsilon_{t+k} + \sum_{j=1}^{\infty} a^{j-1}\xi_{t+k-j}$, for any $k = 1,\dots,\ell-1$. Substituting these equalities into Eq. (9), we have:

$$\begin{aligned}\varphi_X^{(\ell)}(\mathbf{r};\theta) &= E\left\{\exp\left[\mathrm{i}\Big(\sum_{j=1}^{\ell} r_j\varepsilon_{t+j-1} + \sum_{j=2}^{\ell} r_j a^{j-2}\xi_t + \sum_{j=1}^{\infty}\sum_{k=1}^{\ell} r_k a^{j+k-2}\xi_{t-j}\Big)\right]\right\} \\ &= \prod_{j=2}^{\ell}\varphi_\varepsilon(r_j)\,\varphi_\xi\left(r_1 + \sum_{j=2}^{\ell} r_j a^{j-2}\right)\prod_{j=1}^{\infty}\varphi_\xi\left(\sum_{k=1}^{\ell} r_k a^{j+k-2}\right).\end{aligned}$$

From here and Eq. (15), immediately follows Eq. (11). □

Now, denote as $\mathbf{X}_T := \{X_1,\dots,X_T\}$ some realization of length $T\in\mathbb{N}$ of the NIN–ARMA$(1,1)$ model (X_t), and the appropriate ℓ-dimensional ECF of the sample $\mathbf{X}_T$ as

$$\widetilde{\varphi}_T^{(\ell)}(\mathbf{r}) := \frac{1}{T-\ell+1}\sum_{t=1}^{T-\ell+1}\exp\left(\mathrm{i}\mathbf{r}'\mathbf{X}_t^{(\ell)}\right).$$

The estimates based on the ECF method (we called them *the ECF estimates*) are obtained by a minimization, with respect to the parameter $\theta = (a,b,m_c,\sigma^2)'$, of the following objective function:

$$S_T^{(\ell)}(\theta) := \int\cdots\int_{\mathbb{R}^\ell} g(\mathbf{r})\left|\varphi_X^{(\ell)}(\mathbf{r};\theta) - \widetilde{\varphi}_T^{(\ell)}(\mathbf{r})\right|^2 \mathbf{dr}.$$

Here, $\varphi_X^{(\ell)}(\mathbf{r};\theta)$ is the CF of order ℓ, defined by Eq. (9), $\mathbf{dr} := \mathrm{d}r_1\cdots\mathrm{d}r_\ell$ and $g:\mathbb{R}^\ell\to\mathbb{R}^+$ is a some weight function. In other words, ECF estimates represent the solutions of the minimization equation

$$\hat{\theta}_T^{(\ell)} = \arg\min_{\theta\in\Theta} S_T^{(\ell)}(\theta),$$

where $\Theta = (0,1)^3\times(0,+\infty)$ is the parameter space of the non-trivial, stationary NIN–ARMA$(1,1)$ model. Using some general results of the ECF-asymptotic theory [9]-[12], similarly as in [13], the strong consistency and asymptotic normality (AN) of the ECF estimates can be shown, as it follows:

Theorem 3.2. *Let θ_0 be the true value of the parameter $\theta\in\Theta$. and let $\hat{\theta}_T^{(\ell)}$, $T=1,2,\dots$ be solutions of the Eq.* (16). *Additionally, we are assuming satisfaction of the following regularity conditions:*

(i) *There exists the set $\Theta' = (0,1)^3\times(0,M_{\sigma^2})\subset\Theta$, where M_{σ^2} is chosen sufficiently large, so that $\theta_0, \hat{\theta}_T^{(\ell)}\in\Theta'$ for all $T\ge T_0>0$;*

(ii) $\dfrac{\partial^2 S_T^{(\ell)}(\theta_0)}{\partial\theta\,\partial\theta'}$ *is a regular matrix;*

(iii) $\dfrac{\partial\varphi_X^{(\ell)}(\mathbf{r};\theta_0)}{\partial\theta}\cdot\dfrac{\partial\varphi_X^{(\ell)}(\mathbf{r};\theta_0)}{\partial\theta'}$ *is a non-zero matrix, uniformly bounded by the strictly positive, g-integrable function $h:\mathbb{R}^\ell\to\mathbb{R}^+$.*

Then, $\hat{\theta}_T^{(\ell)}$ is strictly consistent and is AN estimator for parameter θ.

Remark 3.1. In following, we use the estimation procedure based on two-dimensional CF of the RVs $\mathbf{X}_t^{(2)} := (X_t,X_{t+1})'$. In this case, the objective function $S_T^{(2)}$ represents a double integral with respect to the weight function $g:\mathbb{R}^2\to\mathbb{R}^+$, which can be numerically approximated by using some cubature formulas. Note that an explicit expression of the two-dimensional CF of (X_t) can be easily obtained by Eq.(11), while the following real-valued function

$$\widetilde{\varphi}_T(r_1,r_2) := \mathrm{Re}\,\widetilde{\varphi}_T^{(2)}(r_1,r_2) = \frac{1}{T-1}\sum_{t=1}^{T-1}\cos\left(r_1X_t + r_2X_{t+1}\right)$$

represents its ECF estimate. Figure 1. illustrates the graphs of two-dimensional CF and appropriate ECF of the Gaussian NIN–ARMA$(1,1)$ model, when $T=1500$, $a=b=0.5$ and $c=\sigma^2=1$.

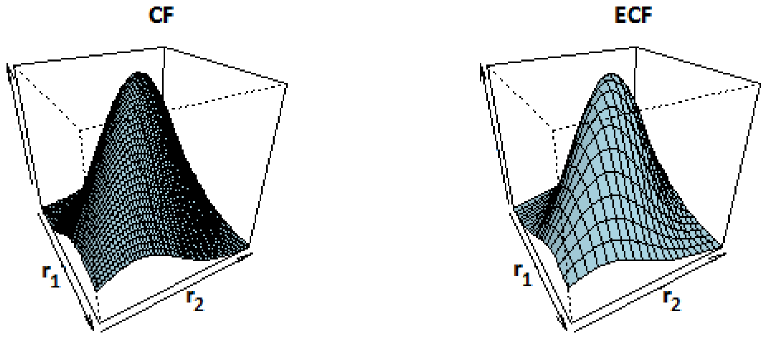

FIGURE 1. Graphs of the two-dimensional CF (panel left) and the appropriate ECF (panel right) of the series $\mathbf{X}_t^{(2)} = (X_t, X_{t+1})'$.

4. Numerical simulations & application of the model

This part is dedicated to implementation of the aforementioned ECF procedure for estimation the parameters $\theta = (a, b, m_c, \sigma^2)'$ of the NIN–ARMA(1,1) model. The ECF estimates are computed according to the minimization of the double integral:

$$S_T^{(2)}(\theta) = \iint_{\mathbb{R}^2} g(r_1, r_2) \left| \varphi_X^{(2)}(r_1, r_2; \theta) - \widetilde{\varphi}_T(r_1, r_2) \right|^2 \mathrm{d}r_1 \, \mathrm{d}r_2, \tag{16}$$

with respect to the exponential weight function $g(r_1, r_2) = \exp\left(-(r_1^2 + r_2^2)\right)$. This function puts more weights around the origin, which is in accordance to the fact that CF in this point contains the most of information about the PDF of estimated model. Moreover, integral in Eq. (16) then can be numerically approximated by using Gauss-Hermitian N-point cubature formula:

$$I(f;g) := \iint_{\mathbb{R}^2} g(r_1, r_2) f(r_1, r_2) \, \mathrm{d}r_1 \, \mathrm{d}r_2 \approx C_N(f) := \sum_{j=1}^{N} \omega_j f(v_{1j}, v_{2j}),$$

where (v_{1j}, v_{2j}) are the cubature nodes, and ω_j are the corresponding weight coefficients. Here, the cubature formula with $N = 81$ nodes, realized using the MATHEMATICA package "`Orthogonal Polynomials`" [14] was used. Thereafter, the objective function given by Eq. (16) is minimized by a Nelder-Mead method, and the whole estimation procedure was done by the original authors' codes written in statistical programming language "`R`".

For comparison, the same estimation procedure was applied on the linear ARMA(1,1) model. In the case of both models, two different sample sizes $T = 150$ (small sample) and $T = 1\,500$ (large sample) have been considered. For all of them have been generated 500 independent realizations $\{X_0, X_1, \ldots, X_T\}$ of the series (X_t) with Gaussian innovations, where $X_0 \stackrel{as}{=} 0$ was set. The obtained numerical results, the means (Mean), minima (Min.), maxima (Max.) and standard estimating errors (SEE) are presented in Table 1. together with values of the objective function $S_T^{(2)}$ as the reference estimation errors. Very similar characteristics of the ECF estimates in all simulated process are evident, as well as that ECF estimates converge and values of SEE and $S_T^{(2)}$ are decreasing with increasing the sample size. Let us notice finally that the estimates of the critical value of the NIN–ARMA model can be easily obtained by solving the equation $P\{\varepsilon_t^2 \geq c\} = m_c$ with respect to c.

Practical application of NIN-ARMA(1,1) model is illustrated by fitting dynamics of two time series, taken from a dataset of *World Ozone and Ultraviolet Radiation Data Centre of Canada (WOUDC)* [15]. The first series, labeled as Series A, refers to a total daily amount of ozone in the atmosphere directly above Bucharest (Romania). The considered data have been obtained by measuring of ozone concentration with Dobson's spectrophotometer between January 1999. and August 2016. The second one, Series B contains data of daily measurements of erythemally-weighted UVB

TABLE 1. Estimated parameters values of ARMA(1,1) and NIN–ARMA(1,1) model. (True parameters are: $a = b = 0.5$, $c = \sigma^2 = 1$, $m_c \approx 0.317$.)

Sample		ARMA Estimates			$S_T^{(2)}$	NIN–ARMA Estimates				$S_T^{(2)}$
		a	b	σ^2		a	b	m_c	σ^2	
T=150	Min.	0.2371	0.2850	0.7014	4.533E-7	0.2300	0.2877	0.1703	0.7358	4.12E-7
	Mean	0.4951	0.5013	1.0123	6.08E-6	0.4920	0.4910	0.3149	1.0481	5.52E-6
	Max.	0.8487	0.5585	1.5487	2.15E-5	0.7698	0.8607	0.3981	1.5643	3.41E-5
	SEE	0.0971	0.0813	0.1940	4.23E-6	0.0952	0.0745	0.0718	0.1902	4.51E-6
T=1500	Min.	0.3580	0.3700	0.8653	3.07E-8	0.3418	0.3861	0.2706	0.8437	3.93E-8
	Mean	0.5000	0.5002	1.0031	5.17E-7	0.5008	0.4995	0.3169	1.0071	5.21E-7
	Max.	0.6801	0.5395	1.1867	3.46E-6	0.5806	0.5749	0.3405	1.1969	2.50E-6
	SEE	0.0547	0.0206	0.0550	3.93E-7	0.0672	0.0160	0.0212	0.0592	4.12E-7

TABLE 2. Estimated parameters values and fitting errors statistics of the actual data.

Sample		Parameters estimates					Fitting errors		
		a	b	m_c	c	σ^2	$S_T^{(2)}$	RMS	AIC
Series A	ARMA	0.0219	0.7770	1.0000	0.0000	0.0715	1.43E-6	0.1714	-4839.46
	NIN–ARMA	0.0289	0.3430	0.1138	0.2608	0.1043	7.26E-7	0.1496	-4864.51
Series B	ARMA	0.2392	0.8140	1.0000	0.0000	0.2790	5.90E-5	0.8751	-1425.16
	NIN–ARMA	0.1330	0.7593	0.3247	0.1728	0.1782	8.20E-7	0.7701	-1429.60

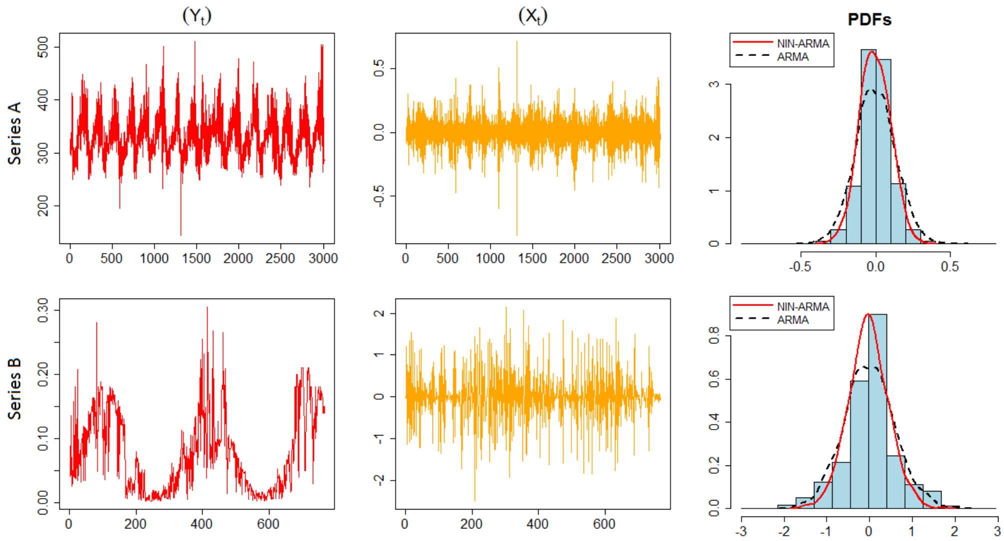

FIGURE 2. Dynamics of the actual time series, along with their empirical fitted PDFs: Series A (graphs above) and Series B (graphs below).

solar radiation over Belgrade (Serbia) in period from March 2009. until August 2011. These data have been obtained by using Solar Light's Model 501 Series Biometer-Radiometer. The sample size of these series are, respectively, $T = 3010$ (large sample) and $T = 767$ (small sample). The both chosen datasets are univariate, discrete time series, measured at daily time intervals. In addition, they are measured as the continuous variables, denoted as (Y_t), using the real positive number scales.

For both of samples we have introduced the so-called log returns $X_t = \ln(Y_t/Y_{t-1})$, which enable condition $E[X_t] = 0$, and thus series (X_t) can be modeled by ARMA or NIN–ARMA model. Table 2. shows the considered Series A and B fitted by ARMA(1,1) and NIN–ARMA(1,1) model, respectively. For checking the efficiency of both of the models we have computed two typical goodness-of-fit statistics: the Root Mean Squares (RMS) of differences between observed and predicted values, as well as the Akaike Information Criterion (AIC). These values, shown in the right side of Table 2, indicate that ECF estimates of the NIN–ARMA model have generally less fitting errors, i.e. proposed model has a higher efficiency. Some of these facts also confirms Figure 2., where the dynamics of the both of actual time series is shown. In this figure also are shown empirical PDFs (histograms) of the Series A and B, which were compared to the PDFs obtained by fitting with the ECF estimates. As it can be easily seen, in both cases the NIN–ARMA model provide better matches to the appropriate empirical PDFs than the corresponding linear ARMA one.

5. **Conclusion**

In this paper is proposed a new, nonlinear stochastic model suitable for empirical analysis of time series which show nonlinear changes in their dynamics. The applicability and suitability of the developed NIN–ARMA model has been checked in fitting of two nonlinear, physically-based data series. First is related to the total daily amount of ozone, while second is daily measurements of the UV-radiation. The ECF estimation method was used in both cases. The results of fitting these two time series show that NIN–ARMA model can be easily applied, with certain modifications, for estimation and fitting the non-linear time series of similar kind as considered ones.

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