ON ROUGH \((m, n)\) BI-\(\Gamma\)-HYPERIDEALS IN \(\Gamma\)-SEMIHYPERGROUPS

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In this paper, we introduced the concept of \((m, n)\) bi-\(\Gamma\)-hyperideals and rough \((m, n)\) bi-\(\Gamma\)-hyperideals in \(\Gamma\)-semihypergroups and some properties of \((m, n)\) bi-\(\Gamma\)-hyperideals in \(\Gamma\)-semihypergroups are presented.

Keywords: \(\Gamma\)-semihypergroups, Rough sets, Rough \((m, n)\) bi-\(\Gamma\)-hyperideals.


1. Introduction

The notion of \((m, n)\)-ideals of semigroups was introduced by Lajos [13, 14]. Later \((m, n)\) quasi-ideals and \((m, n)\) bi-ideals and generalized \((m, n)\) bi-ideals were studied in various algebraic structures.

The notion of a rough set was originally proposed by Pawlak [16] as a formal tool for modeling and processing incomplete information in information systems. Some authors have studied the algebraic properties of rough sets. Kuroki, in [12], introduced the notion of a rough ideal in a semigroup. Anvariyeh et al. [3], introduced Pawlak's approximations in \(\Gamma\)-semihypergroups. Abdullah et al. [1], introduced the notion of \(M\)-hypersystem and \(N\)-hypersystem in \(\Gamma\)-semihypergroups and Aslam et al. [6], studied rough \(M\)-hypersystems and fuzzy \(M\)-hypersystems in \(\Gamma\)-semihypergroups, also see [4, 5, 19]. Yaqoob et al. [18], Applied rough set theory to \(\Gamma\)-hyperideals in left almost \(\Gamma\)-semihypergroups.

The algebraic hyperstructure notion was introduced in 1934 by a French mathematician Marty [15], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups.

In 1986, Sen and Saha [17], defined the notion of a \(\Gamma\)-semigroup as a generalization of a semigroup. One can see that \(\Gamma\)-semigroups are generalizations of semigroups. Many classical notions of semigroups have been extended to \(\Gamma\)-semigroups and a lot of results on \(\Gamma\)-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [7], Chinram and Jirojkul [8], Chinram and Siammai [9], Hila [11]. Then, in [2, 10], Davvaz et al. introduced the notion of \(\Gamma\)-semihypergroup

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as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a $\Gamma$-semigroup. They presented many interesting examples and obtained a several characterizations of $\Gamma$-semihypergroups.

In this paper, we have introduced the notion of $(m, n)$ bi-$\Gamma$-hyperideals and we have applied the concept of rough set theory to $(m, n)$ bi-$\Gamma$-hyperideals, which is a generalization of $(m, n)$ bi-$\Gamma$-hyperideals of $\Gamma$-semihypergroups.

2. Preliminaries

In this section, we recall certain definitions and results needed for our purpose.

**Definition 2.1.** A map $\circ : S \times S \to P^*(S)$ is called hyperoperation or join operation on the set $S$, where $S$ is a non-empty set and $P^*(S)$ denotes the set of all non-empty subsets of $S$. A hypergroupoid is a set $S$ with together a (binary) hyperoperation. A hypergroupoid $(S, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$, $\forall x, y, z \in S$, is called a semihypergroup.

Let $A$ and $B$ be two non-empty subsets of $S$. Then, we define

$$A \Gamma B = \bigcup_{\gamma \in \Gamma} A \gamma B = \bigcup \{a \gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$  

Let $(S, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then, $S$ is a $\Gamma$-semihypergroup. So, every semihypergroup is $\Gamma$-semihypergroup.

Let $S$ be a $\Gamma$-semihypergroup and $\gamma \in \Gamma$. A non-empty subset $A$ of $S$ is called a sub $\Gamma$-semihypergroup of $S$ if $x \gamma y \subseteq A$ for every $x, y \in A$. A $\Gamma$-semihypergroup $S$ is called commutative if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x \gamma y = y \gamma x$.

**Example 2.1.** [2] Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$, we define $\gamma : S \times S \to S$ by $x \gamma y = \left[0, \frac{x + y}{\gamma}\right]$. Then, $\gamma$ is hyperoperation. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have $(x \alpha y) \beta z = \left[0, \frac{\alpha x + \beta y}{\alpha \beta}\right] = x \alpha (y \beta z)$. This means that $S$ is a $\Gamma$-semihypergroup.

**Example 2.2.** [2] Let $(S, \circ)$ be a semihypergroup and $\Gamma$ be a non-empty subset of $S$. We define $x \gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then, $S$ is a $\Gamma$-semihypergroup.

**Definition 2.2.** [2] A non-empty subset $A$ of a $\Gamma$-semihypergroup $S$ is a right (left) $\Gamma$-hyperideal of $S$ if $A \Gamma S \subseteq A$ ($S \Gamma A \subseteq A$), and is a $\Gamma$-hyperideal of $S$ if it is both a right and a left $\Gamma$-hyperideal.

**Definition 2.3.** [2] A sub $\Gamma$-semihypergroup $B$ of a $\Gamma$-semihypergroup $S$ is called a bi-$\Gamma$-hyperideal of $S$ if $B \Gamma S \Gamma B \subseteq B$.

A bi-$\Gamma$-hyperideal $B$ of a $\Gamma$-semihypergroup $S$ is proper if $B \neq S$.

**Lemma 2.1.** In a $\Gamma$-semihypergroup $S$, $(A \Gamma B)^m = A^m \Gamma B^m$ holds if $A \Gamma B = B \Gamma A$ for all $A, B \in S$ and $m$ is a positive integer.

**Proof.** We prove the result $(A \Gamma B)^m = A^m \Gamma B^m$ by induction on $m$. For $m = 1$, $A \Gamma B = A \Gamma B$, which is true. For $m = 2$, $(A \Gamma B)^2 = (A \Gamma B)(A \Gamma B) = A \Gamma (B \Gamma A) \Gamma B = A \Gamma B \Gamma (A \Gamma B) = A \Gamma B \Gamma (B \Gamma A) = A \Gamma B \Gamma B = A \Gamma B$. 

This proves the result for $m = 2$. For $m > 2$, we can write

$$(A \Gamma B)^m = (A \Gamma B)^{m-1} \Gamma B = (A \Gamma B)^{m-1} \Gamma (A \Gamma B).$$

Using the induction hypothesis, we have

$$(A \Gamma B)^m = (A \Gamma B)^{m-1} \Gamma (A \Gamma B) = A \Gamma (B \Gamma A) \Gamma (A \Gamma B) = A \Gamma B \Gamma (A \Gamma B) = A \Gamma B \Gamma (B \Gamma A) = A \Gamma B.$$ 

This completes the proof.
\( A^2 \Gamma B^2 \). Suppose that the result is true for \( m = k \). That is, \((A \Gamma B)^k = A^k \Gamma B^k \). Now for \( m = k + 1 \), we have
\[
(A \Gamma B)^{k+1} = (A \Gamma B)^k \Gamma (A \Gamma B) = (A^k \Gamma B^k) \Gamma (A \Gamma B) = A^k \Gamma (B^k \Gamma A) \Gamma B = (A^k \Gamma A) \Gamma (B^k \Gamma B) = A^{k+1} \Gamma B^{k+1}.
\]
Thus, the result is true for \( m = k + 1 \). By induction hypothesis the result \((A \Gamma B)^m = A^m \Gamma B^m\) is true for all positive integers \( m \).

3. \((m, n)\) Bi-\(\Gamma\)-hyperideals in \(\Gamma\)-semihypergroups

From [14], a subsemigroup \( A \) of a semigroup \( S \) is called an \((m, n)\)-ideal of \( S \) if \( A^m S A^n \subseteq A \).

A subset \( A \) of a \(\Gamma\)-semihypergroup \( S \) is called an \((m, 0)\) \(\Gamma\)-hyperideal \(((0, n)\) \(\Gamma\)-hyperideal) if \( A^m \Gamma S \subseteq A \) \((\Gamma A^n \subseteq A)\). A sub \(\Gamma\)-semihypergroup \( A \) of a \(\Gamma\)-semihypergroup \( S \) is called \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \), if \( A \) satisfies the condition
\[
A^m \Gamma S \Gamma A^n \subseteq A,
\]
where \( m, n \) are non-negative integers \((A^m \) is suppressed if \( m = 0 \). Here if \( m = n = 1 \) then \( A \) is called bi-\(\Gamma\)-hyperideal of \( S \). By a proper \((m, n)\) bi-\(\Gamma\)-hyperideal we mean an \((m, n)\) bi-\(\Gamma\)-hyperideal, which is a proper subset of \( S \).

**Example 3.1.** Let \( (S, \circ) \) be a semihypergroup and \( \Gamma \) be a non-empty subset of \( S \). Define a mapping \( S \times \Gamma \times S \to \mathcal{P}(S) \) by \( x \gamma y = x \circ y \) for every \( x, y \in S \) and \( \gamma \in \Gamma \). By Example 2.2, we know that \( S \) is a \(\Gamma\)-semihypergroup. Let \( B \) be an \((m, n)\) bi-\(\Gamma\)-ideal of the \(\Gamma\)-semihypergroup \( S \). Then, \( B^m \circ S \circ B^n \subseteq B \). So, \( B^m \Gamma S \Gamma B^n = B^m \circ S \circ B^n \subseteq B \). Hence, \( B \) is an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \).

**Example 3.2.** Let \( S = [0, 1] \) and \( \Gamma = \mathbb{N} \). Then, \( S \) together with the hyperoperation \( x \gamma y = \left[ 0, \frac{x+y}{2} \right] \) is a \(\Gamma\)-semihypergroup. Let \( t \in [0, 1] \) and set \( T = [0, t] \). Then, clearly it can be seen that \( T \) is a sub \(\Gamma\)-semihypergroup of \( S \). Since \( T^m \Gamma S = [0, t^m] \subseteq [0, t] = T \), \( S T^m = [0, t^m] \subseteq [0, t] = T \), so \( T \) is an \((m, 0)\) \(\Gamma\)-hyperideal \(((0, n)\) \(\Gamma\)-hyperideal) of \( S \). Since \( T^m \Gamma S T^m = [0, t^m + t] \subseteq [0, t] = T \), then \( T \) is an \((m, n)\) bi-\(\Gamma\)-hyperideal of \(\Gamma\)-semi hypergroup \( S \).

**Example 3.3.** Let \( S = [-1, 0] \) and \( \Gamma = \{-1, -2, -3, \cdots \} \). Define the hyperoperation \( x \gamma y = \left[ \frac{x+y}{2}, 0 \right] \) for all \( x, y \in S \) and \( \gamma \in \Gamma \). Then, clearly \( S \) is a \(\Gamma\)-semihypergroup. Let \( \lambda \in [-1, 0] \) and the set \( B = [\lambda, 0] \). Then, clearly \( B \) is a sub \(\Gamma\)-semi hypergroup of \( S \). Since \( B^m \Gamma S = [\lambda, 2^m, 0] \subseteq [\lambda, 0] = B \), \( S \Gamma B^n = [\lambda, 2^n, 0] \subseteq [\lambda, 0] = B \), so \( B \) is an \((m, 0)\) \(\Gamma\)-hyperideal \(((0, n)\) \(\Gamma\)-hyperideal) of \( S \). Since \( B^m \Gamma S \Gamma B^n = [\lambda, 2^{m+n}, 0] \subseteq [\lambda, 0] = B \), then \( B \) is an \((m, n)\) bi-\(\Gamma\)-hyperideal of \(\Gamma\)-semi hypergroup \( S \).

**Proposition 3.1.** Let \( S \) be a \(\Gamma\)-semi hypergroup, \( B \) be a sub \(\Gamma\)-semi hypergroup of \( S \) and let \( A \) be an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \). Then, the intersection \( A \cap B \) is an \((m, n)\) bi-\(\Gamma\)-hyperideal of \(\Gamma\)-semi hypergroup \( B \).

**Proof.** The intersection \( A \cap B \) evidently is a sub \(\Gamma\)-semi hypergroup of \( S \). We show that \( A \cap B \) is an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( B \), for this
\[
(A \cap B)^m \Gamma B \Gamma (A \cap B)^n \subseteq A^m \Gamma S \Gamma A^n \subseteq A, \tag{1}
\]
because of $A$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. Secondly

$$(A \cap B)^n \Gamma B \Gamma (A \cap B)^n \subseteq B^n \Gamma B \Gamma B^n \subseteq B.$$ \hfill (2)

Therefore, (1) and (2) imply that $(A \cap B)^m \Gamma B \Gamma (A \cap B)^n \subseteq A \cap B$, that is, the intersection $A \cap B$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $B$.

**Theorem 3.1.** Suppose that $\{A_i : i \in I\}$ be a family of $(m, n)$ bi-$\Gamma$-hyperideals of a $\Gamma$-semihypergroup $S$. Then, the intersection $\bigcap_{i \in I} A_i \neq \emptyset$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.

**Proof.** Let $\{A_i : i \in I\}$ be a family of $(m, n)$ bi-$\Gamma$-hyperideals in a $\Gamma$-semihypergroup $S$. We know that the intersection of sub $\Gamma$-semihypergroups is a sub $\Gamma$-semihypergroup. Let $B = \bigcap_{i \in I} A_i$. Now we have to show that $B = \bigcap_{i \in I} A_i$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. Here we need only to show that $B^n \Gamma S B^n \subseteq B$. Let $x \in B^n \Gamma S B^n$. Then, $x = a_1^m a_2^m \alpha S \beta a_2^m$ for some $a_1^m, a_2^m \subseteq B$, $\alpha, \beta \in \Gamma$. Thus, for any arbitrary $i \in I$ as $a_1^m, a_2^m \subseteq B_i$. So, $x \in B_i^n \Gamma S B_i^n$. Since $B_i$ is an $(m, n)$ bi-$\Gamma$-hyperideal so $B_i^n \Gamma S B_i^n \subseteq B_i$ and therefore $x \in B_i$. Since $i$ was chosen arbitrarily so $x \in B_i$ for all $i \in I$ and hence $x \in B$. So, $B^n \Gamma S B^n \subseteq B$ and hence $B = \bigcap_{i \in I} A_i$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. \hfill $\square$

It is obvious that the intersection of two or more $(m, 0)$ $\Gamma$-hyperideals $((0, n)$ $\Gamma$-hyperideals) is an $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal). Similarly, the union of two or more $(m, 0)$ $\Gamma$-hyperideals $((0, n)$ $\Gamma$-hyperideals) is an $(m, 0)$ $\Gamma$-hyperideal

$((0, n)$ $\Gamma$-hyperideal).

**Theorem 3.2.** Let $S$ be a $\Gamma$-semihypergroup. If $A$ is an $(m, 0)$ $\Gamma$-hyperideal and also $(0, n)$ $\Gamma$-hyperideal of $S$, then $A$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.

**Proof.** Suppose that $A$ is an $(m, 0)$ $\Gamma$-hyperideal and also $(0, n)$ $\Gamma$-hyperideal of $S$. Then,

$$A^m \Gamma S A^n \subseteq A \Gamma A^n \subseteq S A^n \subseteq A,$$

which implies that $A$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. \hfill $\square$

**Theorem 3.3.** Let $m, n$ be arbitrary positive integers. Let $S$ be a $\Gamma$-semihypergroup, $B$ be an $(m, n)$ bi-$\Gamma$-hyperideal of $S$ and $A$ be a sub $\Gamma$-semihypergroup of $S$. Suppose that $A \Gamma B = B \Gamma A$. Then,

1. $B \Gamma A$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.
2. $A \Gamma B$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.

**Proof.** (1) The suppositions of the theorem imply that

$$(B \Gamma A) \Gamma (B \Gamma A) = (B \Gamma A \Gamma B) \Gamma A = B \Gamma A.$$\hfill \hfill (1)

This shows that $B \Gamma A$ is a sub $\Gamma$-semihypergroup of $S$. On the other hand, as $B$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$, so

$$(B \Gamma A)^\kappa S \Gamma (B \Gamma A)^n = (B^n \Gamma A^n \Gamma S B^n) \Gamma A^n \subseteq B \Gamma A^n \subseteq B \Gamma A.$$\hfill \hfill (2)

Hence, the product $B \Gamma A$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. (2) The proof is similar to (1). \hfill $\square$
Let $S$ be a $\Gamma$-semihypergroup and for a positive integer $n$, $B_1, B_2, \cdots, B_n$ be $(m, n)$ bi-$\Gamma$-hyperideals of $S$. Then, $B_1 \Gamma B_2 \Gamma \cdots \Gamma B_n$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.

Proof. We prove the theorem by induction. By Theorem 3.3, $B_1 \Gamma B_2$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. Next, for $k \leq n$, suppose that $B_1 \Gamma B_2 \Gamma \cdots \Gamma B_k$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. Then, $B_1 \Gamma B_2 \Gamma \cdots \Gamma B_k \Gamma B_{k+1} = (B_1 \Gamma B_2 \Gamma \cdots \Gamma B_k) \Gamma B_{k+1}$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$ by Theorem 3.3. \qed

Theorem 3.5. Let $S$ be a $\Gamma$-semihypergroup, $A$ be an $(m, n)$ bi-$\Gamma$-hyperideal of $S$, and $B$ be an $(m, n)$ bi-$\Gamma$-hyperideal of the $\Gamma$-semihypergroup $A$ such that $B^2 = B \Gamma B = B$. Then, $B$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$.

Proof. It is trivial that $B$ is a sub $\Gamma$-semihypergroup of $S$. Secondly, since $A^m \Gamma S T A^n \subseteq A$ and $B^n \Gamma A \Gamma B^n \subseteq B$, we have $B^m \Gamma S T B^n = B^m \Gamma (B^n \Gamma S T B^n) \Gamma B^n \subseteq B^m \Gamma (A^m \Gamma S T A^n) \Gamma B^n \subseteq B^m \Gamma A \Gamma B^n \subseteq B$. Therefore, $B$ is an $(m, n)$ bi-$\Gamma$-hyperideal of $S$. \qed

4. Lower and Upper Approximations in $\Gamma$-semihypergroups

In what follows, let $S$ denote a $\Gamma$-semihypergroup unless otherwise specified.

Definition 4.1. Let $S$ be a $\Gamma$-semihypergroup. An equivalence relation $\rho$ on $S$ is called regular on $S$ if $(a, b) \in \rho$ implies $(a\gamma x, b\gamma x) \in \rho$ and $(x\gamma a, x\gamma b) \in \rho$, for all $x \in S$ and $\gamma \in \Gamma$.

If $\rho$ is a regular relation on $S$, then, for every $x \in S$, $[x]_{\rho}$ stands for the class of $x$ with the represent $\rho$. A regular relation $\rho$ on $S$ is called complete if $[a]_{\rho} \cap [b]_{\rho} = [a\gamma b]_{\rho}$ for all $a, b \in S$ and $\gamma \in \Gamma$. In addition, $\rho$ on $S$ is called congruence if, for every $(a, b) \in S$ and $\gamma \in \Gamma$, we have $c \in [a]_{\rho} \cap [b]_{\rho} \Rightarrow [c]_{\rho} \subseteq [a\gamma b]_{\rho}$.

Let $A$ be a non-empty subset of a $\Gamma$-semihypergroup $S$ and $\rho$ be a regular relation on $S$. Then, the sets

$$ \text{Apr}_{\rho}(A) = \left\{ x \in S : [x]_{\rho} \subseteq A \right\} \quad \text{and} \quad \text{Apr}_{\rho}^{-1}(A) = \left\{ x \in S : [x]_{\rho} \cap A \neq \emptyset \right\} $$

are called $\rho$-lower and $\rho$-upper approximations of $A$, respectively. For a non-empty subset $A$ of $S$, $\text{Apr}_{\rho}(A) = (\text{Apr}_{\rho}^{-1}(A), \text{Apr}_{\rho}^{-1}(A))$ is called a rough set with respect to $\rho$ if $\text{Apr}_{\rho}(A) \neq \emptyset$.

Theorem 4.1. \cite{3} Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and let $A$ and $B$ be non-empty subsets of $S$. Then,

1. $\text{Apr}_{\rho}(A) \Gamma \text{Apr}_{\rho}(B) \subseteq \text{Apr}_{\rho}(A \Gamma B)$;

2. If $\rho$ is complete, then $\text{Apr}_{\rho}(A) \Gamma \text{Apr}_{\rho}(B) \subseteq \text{Apr}_{\rho}(A \Gamma B)$.

Theorem 4.2. \cite{3} Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. Then,

1. Every sub $\Gamma$-semihypergroup of $S$ is a $\rho$-upper rough sub $\Gamma$-semihypergroup of $S$.

2. Every right (left) $\Gamma$-hyperideal of $S$ is a $\rho$-upper rough right (left) $\Gamma$-hyperideal of $S$. 

Theorem 4.3. [3] Let $\emptyset \neq A \subseteq S$ and let $\rho$ be a complete regular relation on $S$ such that the $\rho$-lower approximation of $A$ is non-empty. Then,

(1) If $A$ is a sub $\Gamma$-semihypergroup of $S$, then $A$ is a $\rho$-lower rough sub $\Gamma$-semihypergroup of $S$.

(2) If $A$ is a right (left) $\Gamma$-hyperideal of $S$, then $A$ is a $\rho$-lower rough right (left) $\Gamma$-hyperideal of $S$.

A subset $A$ of a $\Gamma$-semihypergroup $S$ is called a $\rho$-upper [\rho-lower] rough bi-$\Gamma$-hyperideal of $S$ if $\overline{\text{Apr}}_{\rho}(A)[\underline{\text{Apr}}_{\rho}(A)]$ is a bi-$\Gamma$-hyperideal of $S$.

Theorem 4.4. [3] Let $\rho$ be a regular relation on $S$ and $A$ be a bi-$\Gamma$-hyperideal of $S$. Then,

(1) $A$ is a $\rho$-upper rough bi-$\Gamma$-hyperideal of $S$.

(2) If $\rho$ is complete such that the $\rho$-lower approximation of $A$ is non-empty, then $A$ is a $\rho$-lower rough bi-$\Gamma$-hyperideal of $S$.

Lemma 4.1. Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. Then, for a non-empty subset $A$ of $S$

(1) $(\overline{\text{Apr}}_{\rho}(A))^n \subseteq \overline{\text{Apr}}_{\rho}(A^n)$ for all $n \in \mathbb{N}$.

(2) If $\rho$ is complete, then $(\overline{\text{Apr}}_{\rho}(A))^n \subseteq \overline{\text{Apr}}_{\rho}(A^n)$ for all $n \in \mathbb{N}$.

Proof. (1) Let $A$ be a non-empty subset of $S$, then for $n = 2$, and by Theorem 4.1(1), we get

$$(\overline{\text{Apr}}_{\rho}(A))^2 = \overline{\text{Apr}}_{\rho}(A)\overline{\text{Apr}}_{\rho}(A) \subseteq \overline{\text{Apr}}_{\rho}(A\Gamma A) = \overline{\text{Apr}}_{\rho}(A^2).$$

Now for $n = 3$, we get

$$(\overline{\text{Apr}}_{\rho}(A))^3 = \overline{\text{Apr}}_{\rho}(A)(\overline{\text{Apr}}_{\rho}(A))^2 \subseteq \overline{\text{Apr}}_{\rho}(A)\overline{\text{Apr}}_{\rho}(A^2) \subseteq \overline{\text{Apr}}_{\rho}(A\Gamma A^2) = \overline{\text{Apr}}_{\rho}(A^3).$$

Suppose that the result is true for $n = k - 1$, such that $(\overline{\text{Apr}}_{\rho}(A))^{k-1} \subseteq \overline{\text{Apr}}_{\rho}(A^{k-1})$, then for $n = k$, we get

$$(\overline{\text{Apr}}_{\rho}(A))^k = \overline{\text{Apr}}_{\rho}(A)(\overline{\text{Apr}}_{\rho}(A))^{k-1} \subseteq \overline{\text{Apr}}_{\rho}(A)\overline{\text{Apr}}_{\rho}(A^{k-1}) \subseteq \overline{\text{Apr}}_{\rho}(A\Gamma A^{k-1}) = \overline{\text{Apr}}_{\rho}(A^k).$$

Hence, this shows that $(\overline{\text{Apr}}_{\rho}(A))^n \subseteq \overline{\text{Apr}}_{\rho}(A^n)$ is true for all $n \in \mathbb{N}$. By using Theorem 4.1(2), the proof of (2) can be seen in a similar way. This completes the proof. \qed

5. Rough $(m, n)$ Bi-$\Gamma$-hyperideals in $\Gamma$-semihypergroups

Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$. A subset $A$ of $S$ is called a $\rho$-upper rough $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$ if $\overline{\text{Apr}}_{\rho}(A)$ is an $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$. Similarly, a subset $A$ of a $\Gamma$-semihypergroup $S$ is called a $\rho$-lower rough $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$ if $\underline{\text{Apr}}_{\rho}(A)$ is an $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$.

Theorem 5.1. Let $\rho$ be a regular relation on a $\Gamma$-semihypergroup $S$ and $A$ be an $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$. Then,

(1) $\overline{\text{Apr}}_{\rho}(A)$ is an $(m, 0)$ $\Gamma$-hyperideal $((0, n)$ $\Gamma$-hyperideal) of $S$. 

\[ \text{Proof:} \]
(2) If \( \rho \) is complete, then \( \overline{\text{Apr}_\rho(A)} \) is, if it is non-empty, an \((m, 0)\) \( \Gamma \)-hyperideal of \( S \).

Proof. (1) Let \( A \) be an \((m, 0)\) \( \Gamma \)-hyperideal of \( S \), that is, \( A^m \Gamma S \subseteq A \). Note that \( \overline{\text{Apr}_\rho(S)} = S \). Then, by Theorem 4.1(1) and Lemma 4.1(1), we have

\[
(\overline{\text{Apr}_\rho(A)})^{m} \Gamma S = (\overline{\text{Apr}_\rho(A)})^{m} \Gamma \overline{\text{Apr}_\rho(S)} \subseteq \overline{\text{Apr}_\rho(A^m)} \Gamma \overline{\text{Apr}_\rho(S)} \subseteq \overline{\text{Apr}_\rho(A)}.
\]

This shows that \( \overline{\text{Apr}_\rho(A)} \) is an \((m, 0)\) \( \Gamma \)-hyperideal of \( S \), that is, \( A \) is a \( \rho \)-upper rough \((m, 0)\) \( \Gamma \)-hyperideal of \( S \). Similarly, we can show that the \( \rho \)-upper approximation of a \((0, n)\) \( \Gamma \)-hyperideal is a \((0, n)\) \( \Gamma \)-hyperideal of \( S \).

(2) Let \( A \) be an \((m, 0)\) \( \Gamma \)-hyperideal of \( S \), that is, \( A^m \Gamma S \subseteq A \). Note that \( \overline{\text{Apr}_\rho(S)} = S \). Then, by Theorem 4.1(2) and Lemma 4.1(2), we have

\[
(\text{Apr}_\rho(A))^{m} \Gamma S = (\text{Apr}_\rho(A))^{m} \Gamma \text{Apr}_\rho(S) \subseteq \text{Apr}_\rho(A^m) \Gamma \text{Apr}_\rho(S) \subseteq \text{Apr}_\rho(A).
\]

This shows that \( \text{Apr}_\rho(A) \) is an \((m, 0)\) \( \Gamma \)-hyperideal of \( S \), that is, \( A \) is a \( \rho \)-lower rough \((m, 0)\) \( \Gamma \)-hyperideal of \( S \). Similarly, we can show that the \( \rho \)-lower approximation of a \((0, n)\) \( \Gamma \)-hyperideal is a \((0, n)\) \( \Gamma \)-hyperideal of \( S \). This completes the proof. \( \square \)

A subset \( A \) of a \( \Gamma \)-semihypergroup \( S \) is called a \( \rho \)-upper [\( \rho \)-lower] rough \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \) if \( \overline{\text{Apr}_\rho(A)} \) [\( \text{Apr}_\rho(A) \)] is an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \).

**Theorem 5.2.** Let \( \rho \) be a regular relation on a \( \Gamma \)-semihypergroup \( S \). If \( A \) is an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \), then it is a \( \rho \)-upper rough \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \).

Proof. Let \( A \) be an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \). Then, by Theorem 4.1(1) and Lemma 4.1(1), we have

\[
(\text{Apr}_\rho(A))^m \Gamma S \Gamma (\text{Apr}_\rho(A))^n = (\text{Apr}_\rho(A))^m \Gamma \text{Apr}_\rho(S) \Gamma (\text{Apr}_\rho(A))^n \subseteq \text{Apr}_\rho(A^m) \Gamma \text{Apr}_\rho(A^n) \subseteq \text{Apr}_\rho(A^m \Gamma S \Gamma A^n) \subseteq \text{Apr}_\rho(A).
\]

From this and Theorem 4.2(1), we obtain that \( \overline{\text{Apr}_\rho(A)} \) is an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \), that is, \( A \) is a \( \rho \)-upper rough \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \). This completes the proof. \( \square \)

**Theorem 5.3.** Let \( \rho \) be a complete regular relation on a \( \Gamma \)-semihypergroup \( S \). If \( A \) is an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \), then \( \overline{\text{Apr}_\rho(A)} \) is, if it is non-empty, an \((m, n)\) bi-\( \Gamma \)-hyperideal of \( S \).
Proof. Let \( A \) be an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \). Then, by Theorem 4.1(2) and Lemma 4.1(2), we have

\[
(A_{\text{pr}}(A))^m \Gamma S \Gamma (A_{\text{pr}}(A))^n = \frac{(A_{\text{pr}}(A))^m \Gamma A_{\text{pr}}(S) \Gamma (A_{\text{pr}}(A))^n}{\frac{A_{\text{pr}}(A)^m \Gamma A_{\text{pr}}(S) \Gamma A_{\text{pr}}(A)^n}{A_{\text{pr}}(A)^m \Gamma S A^n}} \subseteq \frac{A_{\text{pr}}(A)^m \Gamma A_{\text{pr}}(S) \Gamma A_{\text{pr}}(A)^n}{A_{\text{pr}}(A)^m \Gamma S A^n} \subseteq A_{\text{pr}}(A).
\]

From this and Theorem 4.3(1), we obtain that \( A_{\text{pr}}(A) \) is, if it is non-empty, an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \): This completes the proof. \( \square \)

The following example shows that the converse of Theorem 5.2 and Theorem 5.3 does not hold.

Example 5.1. Let \( S = \{x, y, z\} \) and \( \Gamma = \{\beta, \gamma\} \) be the sets of binary hyperoperations defined below:

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( x )</td>
<td>( {x, y} )</td>
<td>( z )</td>
</tr>
<tr>
<td>( y )</td>
<td>( {x, y} )</td>
<td>( x, y )</td>
<td>( z )</td>
</tr>
<tr>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( {x, y} )</td>
<td>( x, y )</td>
<td>( z )</td>
</tr>
<tr>
<td>( y )</td>
<td>( {x, y} )</td>
<td>( y )</td>
<td>( z )</td>
</tr>
<tr>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
</tbody>
</table>

Clearly \( S \) is a \(\Gamma\)-semihypergroup. Let \( \rho \) be a complete regular relation on \( S \) such that the \( \rho \)-regular classes are the subsets \( \{x, y\}, \{z\} \). Now for \( A = \{x, z\} \subseteq S \), \( A_{\text{pr}}(A) = \{x, y, z\} \) and \( A_{\text{pr}}(A) = \{z\} \). It is clear that \( A_{\text{pr}}(A) \) and \( A_{\text{pr}}(A) \) are \((m, n)\) bi-\(\Gamma\)-hyperideals of \( S \), but \( A \) is not an \((m, n)\) bi-\(\Gamma\)-hyperideal of \( S \). Because \( A^m \Gamma S A^n = S \nsubseteq A \).

6. Rough \((m, n)\) Bi-\(\Gamma\)-hyperideals in the Quotient \(\Gamma\)-semihypergroups

Let \( \rho \) be a regular relation on a \(\Gamma\)-semihypergroup \( S \). We put \( \widehat{\Gamma} = \{ \gamma : \gamma \in \Gamma \} \). For every \([a]_\rho, [b]_\rho \in S/\rho\), we define \([a]_\rho \widehat{\gamma} [b]_\rho = \{[z]_\rho : z \in a \gamma b\}\).

Theorem 6.1. ([3, Theorem 4.1]) If \( S \) is a \(\Gamma\)-semihypergroup, then \( S/\rho \) is a \(\widehat{\Gamma}\)-semihypergroup.

Definition 6.1. Let \( \rho \) be a regular relation on a \(\Gamma\)-semihypergroup \( S \). The \(\rho\)-lower approximation and \(\rho\)-upper approximation of a non-empty subset \( A \) of \( S \) can be presented in an equivalent form as shown below:

\[
\text{Apr}_\rho(A) = \left\{ [x]_\rho \in S/\rho : [x]_\rho \subseteq A \right\} \quad \text{and} \quad \overline{\text{Apr}}_\rho(A) = \left\{ [x]_\rho \in S/\rho : [x]_\rho \cap A \neq \emptyset \right\},
\]

respectively.

Theorem 6.2. ([3, Theorems 4.3, 4.4]) Let \( \rho \) be a regular relation on a \(\Gamma\)-semihypergroup \( S \). If \( A \) is a sub \(\Gamma\)-semihypergroup of \( S \). Then,

1. \( \text{Apr}_\rho(A) \) is a sub \(\widehat{\Gamma}\)-semihypergroup of \( S/\rho \).
2. \( \overline{\text{Apr}}_\rho(A) \) is, if it is non-empty, a sub \(\Gamma\)-semihypergroup of \( S/\rho \).
Theorem 6.3. Let \( \rho \) be a regular relation on a \( \Gamma \)-semihypergroup \( S \). If \( A \) is an \( (m, 0) \) \( \Gamma \)-hyperideal of \( (0, n) \) \( \Gamma \)-hyperideal of \( S/\rho \). Then,

1. \( \overline{\text{Apr}_\rho(A)} \) is an \( (m, 0) \) \( \hat{\Gamma} \)-hyperideal of \( (0, n) \) \( \hat{\Gamma} \)-hyperideal of \( S/\rho \).
2. \( \text{Apr}_\rho(A) \) is, if it is non-empty, an \( (m, 0) \) \( \hat{\Gamma} \)-hyperideal of \( (0, n) \) \( \hat{\Gamma} \)-hyperideal of \( S/\rho \).

Proof. (1) Assume that \( A \) is a \( (0, n) \) \( \Gamma \)-hyperideal of \( S \). Let \( [x]_\rho \) and \( [s]_\rho \) be any elements of \( \overline{\text{Apr}_\rho(A)} \) and \( S/\rho \), respectively. Then, \( [x]_\rho \cap A \neq \emptyset \). Hence, \( x \in \overline{\text{Apr}_\rho(A)} \). Since \( A \) is a \( (0, n) \) \( \Gamma \)-hyperideal of \( S \), by Theorem 10(1), \( \overline{\text{Apr}_\rho(A)} \) is a \( (0, n) \) \( \Gamma \)-hyperideal of \( S \). So, for \( \gamma \in \Gamma \), we have \( s\gamma x^n \subseteq \overline{\text{Apr}_\rho(A)} \). Now, for every \( t \in s\gamma x^n \), we have \( [t]_\rho \cap A \neq \emptyset \). On the other hand, from \( t \in s\gamma x^n \), we obtain \( [t]_\rho \in [s]_\rho \gamma [x]_\rho^n \). Therefore, \( [s]_\rho \gamma [x]_\rho^n \subseteq \overline{\text{Apr}_\rho(A)} \). This means that \( \overline{\text{Apr}_\rho(A)} \) is a \( (0, n) \) \( \hat{\Gamma} \)-hyperideal of \( S/\rho \).

The other cases can be seen in a similar way. This completes the proof.

Theorem 6.4. Let \( \rho \) be a regular relation on a \( \Gamma \)-semihypergroup \( S \). If \( A \) is an \( (m, n) \) bi-\( \Gamma \)-hyperideal of \( S \). Then,

1. \( \overline{\text{Apr}_\rho(A)} \) is an \( (m, n) \) bi-\( \hat{\Gamma} \)-hyperideal of \( S/\rho \).
2. \( \text{Apr}_\rho(A) \) is, if it is non-empty, an \( (m, n) \) bi-\( \hat{\Gamma} \)-hyperideal of \( S/\rho \).

Proof. (1) Let \( [x]_\rho \) and \( [y]_\rho \) be any elements of \( \overline{\text{Apr}_\rho(A)} \) and \( [s]_\rho \) be any element of \( S/\rho \). Then,

\[
[x]_\rho \cap A \neq \emptyset \quad \text{and} \quad [y]_\rho \cap A \neq \emptyset.
\]

Hence, \( x \in \overline{\text{Apr}_\rho(A)} \) and \( y \in \overline{\text{Apr}_\rho(A)} \). By Theorem 11, \( \overline{\text{Apr}_\rho(A)} \) is an \( (m, n) \) bi-\( \hat{\Gamma} \)-hyperideal of \( S \). So, for every \( \alpha, \beta \in \Gamma \), we have \( x^m \alpha s\beta y^n \subseteq \overline{\text{Apr}_\rho(A)} \). Now, for every \( t \in x^m \alpha s\beta y^n \), we obtain \( [t]_\rho \in [x]_\rho^m \alpha s[\beta [y]_\rho^n] \). On the other hand, since \( t \in \overline{\text{Apr}_\rho(A)} \), we have \( [t]_\rho \cap A \neq \emptyset \). Thus,

\[
[x]_\rho^m \alpha s[\beta [y]_\rho^n] \subseteq \overline{\text{Apr}_\rho(A)}.
\]

Therefore, \( \overline{\text{Apr}_\rho(A)} \) is an \( (m, n) \) bi-\( \hat{\Gamma} \)-hyperideal of \( S/\rho \).

(2) Let \( [x]_\rho \) and \( [y]_\rho \) be any elements of \( \text{Apr}_\rho(A) \) and \( [s]_\rho \) be any element of \( S/\rho \). Then,

\[
[x]_\rho \subseteq A \quad \text{and} \quad [y]_\rho \subseteq A.
\]

Hence, \( x \in \text{Apr}_\rho(A) \) and \( y \in \text{Apr}_\rho(A) \). By Theorem 12, \( \text{Apr}_\rho(A) \) is an \( (m, n) \) bi-\( \hat{\Gamma} \)-hyperideal of \( S \). So, for every \( \alpha, \beta \in \Gamma \), we have \( x^m \alpha s\beta y^n \subseteq \text{Apr}_\rho(A) \). Then,
for every $t \in x^m a \alpha y^n$, we obtain $[t]_\rho \in [x]_\rho^m a \alpha [y]_\rho^n$. On the other hand, since $t \in \text{Apr}_\rho (A)$, we have $[t]_\rho \subseteq A$. So,

$$[x]_\rho^m a \alpha [y]_\rho^n \subseteq \text{Apr}_\rho (A).$$

Therefore, $\text{Apr}_\rho (A)$ is, if it is non-empty, an $(m, n)$ bi-$\Gamma$-hyperideal of $S/_\rho$. This completes the proof.

7. Conclusion

The relations between rough sets and algebraic systems have been already considered by many mathematicians. In this paper, the properties of $(m, n)$ bi-$\Gamma$-hyperideal in $\Gamma$-semihypergroup are investigated and hence the concept of rough set theory is applied to $(m, n)$ bi-$\Gamma$-hyperideals.

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