STANLEY DEPTH OF CERTAIN CLASSES OF SQUARE-FREE MONOMIAL IDEALS

Mircea Cimpoeaş[1]

Given two finite sequences of positive integers α and β, we associate a square-free monomial ideal $I_{\alpha,\beta}$ in the ring of polynomials $S = K[x_1,\ldots,x_n]$. As a continuation of a previous paper, we study the Stanley depth of $I_{\alpha,\beta}$ in particular cases and also for extensions of the ideal $I_{\alpha,\beta}$.

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Introduction

Let K be a field and $S = K[x_1,\ldots,x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i]$$

of $\mathbb{Z}^n$-graded K-vector spaces, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1,\ldots,x_n\}$ such that $m_iK[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M.

We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vladoiu and Zheng show in [12] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [16], Rinaldo give a computer implementation for this algorithm, in the computer algebra system `CoCoA` [8]. In [2], J. Apel restated a conjecture firstly given by Stanley in [18], namely that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M),$$

for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see Duval et. al. [9].

Stanley depth is an important combinatorial invariant and deserves a thorough study. The explicit computation of the Stanley depth is difficult, from an algorithmic point of view. Also, although the Stanley conjecture was disproved, it is interesting to find large classes of multigraded modules which satisfy the Stanley inequality, i.e. $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$. For a friendly introduction in the thematic of Stanley depth, we refer the reader [13].

Given two sequences of positive integers:

$$\alpha : a_1 < a_2 < \cdots < a_s,\ \beta : b_1 < b_2 < \cdots < b_s,$$

[1]Researcher, Simion Stoilow Institute of Mathematics, Research unit 5, P.O.Box 1-764,Bucharest 014700, Romania, e-mail: `mircea.cimpoeas@imar.ro`

with $a_i \leq b_i$ for all $1 \leq i \leq s$ and $b_s \leq n$, we consider the square-free monomial ideal

$$I_{\alpha,\beta} = (x_{a_1} \cdots x_{b_1}, \ldots, x_{a_s} \cdots x_{b_s}) \subset S = K[x_1, \ldots, x_n].$$

Note that $I_{\alpha,\beta}$ is a natural generalization for the path ideal associated to the path graph, see [6] for further details. In [7] we studied the algebraic and combinatorial invariants of the ideal $I_{\alpha,\beta}$. As a continuation of our paper, we consider several particular cases and extensions of $I_{\alpha,\beta}$.

For $n \geq m \geq 1$, the *m-path ideal of the path graph* of length n is

$$I_{n,m} = (x_1x_2 \cdots x_m, x_2x_3 \cdots x_{m+1}, \ldots, x_{n-m+1} \cdots x_n) \subset S.$$

In [6, Theorem 1.3] we computed $\operatorname{sdepth}(S/I_{n,m})$. We study the ideal

$$I_{n,m,k} = (x_1x_2 \cdots x_m, x_{k+1} \cdots x_{k+m}, \ldots, x_{n-m+1} \cdots x_n),$$

where $n \geq m \geq 1$ and $k|n-m$, which is a generalization of $I_{n,m}$. See Proposition 2.1, Proposition 2.2 and Theorem 2.3.

We consider the following extensions of the sequences α and β:

$$\bar{\alpha} : a_1 < \cdots < a_s < \cdots < a_{s+t},\ \bar{\beta} : b_1 < \cdots < b_{s+t},$$

where $b_1 + n > b_{s+i} > n$ for all $1 \leq i \leq t$ and $a_{s+t} \leq n$. We consider the ideal

$$J_{\bar{\alpha},\bar{\beta}} = I_{\alpha,\beta} + (u_{s+1}, \cdots, u_{s+t}),$$

where $u_{s+i} = x_{a_{s+i}} \cdots x_n x_1 \cdots x_{b_{s+i}-n}$, for all $1 \leq i \leq t$.

The ideal $J_{\bar{\alpha},\bar{\beta}}$ is a natural generalization for the path ideal of a cycle graph, see [5]. In Theorem 2.5, we prove that

$$\operatorname{depth}(S/J_{\bar{\alpha},\bar{\beta}}), \operatorname{sdepth}(S/J_{\bar{\alpha},\bar{\beta}}) \leq \operatorname{sdepth}(S/I_{\alpha,\beta}).$$

In special cases, we obtain precise formulas or sharp bounds for these invariants, see Proposition 2.7, Proposition 2.8 and Proposition 2.9.

1. **Preliminaries**

First, we recall the well known Depth Lemma, see [19, Lemma 1.3.9].

Lemma 1.1. *(Depth Lemma) If* $0 \to U \to M \to N \to 0$ *is a short exact sequence of modules over a local ring* S*, or a Noetherian graded ring with* S_0 *local, then:* (1) $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.

(2) $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.

(3) $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

In [15], Asia Rauf proved the analog of Lemma 1.1(1) for sdepth:

Lemma 1.2. *Let* $0 \to U \to M \to N \to 0$ *be a short exact sequence of* $\mathbb{Z}^n$*-graded* S*-modules. Then:* $\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}$.

We also recall the following well known results. See for instance [15, Corollary 1.3], [4, Proposition 2.7] and [15, Corollary 3.3].

Lemma 1.3. *Let* $I \subset S$ *be a monomial ideal and let* $u \in S$ *a monomial which is not in* I.

(1) $\operatorname{sdepth}(S/(I : u)) \geq \operatorname{sdepth}(S/I)$, $\operatorname{sdepth}(I : u) \geq \operatorname{sdepth}(I)$ *and* $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$.

(2) *If* u *is regular on* S/I*, then* $\operatorname{sdepth}(S/(I, u)) = \operatorname{sdepth}(S/I) - 1$.

Let $0 \leq s \leq n$ be an integer. We consider two sequences of integers $\alpha : a_1 < a_2 < \cdots < a_s$ and $\beta : b_1 < b_2 < \cdots < b_s$ with $1 \leq a_1$, $b_s \leq n$ and $a_i \leq b_i$, for all $1 \leq i \leq s$. If $s = 0$, then α and β are the empty set.

For $s \geq 1$, let $j := j(\alpha, \beta) := \max\{i : a_i \leq b_1\}$. We define

$$\alpha' : a'_1 = a_{j+1} < a'_2 = a_{j+2} < \cdots < a'_{s-j} = a_s,$$

$$\beta' : b'_1 = b_{j+1} < b'_2 = b_{j+2} < \cdots < b'_{s-j} = b_s.$$

If $s = 1$ or $j = 1$, let $\alpha'' = \alpha'$ and $\beta'' = \beta'$. Assume $s > 1$ and $j > 1$.

If $a_{j+1} > b_1 + 1$, we define

$$\alpha'' : a''_1 = b_1 + 1 < a''_2 = a_{j+1} < \cdots < a''_{s-j+1} = a_s,$$

$$\beta'' : b''_1 = b_2 < b''_2 = b_{j+1} < \cdots < b''_{s-j+1} = b_s.$$

If $a_{j+1} = b_1 + 1$, we define

$$\alpha'' : a''_1 = b_1 + 1 < a''_2 = a_{j+2} < \cdots < a''_{s-j} = a_s,$$

$$\beta'' : b''_1 = b_2 < b''_2 = b_{j+2} < \cdots < b''_{s-j} = b_s.$$

Using the sequences $\alpha', \beta', \alpha''$ and β'', we define recursively, the following numbers:

$$\varphi(\alpha,\beta) := \begin{cases} n, & s = 0 \\ \varphi(\alpha'',\beta'') - 1, & s \geq 1 \end{cases}, \quad \psi(\alpha,\beta) := \begin{cases} n, & s = 0 \\ \psi(\alpha',\beta') - 1, & s \geq 1 \end{cases}.$$

We recall several results from [7].

Theorem 1.4. *(*[7, Theorem 1.6]*) For any sequences of positive integers α and β as above:*
(1) $\operatorname{depth}(S/I_{\alpha,\beta}) = \operatorname{sdepth}(S/I_{\alpha,\beta}) = \varphi(\alpha,\beta)$.
(2) $\dim(S/I_{\alpha,\beta}) = \psi(\alpha,\beta)$.

Proposition 1.5. *(*[7, Proposition 1.9]*) Let α and β as above. If $a_{k+2} > b_k + 1$ for all $1 \leq k \leq s-2$, then:*
(1) $\operatorname{sdepth}(S/I_{\alpha,\beta}) = \operatorname{depth}(S/I_{\alpha,\beta}) = n - s$.
(2) $\operatorname{sdepth}(I_{\alpha,\beta}) = n - \left\lfloor \frac{s}{2} \right\rfloor$*, for all* $s \geq 1$.
(3) *If* $b_i \geq a_{i+1}$*, for all* $1 \leq i \leq s-1$*, then* $\dim(S/I_{\alpha,\beta}) = n - \left\lceil \frac{s}{2} \right\rceil$.

Theorem 1.6. *(*[7, Theorem 1.10]*) Let α and β as above. If $a_{k+2} > b_k + 1$ for all $1 \leq k \leq s-2$, then* $\operatorname{sdepth}(S/I^t_{\alpha,\beta}) = \operatorname{depth}(S/I^t_{\alpha,\beta}) = n - s$*, for all* $t \geq 1$.

Proposition 1.7. *(*[7, Proposition 1.11]*) Let α and β as above. If $a_{k+2} = b_k + 1$ for all $1 \leq k \leq s-2$, then* $\operatorname{sdepth}(S/I_{\alpha,\beta}) = \operatorname{depth}(S/I_{\alpha,\beta}) = n - s + \left\lfloor \frac{s}{3} \right\rfloor$.

Corollary 1.8. *(*[7, Corollary 1.13]*) For any $s, t \geq 1$, we have:*
(1) $n - s + \left\lfloor \frac{s}{3} \right\rfloor \geq \operatorname{depth}(S/I^t_{\alpha,\beta}) \geq n - s + \max\{\left\lfloor \frac{s-t+1}{3} \right\rfloor, 0\}$.
(2) $n - s + \left\lfloor \frac{s}{3} \right\rfloor \geq \operatorname{sdepth}(S/I^t_{\alpha,\beta}) \geq n - s + \max\{\left\lfloor \frac{s-t+1}{3} \right\rfloor, 0\}$.

2. Main results

A generalization of the path ideal of the path graph

Let $n \geq m \geq 2$ be two integers. The *path graph* of length n, denoted by P_n, is a graph with the vertex set $V = [n] = \{1,\ldots,n\}$ and the edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$. We denote

$$I_{n,m} = (x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1}, \ldots, x_{n-m+1}x_{n-m+2}\cdots x_n).$$

Note that $I_{n,m}$ is the m-path ideal of the graph P_n, provided with the direction given by $1 < 2 < \ldots < n$, see [11] for further details. In [6, Theorem 1.3] we proved that

$$\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = n + 1 - \left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil. \quad (1)$$

Let $n \geq m > k \geq 1$ be three integers such that $k|n-m$. We consider the ideal $I_{n,m,k} = (x_1\cdots x_m, x_{k+1}\cdots x_{k+m}, \ldots, x_{n-m+1}\cdots x_n)$. We denote $s = (n-m)/k + 1$. Note that $I_{n,m,k} = I_{\alpha,\beta}$, where $\alpha : a_1 < a_2 < \cdots < a_s$, $\beta : b_1 < b_2 < \cdots < b_S$ with $a_j = k(j-1)+1$ and $b_j = k(j-1)+m$, for all $1 \leq j \leq s$. We denote $\varphi(n,m,k) := \varphi(\alpha,\beta)$. Note that, $I_{n,m,1} = I_{n,m}$ is the m-path ideal of the path graph of length n.

As a direct consequence of Proposition 1.5 and Theorem 1.6, we obtain the following result.

Proposition 2.1. *If $m < 2k$, then, the following hold:*
(1) $\operatorname{sdepth}(S/I_{n,m,k}^t) = \operatorname{depth}(S/I_{n,m,k}^t) = n - \frac{n-m}{k}$, *for all* $t \geq 1$.
(2) $\operatorname{sdepth}(I_{n,m,k}) = n - \left\lfloor \frac{n-m}{2k} \right\rfloor$.
(3) $\dim(S/I_{n,m,k}) = n - \left\lceil \frac{n-m}{2k} \right\rceil$.

Note that a similar result to Proposition 2.1(1) was proved in [10, Theorem 1.1(1)]. As a direct consequence of Proposition 1.7 and Corollary 1.8, we get:

Proposition 2.2. *(compare with* [10, Theorem 1.2(1,2)]*) If $m = 2k$, then, the following hold:*
(1) $\operatorname{sdepth}(S/I_{n,m,k}) = \operatorname{depth}(S/I_{n,m,k}) = n - s + \left\lfloor \frac{s}{3} \right\rfloor$.
(2) $\operatorname{sdepth}(S/I_{n,m,k}^t) \geq n - \frac{n-m}{k} + \max\{\left\lfloor \frac{n-m+kt+k}{3k} \right\rfloor, 0\}$, *for all* $t \geq 2$.
(3) $\operatorname{depth}(S/I_{n,m,k}^t) \geq n - \frac{n-m}{k} + \max\{\left\lfloor \frac{n-m+kt+k}{3k} \right\rfloor, 0\}$, *for all* $t \geq 2$.

We conclude this paragraph with the following result, which generalize formula (1).

Theorem 2.3. *Assume $m > 2k$ and let $t = \left\lfloor \frac{m-1}{k} \right\rfloor + 1$. Then, the following hold:*
(1) $\operatorname{sdepth}(S/I_{n,m,k}) = \operatorname{depth}(S/I_{n,m,k}) = n - \left\lceil \frac{(n-m)/k+1}{t+1} \right\rceil - \left\lceil \frac{(n-m)/k}{t+1} \right\rceil$.
(2) $\dim(S/I_{n,m,k}) = n - \left\lceil \frac{(n-m)/k+1}{t} \right\rceil$.

Proof. If $s \leq 1$, then there is nothing to prove. Assume $s \geq 2$. Note that $j = j(\alpha, \beta) = \min\{s, t\}$. If $j = s$, i.e. $s \leq \left\lfloor \frac{m-1}{k} \right\rfloor + 1$, then $\operatorname{sdepth}(S/I_{n,m,k}) = \operatorname{depth}(S/I_{n,m,k}) = n - 2$ and $\dim(S/I_{n,m,k}) = n - 1$, and thus we're done. Assume $j < s$. Since $I' \cong I_{n-kj,m,k}S$, by induction hypothesis and Theorem 1.4(2), it follows that

$$\dim(S/I) = \dim(S/I') - 1 = n - \left\lceil \frac{(n-m-kt)/k+1}{t} \right\rceil - 1 = n - \left\lceil \frac{(n-m)/k+1}{t} \right\rceil.$$

For any $1 \leq i \leq s$ we denote $I_i = (u_i, \ldots, u_s)$, where $u_i = x_{a_i} \cdots x_{b_i}$, for all i. We also denote $I_{s+1} = 0$. Note that $I_i \cong I_{n-k(i-1)}S$, for all $i \leq s$. If $k|m$, then $I'' = (x_{m+1} \cdots x_{m+k}, I_{j+2})$ and $x_{m+1} \cdots x_{m+k}$ is regular on S/I_{j+2}. By induction hypothesis and Theorem 1.4(1) , it follows

$$\operatorname{sdepth}(S/I) = \operatorname{depth}(S/I) = \operatorname{depth}(S/I'') - 1 = \operatorname{depth}(S/I_{j+2}) - 2 =$$
$$= n - \left\lceil \frac{(n-m-k(t+1))/k+1}{t+1} \right\rceil - \left\lceil \frac{(n-m-k(t+1))/k}{t+1} \right\rceil - 2 =$$
$$= n - \left\lceil \frac{(n-m)/k+1}{t+1} \right\rceil - \left\lceil \frac{(n-m)/k}{t+1} \right\rceil.$$

If $k \nmid m$, then $I'' = (x_{m+1} \cdots x_{m+k}, I_{j+1})$. By Theorem 1.4(1), one can easily check that

$$\operatorname{depth}(S/I'') = \operatorname{depth}(S/(I'' : x_{m+2} \cdots x_{m+k})) = \operatorname{depth}(S/I_{j+2}) - 1,$$

and the same for sdepth. As above, we get the required conclusion. □

An extension of the ideals $I_{\alpha,\beta}$

In the following we introduce a new class of ideals. Let $s \geq 2$ be an integer. We consider two integer sequences $\alpha : 1 = a_1 < a_2 < \cdots < a_s$ and $\beta : 1 < b_1 < b_2 < \cdots < b_s$, with $a_i < b_i$, for all i and $b_s \leq n$. We also, consider

$$\bar{\alpha} : a_1 < \cdots < a_s < a_{s+1} < \cdots < a_{s+t},$$
$$\bar{\beta} : b_1 < \cdots < b_s < b_{s+1} < \cdots < b_{s+t},$$

with $b_{s+1} > n$ and $b_{s+t} < n + b_1$.

Let $u_i = (x_{a_i} \cdots x_{b_i})$ for all $1 \leq i \leq s$ and $u_i = x_{a_i} \cdots x_n x_1 \cdots x_{b_i - n}$ for all $s+1 < i < s+t$. Note that $t < b_1$. We denote $I := I_{\alpha,\beta} = (u_1, \cdots, u_s)$. We consider the ideal

$$J := J_{\bar{\alpha},\bar{\beta}} = I_{\alpha,\beta} + (u_{s+1}, \ldots, u_{s+t}).$$

Note that, if there exists some $1 \leq i \leq s-1$ such that $a_{i+1} > b_i$, then, modulo reordering the variables, J can be seen as an ideal of the type described in the first section. Therefore, in the following, we will assume that $a_{i+1} \leq b_i$ for all $1 \leq i \leq s$. In particular, $j(\alpha, \beta) > 1$.

Our computer experiments [8] yields us to the following conjecture.

Conjecture 2.4. *If* $\operatorname{depth}(S/J) = \varphi(\alpha,\beta)$ *then* $\operatorname{sdepth}(S/J) = \varphi(\alpha,\beta)$.

Theorem 2.5. *With the above notations, we have*

$$\varphi(\alpha,\beta) \geq \operatorname{sdepth}(S/J), \operatorname{depth}(S/J) \geq \min_{i=1}^{b_1} \{\operatorname{depth}(S/((J : x_{1+i} \cdots x_{b_1}), x_i))\}.$$

Moreover, Conjecture 2.4 *implies* $\operatorname{sdepth}(S/J) \geq \operatorname{depth}(S/J)$.

Proof. We consider the short exact sequence:

$$0 \longrightarrow S/(J : x_2 \cdots x_{b_1}) \longrightarrow S/J \longrightarrow S/(J, x_2 \cdots x_{b_1}) \longrightarrow 0.$$

Note that $(J : x_2 \cdots x_{b_1}) = (x_1, I'')$, where $I'' = I_{\alpha'',\beta''}$. By Theorem 1.4(1) and Lemma 1.3(1), it follows that $\operatorname{sdepth}(S/J) \leq \operatorname{sdepth}(S/(J : x_2 \cdots x_{b_1})) = \varphi(\alpha,\beta)$. Similarly, we get $\operatorname{sdepth}(S/J) \leq \varphi(\alpha,\beta)$.

If $\operatorname{depth}(S/J) = \operatorname{depth}(S/(J : x_2 \cdots x_{b_1}))$, then by Lemma 1.1(2),

$$\operatorname{sdepth}(S/J) \geq \min\{\varphi(\alpha,\beta), \operatorname{sdepth}(S/(J, x_2 \cdots x_{b_1}))\}.$$

If Conjecture 2.4 holds, then $\operatorname{sdepth}(S/J) = \varphi(\alpha,\beta) = \operatorname{depth}(S/J)$.

If $\operatorname{depth}(S/J) < \varphi(\alpha,\beta) = \operatorname{sdepth}(S/(J : x_2 \cdots x_{b_1}))$, then, by Lemma 1.1, it follows that

$$\operatorname{depth}(S/(J, x_2 \cdots x_{b_1})) = \operatorname{depth}(S/J).$$

Also, by Lemma 1.1(2), it follows that $\operatorname{sdepth}(S/J) \geq \operatorname{depth}(S/J)$. If $t = 1$ and $b_{s+t} = n+1$, then $(J, x_2 \cdots x_{b_1})$ is an ideal of the same type as I, and we can compute its depth and sdepth. Otherwise, $(J, x_2 \cdots x_{b_1}))$ is an ideal of the same type as J, and we can apply the same procedure, i.e. we consider the short exact sequence

$$0 \longrightarrow S/((J : x_3 \cdots x_{b_1}), x_2) \longrightarrow S/(J, x_2 \cdots x_{b_1}) \longrightarrow S/(J, x_3 \cdots x_{b_1}) \longrightarrow 0.$$

Therefore, either

$$\operatorname{sdepth}(S/(J, x_2 \cdots x_{b_1})) = \operatorname{depth}(S/(J, x_2 \cdots x_{b_1})) = \operatorname{depth}(S/((J : x_3 \cdots x_{b_1}), x_2)),$$

either

$$\operatorname{depth}(S/(J, x_2 \cdots x_{b_1})) = \operatorname{depth}(S/(J, x_3 \cdots x_{b_1}))$$

and

$$\operatorname{sdepth}(S/(J, x_2 \cdots x_{b_1})) \geq \operatorname{depth}(S/(J, x_2 \cdots x_{b_1})).$$

This procedure eventually stops, and thus we complete the proof. □

Remark 2.6. Note that, if $\operatorname{depth}(S/J) = \varphi(\alpha,\beta)$, then, from the proof of Theorem 2.5, it follows that $\operatorname{sdepth}(S/J) \geq \varphi(\alpha,\beta) - 1$. Although the Stanley conjecture was disproved for quotient rings, it is still open the conjecture that, if $I \subset S$ is a monomial ideal, then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I) - 1$ and $\operatorname{sdepth}(I) \geq \operatorname{depth}(I)$.

Proposition 2.7. *Let $\bar{\alpha}: a_1 < \cdots < a_s < a_{s+1}$ and $\bar{\beta}: b_1 < \cdots < b_s < b_{s+1}$, as above, such that $b_i + 1 < a_{i+2}$ for all $1 \leq i \leq s-1$, $b_s < n$ and $b_{s+1} - n + 1 < a_2$. Let $J = J_{\bar{\alpha},\bar{\beta}}$. Then:*

$$\operatorname{sdepth}(S/J) = \operatorname{depth}(S/J) = n - s - 1,$$

$$n - \left\lfloor \frac{s}{2} \right\rfloor \geq \operatorname{sdepth}(J) \geq n - \left\lfloor \frac{s+1}{2} \right\rfloor.$$

Proof. Let $t = b_{s+1} - n$ and $v = x_1 \cdots x_t$. We consider the short exact sequence:

$$0 \longrightarrow S/(J:v) \longrightarrow S/J \longrightarrow S/(J,v) \longrightarrow 0.$$

We have $(J : v) = (x_{t+1} \cdots x_{b_1}, u_2, \ldots, u_{s+1})$, where $u_i = x_{a_i} \cdots x_{b_i}$, for all i. Note that $(J : v)$ satisfies the hypothesis of Proposition 1.5 and therefore

$$\operatorname{depth}(S/(J:v)) = \operatorname{sdepth}(S/(J:v)) = n - s - 1.$$

On the other hand, $(J, v) = (v, u_2, \ldots, u_s)$. Therefore, by Proposition 1.5, it follows that

$$\operatorname{sdepth}(S/(J,v)) = \operatorname{depth}(S/(J,v)) = n - s.$$

By Lemma 1.1, Lemma 1.2 and Lemma 1.3(1) it follows that

$$\operatorname{sdepth}(S/J) = \operatorname{depth}(S/J) = n - s - 1.$$

According to [14, Theorem 2.1], we have $\operatorname{sdepth}(J) \geq n - \left\lfloor \frac{s+1}{2} \right\rfloor$. On the other hand, $(J : (x_{a_1} \cdots x_{b_{s+1}-n})(x_{a_2} \cdots x_{b_1}) \cdots (x_{a_s} \cdots x_{b_s}))$ is a complete intersection monomial ideal, generated by s monomials. Therefore, by [17, Theorem 2.4] and Lemma 1.3(1), it follows that $\operatorname{sdepth}(J) \leq n - \left\lfloor \frac{s}{2} \right\rfloor$. □

Proposition 2.8. *Let $\bar{\alpha}: a_1 < \cdots < a_s < a_{s+1}$ and $\bar{\beta}: b_1 < \cdots < b_s < b_{s+1}$, as above, such that $b_i + 1 = a_{i+2}$ for all $1 \leq i \leq s-1$, $b_s = n$ and $b_{s+1} - n + 1 = a_2$. Let $J = J_{\bar{\alpha},\bar{\beta}}$. Then:*

$$n - s + \left\lfloor \frac{s}{3} \right\rfloor \geq \operatorname{sdepth}(S/J) \geq \operatorname{depth}(S/J) = n - s + \left\lfloor \frac{s-1}{3} \right\rfloor,$$

$$\operatorname{sdepth}(J) \geq n - \left\lfloor \frac{s+1}{2} \right\rfloor \geq \operatorname{depth}(J).$$

Proof. If $s = 3$, then it is easy to see that $\operatorname{sdepth}(S/J) = \operatorname{depth}(S/J) = n - 2$. Assume $s \geq 4$. Let $t = b_{s+1} - n$ and $v = x_1 \cdots x_t$. Let $u_i = x_{a_i} \cdots x_{b_i}$ for all $i \leq s$ and $u_{s+1} = x_{a_{s+1}} \cdots x_n x_1 \cdots x_{b_{s+1}-n}$. We consider the short exact sequence:

$$0 \longrightarrow S/(J:v) \longrightarrow S/J \longrightarrow S/(J,v) \longrightarrow 0.$$

Note that $(J : v) = (L, x_{t+1} \cdots x_{b_1}, x_{a_{s+1}} \cdots x_n)$, where $L = (u_3, \ldots, u_{s-1})$ and $x_{t+1} \cdots x_{b_1}$, $x_{a_{s+1}} \cdots x_n$ is a regular sequence on S/L. On the other hand, L satisfies the conditions of Proposition 1.7. Therefore, by Lemma 1.3(2) and Proposition 1.7, it follows that

$$\operatorname{sdepth}(S/(J:v)) = \operatorname{depth}(S/(J:v)) = n - (s-3) + \left\lfloor \frac{s-3}{3} \right\rfloor - 2 = n - s + \left\lfloor \frac{s}{3} \right\rfloor.$$

On the other hand, $(J, v) = (v, U)$, where $U = (u_2, \ldots, u_s)$. Since v is regular on S/U, it follows from Lemma 1.3(2) and Proposition 1.7, that

$$\operatorname{sdepth}(S/(J,v)) = \operatorname{depth}(S/(J,v)) = n - s + \left\lfloor \frac{s-1}{3} \right\rfloor.$$

Thus, by Lemma 1.2, we get

$$\operatorname{depth}(S/J) \geq n - s + \left\lfloor \frac{s-1}{3} \right\rfloor, \ \operatorname{sdepth}(S/J) \geq n - s + \left\lfloor \frac{s-1}{3} \right\rfloor.$$

On the other hand, by Lemma 1.3(1),

$$\operatorname{sdepth}(S/J) \leq n-s+\left\lfloor \frac{s}{3} \right\rfloor,\ \operatorname{depth}(S/J) \leq n-s+\left\lfloor \frac{s}{3} \right\rfloor.$$

In order to complete the proof, it is enough to show that

$$\operatorname{depth}(S/J) = n-s+\left\lfloor \frac{s-1}{3} \right\rfloor.$$

If $s \equiv 1(mod\ 3)$ or $s \equiv 2(mod\ 3)$, then there is nothing to prove. Assume $3|s$. We use induction on $\sum_{i=1}^{s+1} b_i - a_i$. If $b_i = a_i+1$ for all $1 \leq i \leq s+1$, then J is the edge ideal of a cycle graph. Thus, according to [3, Proposition 5.0.6], it follows that

$$\operatorname{depth}(S/J) = n-s+\left\lfloor \frac{s-1}{3} \right\rfloor.$$

If J is not the edge ideal of a cycle graph, by reordering the variables, we can assume that $b_1 > a_1+1$ and $b_2 > a_2+1$. Let $J' = (J : x_{b_1})$.

Note that J' satisfies the hypothesis and therefore, by induction, we get $\operatorname{depth}(S/J') = n-s+\left\lfloor \frac{s-1}{3} \right\rfloor$. On the other hand, $(J, x_{b_1}) \cong (L, x_{b_1})$, where L is generated by $s-1$ generators and satisfies the hypothesis of Proposition 1.7. Thus, by Lemma 1.3(2), it follows that

$$\operatorname{depth}(S/(J,x_{b_1})) = n-(s-2)-1+\left\lfloor \frac{s-2}{3} \right\rfloor = n-s+\left\lfloor \frac{s}{3} \right\rfloor = \operatorname{depth}(S/(J:x_{b_1}))+1.$$

Thus, by Lemma 1.1,

$$\operatorname{depth}(S/J) = \operatorname{depth}(S/(J:x_{b_1})) = n-s+\left\lfloor \frac{s-1}{3} \right\rfloor,$$

as required. The last assertion is a consequence of [14, Theorem 2.1]. □

A generalization of the path ideal of the cycle graph

Let $n > m \geq 2$ be two integers. The *cycle graph* of lenght n, denoted by C_n, is a graph with the vertex set $V = [n]$ and the edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$. We denote

$$J_{n,m} = (x_1x_2\cdots x_m, x_2x_3\cdots x_{m+1}, \ldots, x_{n-m+1}x_{n-m+2}\cdots x_n, \ldots, x_nx_1\cdots x_{m-1}),$$

the m-path ideal of the graph C_n.

Let $p = \left\lfloor \frac{n}{m+1} \right\rfloor$ and $d = n-(m+1)p$. According to [1, Corollary 5.5],

$$pd(S/J_{n,m}) = \begin{cases} 2p+1,\ d \neq 0, \\ 2p,\ d = 0. \end{cases}$$

By Auslander-Buchsbaum formula, see [19, Theorem 2.5.13], it follows that

$$\operatorname{depth}(S/J_{n,m}) = n - pd(S/J_{n,m}) = n - \left\lfloor \frac{n}{m+1} \right\rfloor - \left\lceil \frac{n}{m+1} \right\rceil = \varphi(n-1,m).$$

In [5, Theorem 1.4] we proved that

$$\varphi(n,m) \geq \operatorname{sdepth}(S/J_{n,m}) \geq \operatorname{depth}(S/J_{n,m}) = \varphi(n-1,m).$$

Note that the first inequality follows from Theorem 2.5. Also, if Conjecture 2.4 is true, then, in particular, $\operatorname{sdepth}(S/J_{n,m}) \geq \operatorname{depth}(S/J_{n,m})$.

Let $n \geq m > k \geq 1$ be three integers such that $k|n$. We consider the ideal

$$J_{n,m,k} = (x_1\cdots x_m, x_{k+1}\cdots x_{k+m}, \ldots, x_{n-k+1}\cdots x_{n-k+m}) \subset S,$$

where we denoted, by abuse, $x_{n+i} := x_i$, for $1 \leq i \leq m-k$.

Note that $J_{n,m,k} = J_{\bar{\alpha},\bar{\beta}}$, where

$$\bar{\alpha} : 1 < k+1 < \cdots < n-k+1,\ \bar{\beta} : m < k+m < \cdots < n-k+m.$$

Also, $J_{n,m,1} = J_{n,m}$ is the m-path ideal of the cycle graph of length n.

As a consequence of Proposition 2.7 and Proposition 2.8, we get the following result, which is similar to [10, Theorem 1.1(2,3)].

Proposition 2.9. *With the above notations, we have:*

(1) *If* $2k < m$, *then*

$$\operatorname{sdepth}(S/J_{n,m,k}) = \operatorname{depth}(S/J_{n,m,k}) = n - s - 1,$$
$$n - \left\lfloor \frac{s}{2} \right\rfloor \geq \operatorname{sdepth}(J_{n,m,k}) \geq n - \left\lfloor \frac{s+1}{2} \right\rfloor.$$

(2) *If* $2k = m$, *then*

$$n - s + \left\lfloor \frac{s}{3} \right\rfloor \geq \operatorname{sdepth}(S/J_{n,m,k}) \geq \operatorname{depth}(S/J_{n,m,k}) = n - s + \left\lfloor \frac{s-1}{3} \right\rfloor,$$
$$\operatorname{sdepth}(J_{n,m,k}) \geq n - \left\lfloor \frac{s+1}{2} \right\rfloor \geq \operatorname{depth}(J_{n,m,k}).$$

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