# ON PARTIAL SUMS OF WRIGHT FUNCTIONS

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*In this paper, we find the partial sums of two kinds of normalized Wright functions and the partial sums of Alexander transform of these normalized Wright functions. In view of the importance of these results, their geometric interpretation is also included. Furthermore, we discuss the radii of starlikeness for both the normalizations of the Wright functions.*

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## 1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Consider the Alexander transform given as:

$$\mathbb{A}[f](z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{m=2}^{\infty} \frac{a_m}{m} z^m.$$

The surprise use of Hypergeometric function in the solution of the Bieberbach conjecture has attracted many researchers to study the special functions. Many authors who study geometric functions theory are intersted in some geometric properties such as univalency, starlikeness, convexity and close-to-convexity of special functions. Recently, several researchers have studied the geometric properties of hypergeometric functions [17, 34], Bessel functions [1, 2, 3, 4, 5, 6, 7, 28, 29, 30], Struve functions [20, 36], Lommel functions [11]. This study motivated Prajpat [24] to study some geometric properties of Wright functions

$$W_{\lambda,\mu}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(\lambda m + \mu)}, \ \lambda > -1, \ \mu \in \mathbb{C}.$$

This series is absolutely convergent in $\mathbb{C}$, when $\lambda > -1$ and absolutely convergent in open unit disc $\mathcal{U}$ for $\lambda = -1$. Furthermore these function are entire. The Wright functions were introduced by Wright [35] and have been used in the asymtotic theory of partitions, in the theory of integral transforms of Hankel type and in Mikusinski operational calculus. Recently, Wright functions have been found in the solution of partial differential equations of

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fractional order. It was found that the corresponding Green functions can be expressed in terms of Wright functions [23, 27]. Mainardi [16] involved Wright functions in the solution of fractional diffusion wave equation. Luchko et.al [10, 15] obtained the scale variant solutions of partial differential equations of fractional order in terms of Wright functions. For positive rational number $\lambda$, the Wright functions can be expressed in terms of generalized hypergeometric functions. For some details see [13, section 2.1]. In particular, the functions $W_{1,v+1}(-z^2/4)$ can be expressed in terms of the Bessel functions $J_v$, given as:

$$J_v(z) = \left(\frac{z}{2}\right)^2 W_{1,v+1}(-z^2/4) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+v}}{m!\Gamma(m+v+1)}.$$

The Wright functions generalize various functions like Airy functions, Whittaker functions, entire auxiliary functions, etc. For the details, we refer to [13]. Prajapat discussed some geometric properties of the following normalizations of Wright functions in [24]

$$\begin{aligned} \mathcal{W}_{\lambda,\mu}(z) &= \Gamma(\mu)\, z W_{\lambda,\mu}(z) \\ &= z + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)} z^{m+1}.\ \lambda > -1,\ \mu > 0,\ z \in \mathcal{U}, \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbb{W}_{\lambda,\mu}(z) &= \Gamma(\lambda+\mu)\left[W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)}\right] \\ &= z + \sum_{m=1}^{\infty} \frac{\Gamma(\lambda+\mu)}{(m+1)!\Gamma(\lambda m+\lambda+\mu)} z^{m+1},\quad z \in \mathcal{U}, \end{aligned} \tag{2}$$

where $\lambda > -1, \lambda+\mu > 0$. The Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions is given as $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)...(x+n-1)$. For some further work on Wright functions see [8, 26].

In this note, we study the ratio of a function of the forms (1) and (2) to its sequence of partial sums $(\mathcal{W}_{\lambda,\mu})_n(z) = z + \sum_{m=1}^{n} \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)} z^{m+1}$ when the coefficients of $\mathcal{W}_{\lambda,\mu}$ satisfy certain conditions. We determine the lower bounds of $\Re\left\{\frac{\mathcal{W}_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})_n(z)}\right\}$, $\Re\left\{\frac{(\mathcal{W}_{\lambda,\mu})_n(z)}{\mathcal{W}_{\lambda,\mu}(z)}\right\}$, $\Re\left\{\frac{\mathcal{W}'_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})'_n(z)}\right\}$, $\Re\left\{\frac{(\mathcal{W}_{\lambda,\mu})'_n(z)}{\mathcal{W}'_{\lambda,\mu}(z)}\right\}$,
$\Re\left\{\frac{\mathbb{A}[\mathcal{W}_{\lambda,\mu}](z)}{(\mathbb{A}[\mathcal{W}_{\lambda,\mu}])_n(z)}\right\}$, $\Re\left\{\frac{(\mathbb{A}[\mathcal{W}_{\lambda,\mu}])_n(z)}{\mathbb{A}[\mathcal{W}_{\lambda,\mu}](z)}\right\}$, where $\mathbb{A}[\mathcal{W}_{\lambda,\mu}]$ is the Alexander transform of $\mathcal{W}_{\lambda,\mu}$. Some similar results are obtained for the function $\mathbb{W}_{\lambda,\mu}(z)$. For some works on partial sums, we refer [9, 14, 19, 21, 22, 31, 32, 33].

**Lemma 1.1.** *Let $\lambda, \mu \in \mathbb{R}$ and $\lambda \geq 1$, $\mu > 0$. Then the function $\mathcal{W}_{\lambda,\mu} : \mathcal{U} \to \mathbb{C}$ defined by* (1) *satisfies the following inequalities:*

*(i)*

$$|\mathcal{W}_{\lambda,\mu}(z)| \leq \frac{2\mu^2+3\mu+2}{2\mu^2+\mu},\quad z \in \mathcal{U},$$

*(ii)*

$$\left|\mathcal{W}'_{\lambda,\mu}(z)\right| \leq \frac{2\mu^3+8\mu^2+13\mu+10}{2\mu^3+4\mu^2+2\mu},\quad z \in \mathcal{U},$$

*(iii)*

$$|\mathbb{A}[\mathcal{W}_{\lambda,\mu}](z)| \leq \frac{2\mu^2+2\mu+1}{2\mu^2+\mu},\quad z \in \mathcal{U}.$$

*Proof.* (i) By using the well-known triangle inequalitiy

$$|z_1+z_2| \leq |z_1| + |z_2|$$

with the inequality $\Gamma(\mu+m) \leq \Gamma(\mu+m\lambda)$, $m \in \mathbb{N}$, which is equivalent to $\frac{\Gamma(\mu)}{\Gamma(\lambda m+\mu)} \leq \frac{1}{\mu(\mu+1)\ldots(\mu+m-1)} = \frac{1}{(\mu)_m}$, $m \in \mathbb{N}$ and the inequalities

$$(\mu)_m \geq \mu^m, \; m! \geq 2^{m-1}, \; m \in \mathbb{N},$$

we obtain

$$\begin{aligned} |\mathcal{W}_{\lambda,\mu}(z)| &= \left| z + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)} z^{m+1} \right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)} \\ &\leq 1 + \sum_{m=1}^{\infty} \frac{1}{m!(\mu)_m} \\ &\leq 1 + \frac{1}{\mu} \sum_{m=1}^{\infty} \left( \frac{1}{2(\mu+1)} \right)^{m-1} \\ &= \frac{2\mu^2+3\mu+2}{2\mu^2+\mu}, \quad \mu > -1/2, \quad z \in \mathcal{U}. \end{aligned}$$

(ii) To prove (ii), we use the well-known triangle inequality with the inequality $\frac{\Gamma(\mu)}{\Gamma(\lambda m+\mu)} \leq \frac{1}{\mu(\mu+1)\ldots(\mu+m-1)} = \frac{1}{(\mu)_m}$, $m \in \mathbb{N}$ and the inequalities

$$(\mu+1)_m \geq (\mu+1)^m, \; m! \geq \frac{2(m+1)}{3}, \; m \in \mathbb{N} \setminus \{1\},$$

we have

$$\begin{aligned} |\mathcal{W}'_{\lambda,\mu}(z)| &= \left| 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)(m+1)}{m!\Gamma(\lambda m+\mu)} z^m \right| \leq 1 + \sum_{m=1}^{\infty} \frac{\Gamma(\mu)(m+1)}{m!\Gamma(\lambda m+\mu)} \\ &\leq 1 + \sum_{m=1}^{\infty} \frac{m+1}{m!(\mu)_m} \\ &= 1 + \frac{2}{\mu} + \sum_{m=2}^{\infty} \frac{m+1}{m!(\mu)_m} \\ &\leq 1 + \frac{2}{\mu} + \frac{3}{2\mu(\mu+1)} \sum_{m=2}^{\infty} \left( \frac{1}{\mu+2} \right)^{m-2} \\ &= \frac{2\mu^3+8\mu^2+13\mu+10}{2\mu^3+4\mu^2+2\mu}, \quad \mu > -1, \; z \in \mathcal{U}. \end{aligned}$$

(iii) Making the use of triangle inequality with $\frac{\Gamma(\mu)}{\Gamma(\lambda m+\mu)} \leq \frac{1}{(\mu)_m}$ and the inequalities

$$(\mu+1)_m \geq (\mu+1)^m, \; (m+1)! \geq 2^m, \; m \in \mathbb{N},$$

we have

$$\begin{aligned} |\mathbb{A}[\mathcal{W}_{\lambda,\mu}](z)| &= \left|z+\sum_{m=1}^{\infty}\frac{\Gamma(\mu)}{(m+1)!\Gamma(\lambda m+\mu)}z^{m+1}\right| \\ &\leq 1+\sum_{m=1}^{\infty}\frac{\Gamma(\mu)}{(m+1)!\Gamma(\lambda m+\mu)} \\ &\leq 1+\sum_{m=1}^{\infty}\frac{1}{(m+1)!\,(\mu)_m} \\ &\leq 1+\frac{1}{2\mu}\sum_{m=1}^{\infty}\left(\frac{1}{2(\mu+1)}\right)^{m-1} \\ &= \frac{2\mu^2+2\mu+1}{2\mu^2+\mu}, \quad \mu>-1/2, \quad z\in\mathcal{U}. \end{aligned}$$

□

**Lemma 1.2.** *Let* $\lambda$, $\mu\in\mathbb{R}$ *and* $\lambda\geq 1$, $M=\lambda+\mu>0$. *Then the function* $\mathbb{W}_{\lambda,\mu}:\mathcal{U}\to\mathbb{C}$ *defined by* (2) *satisfies the following inequalities:*

*(i) If* $M>-\frac{1}{2}$, *then*

$$|\mathbb{W}_{\lambda,\mu}(z)|\leq\frac{2M^2+3M+2}{2M^2+M}, \quad z\in\mathcal{U}.$$

*(ii) If* $M>0$, *then*

$$\left|\mathbb{W}'_{\lambda,\mu}(z)\right|\leq\frac{M^2+2M+2}{M^2}, \quad z\in\mathcal{U}.$$

*Proof.* (i) By using the well-known triangle inequality

$$|z_1+z_2|\leq|z_1|+|z_2|$$

with the inequality $\Gamma(\lambda+\mu+m)\leq\Gamma(m\lambda+\lambda+\mu)$, $m\in\mathbb{N}$, which is equivalent to $\frac{\Gamma(\lambda+\mu)}{\Gamma(m\lambda+\lambda+\mu)}\leq\frac{1}{(\lambda+\mu)(\lambda+\mu+1)\ldots(\lambda+\mu+m-1)}=\frac{1}{(\lambda+\mu)_m}$, $m\in\mathbb{N}$ and the inequalities

$$(\lambda+\mu+1)_m\geq(\lambda+\mu+1)^m, \; m!\geq 2^{m-1}, \; m\in\mathbb{N},$$

we obtain

$$\begin{aligned} |\mathbb{W}_{\lambda,\mu}(z)| &= \left|z+\sum_{m=1}^{\infty}\frac{\Gamma(\lambda+\mu)}{m!\Gamma(\lambda m+\lambda+\mu)}z^{m+1}\right|\leq 1+\sum_{m=1}^{\infty}\frac{\Gamma(\lambda+\mu)}{m!\Gamma(\lambda m+\lambda+\mu)} \\ &\leq 1+\sum_{m=1}^{\infty}\frac{1}{m!\,(\lambda+\mu)_m} \\ &\leq 1+\frac{1}{M}\sum_{m=1}^{\infty}\left(\frac{1}{2(M+1)}\right)^{m-1} \\ &= \frac{2M^2+3M+2}{2M^2+M}, \quad M>-1/2, \quad z\in\mathcal{U}. \end{aligned}$$

(ii) By using the well-known triangle inequality with the inequality $\frac{\Gamma(\lambda+\mu)}{\Gamma(m\lambda+\lambda+\mu)}\leq\frac{1}{(\lambda+\mu)(\lambda+\mu+1)\ldots(\lambda+\mu+m-1)}=\frac{1}{(\lambda+\mu)_m}$, $m\in\mathbb{N}$ and the inequalities

$$(\lambda+\mu+1)_m\geq(\lambda+\mu+1)^m, \; m!\geq\frac{m+1}{2}, \; m\in\mathbb{N},$$

we have

$$\begin{aligned}\left|\mathbb{W}'_{\lambda,\mu}(z)\right| &= \left|1+\sum_{m=1}^{\infty}\frac{\Gamma(\lambda+\mu)(m+1)}{m!\Gamma(\lambda m+\lambda+\mu)}z^m\right| \le 1+\sum_{m=1}^{\infty}\frac{\Gamma(\lambda+\mu)(m+1)}{m!\Gamma(\lambda m+\lambda+\mu)}\\ &\le 1+\sum_{m=1}^{\infty}\frac{m+1}{m!(\lambda+\mu)_m}\\ &\le 1+\frac{2}{M}\sum_{m=1}^{\infty}\left(\frac{1}{M+1}\right)^{m-1}\\ &= \frac{M^2+2M-2}{M^2},\quad M>0,\ z\in\mathfrak{U}.\end{aligned}$$

□

## 2. Partial Sums of $\mathcal{W}_{\lambda,\mu}(z)$

**Theorem 2.1.** *Let $\lambda$, $\mu \in \mathbb{R}$ such that $\lambda \ge 1$, $\mu > 1.280776406\cdots$. Then*

$$Re\left\{\frac{\mathcal{W}_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})_n(z)}\right\} \ge \frac{2\mu^2-\mu-2}{2\mu^2+\mu},\quad z\in\mathfrak{U},\tag{3}$$

*and*

$$Re\left\{\frac{(\mathcal{W}_{\lambda,\mu})_n(z)}{\mathcal{W}_{\lambda,\mu}(z)}\right\} \ge \frac{2\mu^2+\mu}{2\mu^2+3\mu+2},\quad z\in\mathfrak{U}.\tag{4}$$

*Proof.* By using (i) of Lemma 1.1, it is clear that

$$1+\sum_{m=1}^{\infty}|a_m| \le \frac{2\mu^2+3\mu+2}{2\mu^2+\mu},$$

which is equivalent to

$$\frac{2\mu^2+\mu}{2\mu+2}\sum_{m=1}^{\infty}|a_m| \le 1,$$

where $a_m = \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)}$. Now, we may write

$$\begin{aligned}&\frac{2\mu^2+\mu}{2\mu+2}\left\{\frac{\mathcal{W}_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})_n(z)}-\frac{2\mu^2-\mu-2}{2\mu^2+\mu}\right\}\\ &= \frac{1+\sum_{m=1}^{n}a_m z^m+\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}a_m z^m}{1+\sum_{m=1}^{n}a_m z^m}\\ &=: \frac{1+w(z)}{1-w(z)}.\end{aligned}$$

Then it is clear that

$$w(z) = \frac{\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}a_m z^m}{2+2\sum_{m=1}^{n}a_m z^m+\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}a_m z^m}$$

and

$$|w(z)| \le \frac{\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}|a_m|}{2-2\sum_{m=1}^{n}|a_m|-\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}|a_m|}.$$

This implies that $|w(z)| \leq 1$ if and only if

$$2\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}|a_m| \leq 2-2\sum_{m=1}^{n}|a_m|.$$

Which further implies that

$$\sum_{m=1}^{n}|a_m| + \left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}|a_m| \leq 1. \tag{5}$$

It suffices to show that the left hand side of (5) is bounded above by $\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum\limits_{m=1}^{\infty}|a_m|$, which is equivalent to

$$\frac{2\mu^2-\mu-2}{2\mu+2}\sum_{m=1}^{n}|a_m| \geq 0.$$

To prove $(4)$, we write

$$\begin{aligned}
&\frac{2\mu^2+3\mu+2}{2\mu+2}\left\{\frac{(\mathcal{W}_{\lambda,\mu})_n(z)}{\mathcal{W}_{\lambda,\mu}(z)} - \frac{2\mu^2+\mu}{2\mu^2+3\mu+2}\right\} \\
&= \frac{1+\sum\limits_{m=1}^{n}a_m z^m - \left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum\limits_{m=n+1}^{\infty}a_m z^m}{1+\sum\limits_{m=1}^{\infty}a_m z^m} \\
&= \frac{1+w(z)}{1-w(z)}.
\end{aligned}$$

Therefore

$$|w(z)| \leq \frac{\left(\frac{2\mu^2+3\mu+2}{2\mu+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}{2-2\sum\limits_{m=1}^{n}|a_m| - \left(\frac{2\mu^2-\mu-2}{2\mu+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}|a_m| + \left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum_{m=n+1}^{\infty}|a_m| \leq 1. \tag{6}$$

Since the left hand side of (6) is bounded above by $\left(\frac{2\mu^2+\mu}{2\mu+2}\right)\sum\limits_{m=1}^{\infty}|a_m|$, this completes the proof. $\square$

**Theorem 2.2.** *Let $\lambda$, $\mu \in \mathbb{R}$, with $\lambda \geq 1$ and $\mu > 2.542886....$ Then*

$$Re\left\{\frac{\mathcal{W}'_{\lambda,\mu}(z)}{(\mathcal{W}_{\lambda,\mu})'_n(z)}\right\} \geq \frac{2\mu^3-9\mu-10}{2\mu^3+4\mu^2+2\mu}, \quad z \in \mathcal{U}, \tag{7}$$

$$Re\left\{\frac{(\mathcal{W}_{\lambda,\mu})'_n(z)}{\mathcal{W}'_{\lambda,\mu}(z)}\right\} \geq \frac{\mu^2}{\mu^2+2\mu+2}, \quad z \in \mathcal{U}. \tag{8}$$

*Proof.* From part (ii) of Lemma 1.1, we observe that

$$1+\sum_{m=1}^{\infty}(m+1)|a_m| \leq \frac{2\mu^3+8\mu^2+13\mu+10}{2\mu^3+4\mu^2+2\mu},$$

where $a_m = \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)}$. This implies that

$$\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum_{m=1}^{\infty}(m+1)\left|a_m\right| \le 1.$$

Consider

$$\begin{aligned} &\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\left\{\frac{\mathcal{W}'_{\lambda,\mu}(z)}{\left(\mathcal{W}_{\lambda,\mu}\right)'_n(z)}-\frac{2\mu^3-9\mu-10}{2\mu^3+4\mu^2+2\mu}\right\} \\ =\ & \frac{1+\sum\limits_{m=1}^{n}(m+1)a_m z^m+\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum\limits_{m=n+1}^{\infty}(m+1)a_m z^m}{1+\sum\limits_{m=1}^{n}(m+1)a_m z^m} \\ =\ & \frac{1+w(z)}{1-w(z)}. \end{aligned}$$

Therefore

$$|w(z)| \le \frac{\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|}{2-2\sum\limits_{m=1}^{n}(m+1)\left|a_m\right|-\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|} \le 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}(m+1)\left|a_m\right|+\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum_{m=n+1}^{\infty}(m+1)\left|a_m\right| \le 1. \tag{9}$$

It suffices to show that the left hand side of (9) is bounded above by $\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\right)\sum\limits_{m=1}^{\infty}\left|a_m\right|(m+1)$. Which is equivalent to $\left(\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}-1\right)\sum\limits_{m=1}^{n}(m+1)\left|a_m\right| \ge 0$.

To prove the result (8) , we write

$$\begin{aligned} &\left(\frac{2\mu^3+8\mu^2+13\mu+10}{4\mu^2+11\mu+10}\right)\left\{\frac{\left(\mathcal{W}_{\lambda,\mu}\right)'_n(z)}{\mathcal{W}'_{\lambda,\mu}(z)}-\frac{2\mu^3+4\mu^2+2\mu}{2\mu^3+8\mu^2+13\mu+10}\right\} \\ =\ & \frac{1+w(z)}{1-w(z)}. \end{aligned}$$

Therefore

$$|w(z)| \le \frac{\left(\frac{2\mu^3+8\mu^2+13\mu+10}{4\mu^2+11\mu+10}\right)\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|}{2-2\sum\limits_{m=1}^{n}(m+1)\left|a_m\right|-\frac{2\mu^3-9\mu-10}{4\mu^2+11\mu+10}\sum\limits_{m=n+1}^{\infty}(m+1)\left|a_m\right|} \le 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}\left|a_m\right|(m+1)+\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\sum_{m=n+1}^{\infty}(m+1)\left|a_m\right| \le 1. \tag{10}$$

It suffices to show that the left hand side of (10) is bounded above by $\frac{2\mu^3+4\mu^2+2\mu}{4\mu^2+11\mu+10}\sum\limits_{m=1}^{\infty}(m+1)\left|a_m\right|$, the proof is complete. □

**Theorem 2.3.** *Let* $\lambda,\ \mu \in \mathbb{R}$, *with* $\lambda \geq 1$ *and* $\mu > 0.70710678....$ *Then*

$$Re\left\{\frac{\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right](z)}{\left(\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right]\right)_n(z)}\right\} \geq \frac{2\mu^2-1}{2\mu^2+\mu}, \qquad z \in \mathcal{U}, \tag{11}$$

*and*

$$Re\left\{\frac{\left(\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right]\right)_n(z)}{\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right](z)}\right\} \geq \frac{2\mu^2+\mu}{2\mu^2+2\mu+1}, \qquad z \in \mathcal{U}, \tag{12}$$

*where* $\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right]$ *is the Alexander transform of* $\mathcal{W}_{\lambda,\mu}$.

*Proof.* To prove $(11)$, we consider from part (iii) of Lemma 1.1 so that

$$1+\sum_{m=1}^{\infty}\frac{|a_m|}{(m+1)} \leq \frac{2\mu^2+2\mu+1}{2\mu^2+\mu},$$

which is equvalent to

$$\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum_{m=1}^{\infty}\frac{|a_m|}{(m+1)} \leq 1,$$

where $a_m = \frac{\Gamma(\mu)}{m!\Gamma(\lambda m+\mu)}$. Now, we write

$$\begin{aligned}&\left(\frac{2\mu^2+\mu}{\mu+1}\right)\left\{\frac{\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right](z)}{\left(\mathbb{A}\left[\mathcal{W}_{\lambda,\mu}\right]\right)_n(z)}-\frac{2\mu^2-1}{2\mu^2+\mu}\right\}\\ &= \frac{1+\sum\limits_{m=1}^{n}\frac{a_m}{(m+1)}z^m+\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum\limits_{m=n+1}^{\infty}\frac{a_m}{(m+1)}z^m}{1+\sum\limits_{m=1}^{n}\frac{a_m}{(m+1)}z^m}\\ &= \frac{1+w(z)}{1-w(z)},\end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum\limits_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)}}{2-2\sum\limits_{m=1}^{n}\frac{|a_m|}{(m+1)}-\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum\limits_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)}} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}\frac{|a_m|}{(m+1)}+\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum_{m=n+1}^{\infty}\frac{|a_m|}{(m+1)} \leq 1. \tag{13}$$

It suffices to show that the left hand side of (13) is bounded above by $\left(\frac{2\mu^2+\mu}{\mu+1}\right)\sum\limits_{m=1}^{\infty}\frac{|a_m|}{(m+1)}$, which is equivalent to $\left(\frac{2\mu^2-1}{\mu+1}\right)\sum\limits_{m=1}^{\infty}\frac{|a_m|}{(m+1)} \geq 0$. This completes the proof.

The proof of (12) is similar to the proof of Theorem 2.1. □

**Remark 2.4.** For $\lambda = 1$, $\mu = 5/2$ we get $\mathcal{W}_{1,5/2}(-z) = \frac{3}{4}\left(\frac{\sin(2\sqrt{z})}{2\sqrt{z}}-\cos(2\sqrt{z})\right)$, and for $n = 0$, we have $\left(\mathcal{W}_{1,5/2}\right)_0(z) = z$, so,

$$Re\left(\frac{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}{2z\sqrt{z}}\right) \geq \frac{8}{15} \cong 0.53333... \quad (z \in \mathcal{U}), \tag{14}$$

and

$$Re\left(\frac{2z\sqrt{z}}{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}\right) \geq \frac{15}{22} \cong 0.681818... \quad (z \in \mathcal{U}). \tag{15}$$

The image domains of $f(z) = \frac{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}{2z\sqrt{z}}$ and $g(z) = \frac{2z\sqrt{z}}{\sin(2\sqrt{z})-2\sqrt{z}\cos(2\sqrt{z})}$ are shown in the Figure below.

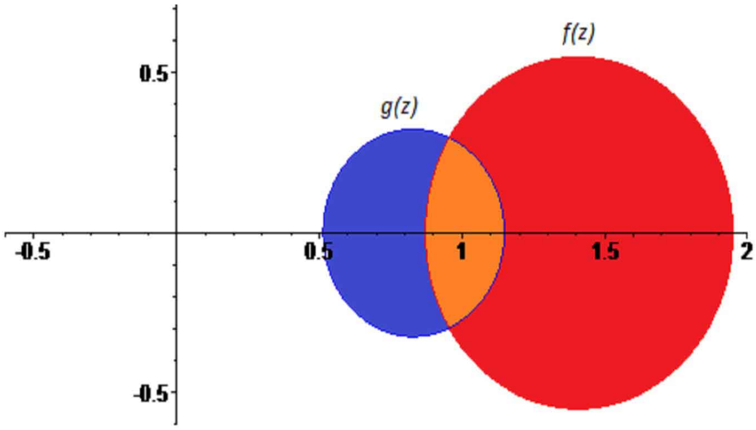


FIGURE 1

## 3. Partial Sums of $\mathbb{W}_{\lambda,\mu}(z)$

**Theorem 3.1.** *Let* $\lambda$, $\mu \in \mathbb{R}$, *with* $\lambda \geq 1$ *and* $M = \mu + \lambda > 0$. *Then*

$$Re\left\{\frac{\mathbb{W}_{\lambda,\mu}(z)}{\left(\mathbb{W}_{\lambda,\mu}\right)_n(z)}\right\} \geq \frac{2M^2 - M - 2}{2M^2 + M}, \quad z \in \mathcal{U}, \tag{16}$$

*and*

$$Re\left\{\frac{\left(\mathbb{W}_{\lambda,\mu}\right)_n(z)}{\mathbb{W}_{\lambda,\mu}(z)}\right\} \geq \frac{2M^2 + M}{2M^2 + 3M + 2}, \quad z \in \mathcal{U}, \tag{17}$$

*where* $\mathbb{W}_{\lambda,\mu}(z)$ *is the normalized Wright function.*

*Proof.* By using Lemma 1.2 (i), It is clear that

$$1 + \sum_{m=1}^{\infty} |a_m| \leq \frac{2M^2 + 3M + 2}{2M^2 + M},$$

where $a_m = \frac{\Gamma(\lambda+\mu)}{(m+1)!\Gamma(\lambda m+\lambda+\mu)}$. This implies that

$$\left(\frac{2M^2 + M}{2M + 2}\right) \sum_{m=1}^{\infty} |a_m| \leq 1.$$

Now we may write

$$\begin{aligned}
&\left(\frac{2M^2 + M}{2M + 2}\right)\left\{\frac{\mathbb{W}_{\lambda,\mu}(z)}{\left(\mathbb{W}_{\lambda,\mu}\right)_n(z)} - \frac{2M^2 - M - 2}{2M^2 + M}\right\} \\
&= \frac{1 + \sum\limits_{m=1}^{n} a_m z^m + \left(\frac{2M^2+M}{2M+2}\right) \sum\limits_{m=n+1}^{\infty} a_m z^m}{1 + \sum\limits_{m=1}^{n} a_m z^m} \\
&= \frac{1 + w(z)}{1 - w(z)}.
\end{aligned}$$

It is clear that

$$w(z)=\frac{\{2(\lambda+\mu)-1\}\sum\limits_{m=n+1}^{\infty}a_m z^m}{2+2\sum\limits_{m=1}^{n}a_m z^m+\{2(\lambda+\mu)-1\}\sum\limits_{m=n+1}^{\infty}a_m z^m},$$

and

$$|w(z)|\leq\frac{\left(\frac{2M^2+M}{2M+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}{2-2\sum\limits_{m=1}^{n}|a_m|-\left(\frac{2M^2+M}{2M+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}.$$

This implies that $|w(z)|\leq 1$ if and only if

$$\sum_{m=1}^{n}|a_m|+\left(\frac{2M^2+M}{2M+2}\right)\sum_{m=n+1}^{\infty}|a_m|\leq 1. \tag{18}$$

It suffices to show that the left hand side of (18) is bounded above by $\left(\frac{2M^2+M}{2M+2}\right)\sum\limits_{m=1}^{\infty}|a_m|$, which is equivalent to $\left(\frac{2M^2+M}{2M+2}-1\right)\sum\limits_{m=1}^{\infty}|a_m|\geq 0.$

To prove (17), we consider that

$$\frac{2M^2+3M+2}{2M+2}\left\{\frac{(\mathbb{W}_{\lambda,\mu})_n(z)}{\mathbb{W}_{\lambda,\mu}(z)}-\frac{2M^2+M}{2M^2+3M+2}\right\}.$$

$$=\frac{1+\sum\limits_{m=1}^{n}a_m z^m-\left(\frac{2M^2+M}{2M+2}\right)\sum\limits_{m=n+1}^{\infty}a_m z^m}{1+\sum\limits_{m=1}^{\infty}a_m z^m}$$

$$=\frac{1+w(z)}{1-w(z)}.$$

Therefore

$$|w(z)|\leq\frac{\left(\frac{2M^2+3M+2}{2M+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}{2-2\sum\limits_{m=1}^{n}|a_m|-\left(\frac{2M^2-M-2}{2M+2}\right)\sum\limits_{m=n+1}^{\infty}|a_m|}.$$

The last inequality is equivalent to

$$\sum_{m=1}^{n}|a_m|+\left(\frac{2M^2+M}{2M+2}\right)\sum_{m=n+1}^{\infty}|a_m|\leq 1. \tag{19}$$

Since the left hand side of (19) is bounded above by $\left(\frac{2M^2+M}{2M+2}\right)\sum\limits_{m=1}^{\infty}|a_m|$, the proof is complete. □

Similarly, we have the following result.

**Theorem 3.2.** *Let* $\lambda,\ \mu\in\mathbb{R}$, *with* $\lambda\geq 1$ *and* $\mu+\lambda>0$. *Then*

$$Re\left\{\frac{\mathbb{W}'_{\lambda,\mu}(z)}{(\mathbb{W}_{\lambda,\mu})'_n(z)}\right\}\geq\frac{M^2-2M-2}{M^2},\quad z\in\mathcal{U}, \tag{20}$$

*and*

$$Re\left\{\frac{(\mathbb{W}_{\lambda,\mu})'_n(z)}{\mathbb{W}'_{\lambda,\mu}(z)}\right\}\geq\frac{M^2}{M^2+2M+2},\quad z\in\mathcal{U}, \tag{21}$$

*where* $\mathbb{W}_{\lambda,\mu}(z)$ *is the normalized Wright function.*

*Proof.* Proof is similar to the Theorem 2.2. □

**Remark 3.1.** Recently Ravichandran [25] presented a survey article on geometric properties of partial sums of univalent functions. Using Noshiro Warsc-
hawski Theorem [12] for $n = 0$ in the inequalities (7) of Theorem 2.2 and (20) of Theorem 3.2, the functions $\mathcal{W}_{\lambda,\mu}(z)$ and $\mathbb{W}_{\lambda,\mu}(z)$ are univalent and also close to convex. Noshiro [18] showed that the radius of starlikness of $f_n$ ( the partial sums of the function $f \in \mathcal{A}$) is $1/M$ if $f$ satisfies the inequality $|f'(z)| \leq M$. This implies that by using the parts (ii) of Lemma 1.1 and Lemma 2.1, the radii of starlikeness of the functions $\left(\mathcal{W}_{\lambda,\mu}\right)_n(z)$ and $\left(\mathbb{W}_{\lambda,\mu}\right)_n(z)$ are $\frac{\mu^2}{\mu^2+2\mu+2}$ and $\frac{M^2}{M^2+2M+2}$ respectively.

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