UAV FUZZY LOGIC CONTROL SYSTEM STABILITY ANALYSIS IN THE SENSE OF LYAPUNOV

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This article presents the stability analysis in the sense of Lyapunov for a fuzzy logic control system of an unmanned aerial aircraft (UAV). The first section presents an introduction to linear dynamic systems and their stability analysis and the second section presents the equations of motion an aircraft in small perturbations hypothesis. In the third section the fuzzy logic controller is presented and in the forth section the analysis of the stability of the system is presented. In the fifth section the experimental simulation results for a specific aircraft are presented and in the sixth the conclusions are stated.

Keywords: Lyapunov stability analysis, fuzzy logic, dynamic system, unmanned aerial vehicle

1. Introduction

A linear dynamic system is a dynamic system based on the use of linear operator. The mathematical model of a linear dynamic system is represented by a linear differential equation of first order.

$$\frac{d}{dt}x = A \cdot x \tag{1}$$

where $x \in \Re^n$ is the state vector and A is a $n \times n$ constant matrix. The linear systems are commonly used in control theory, e.g. state space representation of a physical system, represented by (2) – continuous time-invariant/autonomous system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2}$$

where $x \in \Re^n$ is the state vector, $y \in \Re^q$ is the output vector, $u \in \Re^p$ is the control vector, A is the system matrix, B is the control matrix, C is the output matrix and D is feedback matrix – $\dim(A) = n \times n, \dim(B) = n \times p, \dim(C) = q \times n$ $\dim(D) = q \times p$ [1].

The stability of the origin for a linear system can be determined by the Routh–Hurwitz stability criterion by determining the eigenvalues of a matrix, i.e. the roots of its characteristic polynomial. A polynomial in one variable with real coefficients is called a Hurwitz polynomial if the real parts of all roots are strictly

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negative. The Routh–Hurwitz theorem implies a characterization of Hurwitz polynomials by means of an algorithm that avoids computing the roots [2]. For general dynamic systems, a general way to establish Lyapunov stability or asymptotic stability is by means of Lyapunov functions [3].

The purpose of the paper is to present a stability analysis for a fuzzy logic controller [4] for an unmanned aerial vehicle in the sense of Lyapunov.

2. Aircraft equations of motion

Experience has shown that in many cases, the dynamic behavior of airplanes can be satisfactorily represented by assuming that the perturbations away from steady state flight are small. These equations are called the small perturbation equations and may be presented in two independent sets – longitudinal and lateral-directional equations [5].

The perturbed longitudinal equations with dimensional stability derivatives:

$$\begin{aligned} &\dot{u} = -g\theta\cos\theta_1 + X_u u + X_{T_u} u + X_\alpha + X_{\delta_e}\delta_e \\ &U_1\dot{\alpha} - U_1\dot{\theta} = -g\theta\sin\theta_1 + Z_u u + Z_\alpha\alpha + Z_{\dot{\alpha}}\dot{\alpha} + Z_q\dot{\theta} + Z_{\delta_e}\delta_e \\ &\ddot{\theta} = M_u u + M_{T_u} u + M_\alpha\alpha + M_{T_\alpha}\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q\dot{\theta} + M_{\delta_e}\delta_e \\ &q = \dot{\theta}, w = U_1\alpha \end{aligned} \tag{3}$$

The perturbed lateral-directional equations with dimensional stability derivatives:

$$\begin{aligned} &U_1\dot{\beta} + U_1\dot{\psi} = g\phi\cos\theta_1 + Y_\beta\beta + Y_p\dot{\phi} + Y_r\dot{\psi} + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r \\ &\ddot{\phi} - \overline{A}_1\ddot{\psi} = L_\beta\beta + L_p\dot{\phi} + L_r\dot{\psi} + L_{\delta_a}\delta_a + L_{\delta_r}\delta_r \\ &\ddot{\psi} - \overline{B}_1\ddot{\phi} = N_\beta\beta + N_{T_\beta}\beta + N_p\dot{\phi} + N_r\dot{\psi} + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r \\ &p = \dot{\phi}, r = \dot{\psi}, v = U_1\beta \end{aligned} \tag{4}$$

Where θ_1 - steady state pitch attitude angle; α, β - angle of attack and side slip

θ, ϕ, ψ - aircraft attitude; $\delta_a, \delta_e, \delta_r$ - aileron, elevator and rudder deflections

$\overline{q}$ - adimensional dynamic pressure $\overline{A}_1 = \dfrac{I_{xz}}{I_{xx}}, \overline{B}_1 = \dfrac{I_{xz}}{I_{zz}}$

The dynamic stability analysis can be more easily predicted from an approximation to the equations (3) and (4) as follows:

- for longitudinal equations, short period and phugoid approximations
- for lateral-directional equations, dutch-roll and spiral approximations.

UAV stability using transfer functions

The classic approach to analyze the stability of an aircraft is to determine the **open-loop transfer functions** based of the equations of motion for every mode.

The equations set expressed in (3) and (4) can be written in a matrix format given by for both longitudinal and lateral-directional modes.

By applying Laplace transform on equations (3) and (4), the system equations are [5]:

$$\begin{bmatrix} s-X_u-X_{T_u} & -X_\alpha & g\cos\theta_1 \\ -Z_u & s(U_1-Z_{\dot\alpha})-Z_\alpha & -(Z_q+U_1)s+g\sin\theta_1 \\ -(M_u+M_{T_u}) & -(M_{\dot\alpha}s+M_\alpha+M_{T_\alpha}) & s^2-M_q s \end{bmatrix}\begin{bmatrix} u(s)/\delta_e(s) \\ \alpha(s)/\delta_e(s) \\ \theta(s)/\delta_e(s) \end{bmatrix}=\begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix}$$

$$\begin{bmatrix} sU_1-Y_\beta & -(sY_p+g\cos\theta_1) & s(U_1-Y_r) \\ -L_\beta & s^2-L_p s & -(s^2\overline{A}_1+sL_r) \\ -N_\beta-N_{T_\beta} & -(s^2\overline{B}_1+N_p s) & s^2-sN_r \end{bmatrix}\begin{bmatrix} \beta(s)/\delta(s) \\ \phi(s)/\delta(s) \\ \psi(s)/\delta(s) \end{bmatrix}=\begin{bmatrix} Y_\delta \\ L_\delta \\ N_\delta \end{bmatrix} \quad (5)$$

Using the approximations stated above, one can write the transfer functions for every mode, as follows:

– short period approximation transfer functions:

$$\begin{bmatrix} sU_1-Z_\alpha & -U_1 \\ -(M_{\dot\alpha}s+M_\alpha) & s^2-M_q s \end{bmatrix}\begin{bmatrix} \alpha(s)/\delta_e(s) \\ \theta(s)/\delta_e(s) \end{bmatrix}=\begin{bmatrix} Z_{\delta_e} \\ M_{\delta_e} \end{bmatrix} \quad (6)$$

– phugoid approximation transfer functions:

$$\begin{bmatrix} s-X_u & g \\ -Z_u & -U_1 s \end{bmatrix}\begin{bmatrix} u(s)/\delta_e(s) \\ \theta(s)/\delta_e(s) \end{bmatrix}=\begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \end{bmatrix} \quad (7)$$

– dutch-roll approximation transfer functions:

$$\begin{bmatrix} sU_1-Y_\beta & s(U_1-Y_r) \\ -N_\beta & s^2-N_r s \end{bmatrix}\begin{bmatrix} \beta(s)/\delta(s) \\ \psi(s)/\delta(s) \end{bmatrix}=\begin{bmatrix} Y_\delta \\ N_\delta \end{bmatrix} \quad (8)$$

– spiral and roll approximations transfer functions:

$$\begin{bmatrix} -L_\beta & -s(s\overline{U}_1+L_r) \\ -N_\beta & s^2-N_r s \end{bmatrix}\begin{bmatrix} \beta(s)/\delta(s) \\ \psi(s)/\delta(s) \end{bmatrix}=\begin{bmatrix} L_\delta \\ N_\delta \end{bmatrix} \quad (9)$$

$$\frac{\phi(s)}{\delta_a(s)}=\frac{L_{\delta_a}}{s^2-sL_p}$$

Next, the roots of the characteristic equation are determined or Routh-Hurwitz criterion is applied after computing the coefficients of the characteristic equation for a given aircraft and flight condition [2].

For closed-loop systems, the **closed-loop system transfer function** is determined (the controller’s transfer function must be determined) and then the behavior of the roots of the characteristic equations as a function of gain K is studied by means of root-locus or Bode methods [5].

3. Fuzzy logic controller for an UAV

In order to study how an airplane or UAV responds to actuation of the primary controls – elevator, aileron, rudder and throttle – the state and the control vectors for longitudinal and lateral motions must be defined. The state of the airplane may include position and velocity vectors relative to a reference frame, airplane attitude (θ,φ,ψ), rotation rates (p,q,r), aerodynamic angles – angle of attack and sideslip (α,β) and acceleration components. The control vector is defined by:

$$c=[\delta_a,\delta_e,\delta_r,\delta_t]$$

where δ_a is aileron deflection, δ_e is elevator deflection, δ_r is rudder deflection and δ_t is throttle deflection.

The process of using state information to govern the control inputs is known as closing the loop, and the resulting system as a closed-loop control or feedback control. Fig. 1 presents a general block diagram describing the feedback situation in a flight control system.

A classic fuzzy logic system is built in two phases [9]: the first phase is defining the systems variables (linguistic variables), the fuzzy database and the inference engine (rules database). The second phase is building the fuzzy logic controller which reads the sensors raw data, pre-process the raw data which transforms into fuzzy logic specific data in order to perform the fuzzy logic operations: fuzzification (transforming the fuzzy logic specific data into degrees of membership), inferencing the data (the rules are applied over the data) and defuzzification.

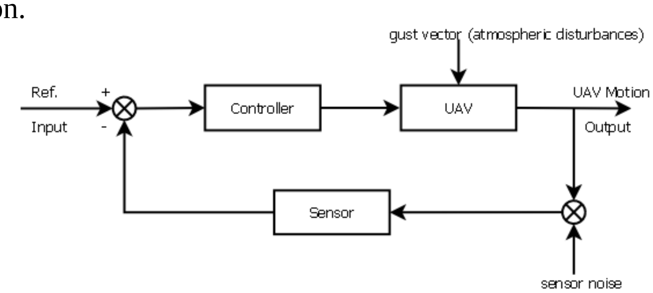

Fig. 1. UAV closed loop block diagram

The relative angle (ρ) is defined as the difference between the aircraft's current heading – h_c and the destination heading – h_d . The deflection angle (δ) is defined as the angle formed by the destination direction and aircraft's current position vector. Therefore, one can write $\rho\in[0,360)$, $\delta\in[0,360)$ - Fig. 2.

Once the fuzzy sets are defined the linguistic variables can be defined; a number of eight linguistic variables: Altitude, Distance, DeflectionAngle, RelativeAngle, YokeX, YokeY, Rudder, and ThrottleLever. These linguistic variables are of two kinds: **conditional** and **action** linguistic variable. Altitude, Distance, DeflectionAngle, and RelativeAngle are condition variables; YokeX, YokeY, Rudder, and ThrottleLever are action variables. The Altitude linguistic variable is defined using three fuzzy sets: Low, Medium and High. The Distance linguistic variable is also defined using three fuzzy sets: Near, Medium and Far.

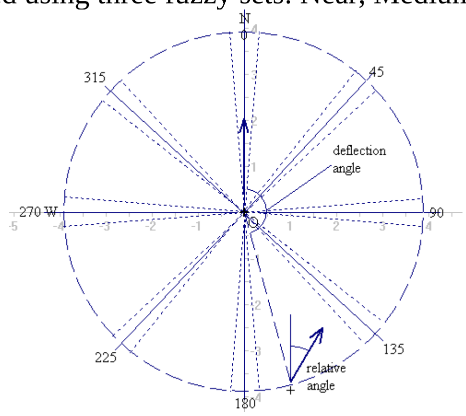

Fig. 2. Space partitioning

The specific angles – DeflectionAngle and RelativeAngle – are defined using 17 fuzzy sets – Fig. 2. Yoke (X and Y), Rudder are defined using three fuzzy sets – Positive, Zero and Negative and the ThrottleLever linguistic variable is defined using three fuzzy sets: Idle, Medium, Full. The rules for Altitude conditional variable are defined – for Distance variable, three rules are also defined. All these rules are added to the inference system together with the method of obtaining the output value for each input value: CentroidDefuzzier.

The simulation environment used in this research is *Microsoft® ESP™/Lockhead Martin Prepar3D* and uses the following equations of motions [5]:

$$
\begin{aligned}
& m(\dot{U} - VR + WQ) = -mg\sin\theta + F_{A_X} + F_{T_X} \\
& m(\dot{V} + UR - WP) = mg\sin\phi\cos\theta + F_{A_Y} + F_{T_Y} \\
& m(\dot{W} - UQ + VP) = mg\cos\phi\cos\theta + F_{A_Z} + F_{T_Z} \\
& I_{XX}\dot{P} - I_{XZ}\dot{R} - I_{XZ}PQ + (I_{ZZ} - I_{YY})RQ = L_A + L_T \\
& I_{YY}\dot{Q} - (I_{XX} - I_{ZZ})PR + I_{XZ}(P^2 - R^2) = M_A + M_T \\
& I_{ZZ}\dot{R} - I_{XZ}\dot{P} + (I_{YY} - I_{XX})PQ + I_{XZ}QR = N_A + N_T \\
& P = \dot{\Phi} - \dot{\Psi}\sin\Theta \\
& R = \dot{\Theta}\cos\Phi + \dot{\Psi}\cos\Theta\sin\Phi \\
& Q = \dot{\Psi}\cos\Theta\cos\Phi - \dot{\Theta}\sin\Phi
\end{aligned}
\tag{10}
$$

From the mathematical point of view, this can be seen as a system of differential equations with the unknowns U, V, W, Ψ, Θ, Φ, P, Q, R. Using an elimination process, the number of unknowns can be reduce to six: ether U, V, W, P, Q, R or U, V, W, Ψ, Θ, Φ.

In both cases, the solutions can be determined by numercal integration and in the case of *Microsoft® ESP™/Lockhead Martin Prepar3D* the numerical method used is a modified version of Euler's method:

$$P_n = P_{n-1} + \left(\frac{\dot{P}_{n-1} + \dot{P}_n}{2} \right) \Delta T \tag{11}$$

In order to find the position and attitude of an aircraft, Hamilton's quaterion method is used (from the coordonate system of the aircraft to the coordonate system of the Earth) for avoiding the singularites at $\pm 90^\circ$ [7].

4. Fuzzy logic control system Lyapunov stability analysis

In order to analyze the stability of the fuzzy logic control system described above, one must consider the following theorems:

Theorem 1 (*Asymptotic stability theorem*): *Let x = 0 be an equilibrium point of* $\dot{x} = f(x), f : D \to \Re^n$, *and let* $V : D \to \Re$ *be a continuous differentiable function such that:*

(i) V(0) = 0, V(x) > 0 in D – {0}, $\dot{V}(x) < 0$ *in D – {0},*

thus x= 0 is asymptotically stable.

Proof: See [6].

Theorem 2: *A function g(x) is the gradient of the scalar function V(x) if and only if the matrix*

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_2} \\ \cdots & \cdots & & \cdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

is symmetric.

Proof: See [6].

The essence of the method is to assume that the *gradient* of the (unknown) Lyapunov function *V(.)* is unknown up to some free parameters. Then by integrating the assumed gradient, the Lyapunov function is found.

Theorem 3: *If **P** is a positive definite matrix and:*

1. $V(x) = x^T P x \to \infty$ *as* $\|x\| \to \infty$, *V(0)=0,*
2. $\dot{V}(x) \le 0, \forall x \in X \subset \Re^n$ *for all fuzzy logic subsystems,*
3. *the set* $\{x \in X \mid \dot{V}(x) = 0\}$ *contains no trajectory of the system except the trivial trajectory x(t)=0 for* $t \ge 0$,

then the fuzzy logic control system with AND-SUM-COG fuzzy logic controller (of Mamdani type) and the process defined in (2), is globally asymptotically stable in origin.

Proof: See [8].

Using the variable gradient method, assume $\nabla V(x) = g(x)$. Thus:

$$\nabla V(x) = g(x) \Rightarrow g(x)dx = \nabla V(x)dx = dV(x) \Rightarrow$$

$$V(x_a) - V(x_b) = \int_{x_a}^{x_b} \nabla V(x)dx = \int_{x_a}^{x_b} g(x)dx \tag{12}$$

The free parameters in the function *g(x)* are constraint to satisfy certain symmetry conditions from Theorem 2.

For the four equations sets (6) – (9), one must determine the numerical values of the coefficients based on the flight condition and corresponding airplane configuration, airplane mass and mass distribution (pitching moment of inertia), dimensionless stability derivatives and the dimensional stability derivatives.

Short period approximation

The equation for short period approximation can be written in the following form $\dot{x} = f(x) + b(x) \cdot u$.

Consider the system in equation (1), $\dot{x} = f(x)$:

$$\begin{aligned} \dot{x}_1 &= c_{11}x_1 + c_{12}x_2 \\ \dot{x}_2 &= c_{21}x_1 + c_{22}x_2 \\ x_1 &= \alpha, x_2 = \dot{\theta} = q \end{aligned} \tag{13}$$

To study the stability of the equilibrium point $(\alpha = 0, \theta = 0)$, the Lyapunov function is found as follows:

Step 1. For this system, consider a candidate gradient:

$g(x) = \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} h_{11}x_1 + h_{12}x_2 & h_{21}x_1 + h_{22}x_2 \end{bmatrix}$.

Step 2. Impose the symmetry conditions. Thus:

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} \Leftrightarrow \frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j} \tag{14}$$

To simplify the solution, assume g_i^j are constant and since $\frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j}, h_{12} = h_{21} = k \Rightarrow g(x) = \begin{bmatrix} h_{11}x_1 + kx_2 & kx_1 + h_{22}x_2 \end{bmatrix}$.

If $k = 0 \Rightarrow g(x) = \begin{bmatrix} h_{11}x_1 & h_{22}x_2 \end{bmatrix}$.

Step 3. Find $\dot{V}(x)$:

$$\dot{V}(x) = g(x) \cdot f(x) = \dot{V}(x) = h_{11}c_{11}x_1^2 + (h_{11}c_{12} + h_{22}c_{21})x_1x_2 + h_{22}c_{22}x_2^2. \tag{15}$$

Step 4. Now, find *V(x)* by integration:

$$V(x)=\int_0^{x_1} h_{11}s_1 ds_1+\int_0^{x_2} h_{22}s_2 ds_2=\frac{1}{2}\left(h_{11}x_1^2+h_{22}x_2^2\right) \tag{16}$$

Step 5. Verify $\dot{V}<0$ and $V>0$.

$V>0 \Leftrightarrow h_{11},h_{22}>0$. Assume, $h_{11}=h_{22}=1$ and $V(x)=\frac{1}{2}\left(x_1^2+x_2^2\right)$. Therefore: $\dot{V}(x)=c_{11}x_1^2+(c_{12}+c_{21})x_1x_2+c_{22}x_2^2$.

In order to satisfy the hypothesis from Theorem 3, one should find the matrix ***P*** and verify that is positive defined.

$$\begin{gathered}V(x)=x^TPx=\frac{1}{2}\left(x_1^2+x_2^2\right)\Leftrightarrow\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{12}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{2}\left(x_1^2+x_2^2\right)\Rightarrow\\ p_{11}=p_{22}=\frac{1}{2},p_{12}=p_{21}=0\Rightarrow P=\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}>0\end{gathered} \tag{17}$$

Therefore, $V(x)$ is positive and for $\|x\|\to\infty$, then $V(x)=x^TPx\to\infty$.

Considering the configuration of Cessna 172 Skyhawk and cruise at 1240 meters (5000 feet), the coefficients are:

$$c_{11}=\frac{Z_\alpha}{U_1}=-3.704<0, c_{12}+c_{21}=1+M_\alpha+\frac{Z_\alpha}{U_1}=-3.3827<0,$$
$$c_{22}=M_{\dot{\alpha}}+M_q=-6.1245<0$$

From the fuzzy control rule base (and from real-time data of the flight dynamic model), can be deduced that the variables α and $\dot{\theta}$ have the same sign and opposite sign with respect to δ_e (can also be seen from recorded flight data). Hence, $\dot{V}(x)<0$.

The condition 3 of the Theorem 3 holds by assuming that $x_2(t)=0, x_1(t)\neq 0$. This means that $\dot{x}_2\neq 0$ which means that x_2 cannot stay constant. Therefore, $x(t)=0$ is the only trajectory for which $\dot{V}(x)<0$.

Also, the fuzzy logic control system described in 4 has a Mamdani type FLC with AND-SUM-COG defuzzification method.

The equations for dutch-roll and spiral modes have the same design; therefore the Lyapunov function will be the same – equilibrium point $(\beta=0,\psi=0)$.

Considering the same configuration, the coefficients are:

$$c_{11}=\frac{Y_\beta}{U_1}=-0.2479<0, c_{12}+c_{21}=Y_r-U_1+N_\beta=-1.8424<0$$
$$c_{22}=N_r=-1.2583<0$$
- dutch roll

$$c_{11} = -\frac{\overline{A}_1 N_r + L_r}{\overline{A}_1 N_\beta + L_\beta} N_\beta = -0.881 < 0, c_{12} + c_{21} = -\frac{\overline{A}_1 N_r + L_r}{\overline{A}_1 N_\beta + L_\beta} N_\beta + N_\beta = -1.9882 < 0$$

$$c_{22} = N_r = -2.6109 < 0, \overline{A}_1 = 0, \alpha = 0^\circ$$

Phugoid approximation

The equations for phugoid approximation can be written as follows:

$$\begin{aligned} \dot{x}_1 &= c_{11}x_1 + c_{12}x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= c_{31}x_1 + c_{33}x_3 \\ x_1 &= u, x_2 = \theta, x_3 = \dot{\theta} = q \end{aligned} \tag{18}$$

To study the stability of the equilibrium point $(\alpha = 0, u - u_0 = 0)$, the Lyapunov function is found as follows:

Step 1. Consider a candidate gradient:

$$g(x) = [g_1 \quad g_2 \quad g_3] = \begin{bmatrix} h_{11}x_1 + h_{12}x_2 + h_{13}x_3 \\ h_{21}x_1 + h_{22}x_2 + h_{23}x_3 \\ h_{31}x_1 + h_{32}x_2 + h_{33}x_3 \end{bmatrix}^T$$

Step 2. Impose the symmetry conditions. Thus:

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} \Leftrightarrow \frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j} \tag{19}$$

To simplify the solution, assume g_i^j are constant and since $\frac{\partial g_j}{\partial x_i} = \frac{\partial g_i}{\partial x_j}$, $h_{12} = h_{13} = h_{21} = h_{23} = h_{31} = h_{32} = k \Rightarrow$

$$g(x) = [h_{11}x_1 + kx_2 + kx_3 \quad kx_1 + h_{22}x_2 + kx_3 \quad kx_1 + kx_2 + h_{33}x_3].$$

If $k = 0 \Rightarrow g(x) = [h_{11}x_1 \quad h_{22}x_2 \quad h_{33}x_3]$.

Step 3. Find $\dot{V}(x)$:

$$\begin{aligned} \dot{V}(x) &= \nabla V \cdot f(x) = g(x) \cdot f(x) = [h_{11}x_1 \quad h_{22}x_2 \quad h_{33}x_3] \cdot f(x) \Rightarrow \\ \dot{V}(x) &= h_{11}c_{11}x_1^2 + h_{11}c_{12}x_1x_2 + h_{22}x_2x_3 + h_{33}c_{31}x_1x_3 + h_{33}c_{23}x_3^2. \end{aligned} \tag{20}$$

Step 4. Now, find *V(x)* by integration:

$$V(x) = \int_0^{x_1} h_{11}s_1 ds_1 + \int_0^{x_2} h_{22}s_2 ds_2 + \int_0^{x_3} h_{33}s_3 ds_3 = \frac{1}{2}\left(h_{11}x_1^2 + h_{22}x_2^2 + h_{33}x_3^2\right) \tag{21}$$

Step 5. Verify $\dot{V} < 0$ and $V > 0$.

$V > 0 \Leftrightarrow h_{11}, h_{22}, h_{33} > 0$. Assume $h_{11} = h_{22} = h_{33} = 1$ thus, $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$.

Therefore: $\dot{V}(x) = c_{11}x_1^2 + c_{12}x_1x_2 + x_2x_3 + c_{31}x_1x_3 + c_{33}x_3^2$.

In order to satisfy the hypothesis from Theorem 3, one should find the matrix ***P*** and verify that is positive defined.

$$V(x) = x^T P x = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \Leftrightarrow$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) \Rightarrow \quad (22)$$

$$p_{11} = p_{22} = p_{22} = \frac{1}{2}, p_{ij} = 0 \Rightarrow P = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} > 0$$

Therefore, *V(x)* is positive and for $\|x\| \to \infty$, then $V(x) = x^T P x \to \infty$.

Considering the configuration of Cessna 172 Skyhawk and cruise at 1240 meters (5000 feet), the coefficients are:

$c_{11} = X_u = -0.0293 < 0, c_{12} = -g < 0, c_{31} = M_u + M_{T_u} = 0, c_{33} = M_q = -4.3150 < 0$

From the fuzzy control rule base (and from real-time data of the flight dynamic model), can be deduced that the variables α and $\dot{\theta}$ have the same sign and opposite sign with respect to δ_e.

Hence, $\dot{V}(x) < 0$.

The condition 3 of the Theorem 3 holds by assuming that $x_3(t) = 0, x_1(t) \neq 0, x_2(t) \neq 0$. This means that $\dot{x}_3 \neq 0$ which means that x_3 cannot stay constant. Therefore, $x(t) = 0$ is the only trajectory for which $\dot{V}(x) < 0$.

Therefore, the fuzzy logic control system with the Mamdani type FLC is globally asymptotically stable in the equilibrium point.

The values of the coefficients were determined by using the Advanced Aircraft Analysis software program [10].

5. Experimental results

The Fuzzy Logic model was able to control the UAV even if the atmospheric conditions were enabled: light snow, maximum visibility (100 m), cloud ceiling at 1500m, wind speed 10m/s and wind direction west. Fig. 3 and Fig. 4 present a comparison between fuzzy logic control system and an automatic control system given the start and end location the airports (given ICAO codes):

Table 1

SBNT airport location

Parameter	Value
Latitude	05° 54' 41.10" S
Longitude	35° 14' 51.78" W
Heading	321°

Table 2

Target SBJP airport location

Parameter	Value
Latitudine	08' 54.17" S
Longitude	34° 57' 02.45" W
Heading	137°

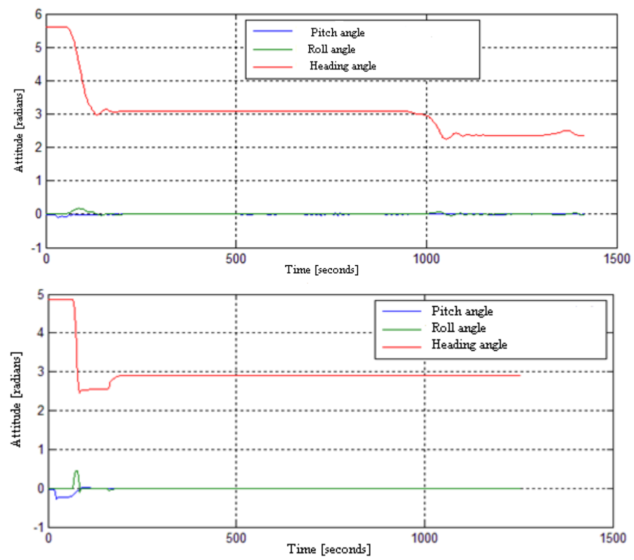

Fig. 3. Fuzzy (upper) vs. automated control system - attitude (radians)

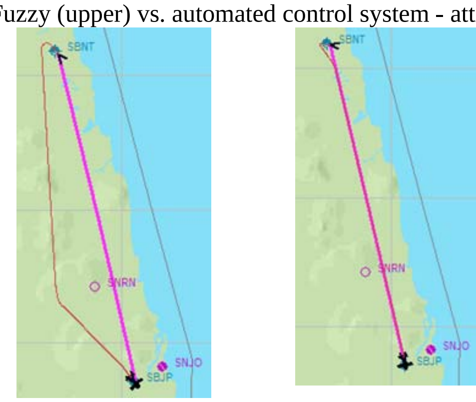

Fig. 4. Fuzzy (left) vs. automatic control system - trajectory (long-lat)

6. Conclusions

The classic approach of analyzing the stability of an UAV is by determine the transfer functions for both longitudinal and lateral-directional modes and this method is only applicable to linear time-invariant systems

The present paper presents a different approach in analyzing the stability of an UAV in the sense of Lyapunov by implement fuzzy logic control system with Mamdani type FLC. The approach used Lyapunov stability theorems, the variable gradient method to find the Lyapunov function and a stability theorem for fuzzy control systems with Mamdani type fuzzy controller.

An advantage of the current design of the fuzzy control system is extensibility. Future work is concentrated on implementing the A* algorithm to extend the model in order to determine the destination location (and a possible physical trajectory) in order to avoid certain zones in real-time. Other advantage over automatic control (as can be seen in Fig 3 and Fig 4.) is that the variations of the attitude angles are not as smooth and a preprogrammed path must be in place in order to satisfy the initial conditions (the predefined heading).

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