U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 4, 2018 ISSN 1223-7027

ANALYTICAL STUDY OF SOLITONS TO BENJAMIN-BONA-MAHONY-PEREGRINE EQUATION WITH POWER LAW NONLINEARITY BY USING THREE METHODS

M.S. OSMAN[1], Hadi REZAZADEH[2], Mostafa ESLAMI[3], Ahmad NEIRAMEH[4], Mohammad MIRZAZADEH[5]

In this paper, the unified, improved Riccati sub-equation and modified Kudryashov methods are used to construct travelling wave solutions of the Benjamin-Bona-Mahony-Peregrine equation with power law nonlinearity (BBMP equation with power law nonlinearity). Several solutions are determined including soliton wave solutions, solitary wave solutions, elliptic wave solutions and periodic wave solutions. Itis shown that these methodologies are very powerful mathematical tools for obtaining exact travelling wave solutions of nonlinear evolution equations.

Keywords: Power law nonlinearity, The unified method, The improved sub-equation method, The modified Kudryashov method, Travelling wave solutions.

AMS Subject Classification: 83C15, 35C07.

1. Introduction

The investigation of travelling wave solutions of nonlinear evolution equations (NLEEs) are growing rapidly because mathematical modeling of many physical systems leads to NLEEs. They are encountered in a variety of engineering and scientific aplications such as the fluid dynamics, plasma physics, optical fibers, chemical physics, biomathematics, oceans engineering, geochemistry and many other scientific fields. There are several analytic techniques for solving NLEEs and for constructing travelling wave solutions. In between these methods are the extended G'/G – expansion method [1], the extended trial equation method [2,3],

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt, and Department of Mathematics, University College of Umluj, Tabuk, Saudi Arabia, e-mail: Mofatzi@sci.cu.edu.eg
[2] Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran, (Corresponding author): e-mail: H.rezazadeh@ausmt.ac.ir
[3] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, e-mail: Mostafa.eslami@umz.ac.ir
[4] Department of Mathematics, Faculty of Sciences, Gonbad Kavous University, Gonbad, Iran, e-mail: A.neirameh@gmail.com
[5] Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar-Vajargah, Iran, e-mail: Mirzazadehs2@guilan.ac.ir

the Riccati-Bernoulli's sub-ODE method [4], the first integral method [5,6], the truncated expansion method [7], the sub-equation method [8,9], the modified simple equation method [10], the extended Jacobi's elliptic function method [11], the functional variable method [12, 13], the sine-Gordon expansion method [14], the new extended direct algebraic method [15], and many others [16,17].

In this article, we use the unified, improved Riccati sub-equation and modified Kudryashov approaches to find traveling wave solutions of the BBMP equation with power law nonlinearity of the form [18]:

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+bu^{n}\frac{\partial u}{\partial x}+c\frac{\partial^{3}u}{\partial^{2}x\partial t}=0, \tag{1}$$

where a,b,c and n are nonzero constants. The exponent n represents the power law nonlinearity parameter and it is necessary to have $n\neq 0$, as these values will place Eq. (1) beyond the linear regime. Here in Eq. (1) the first term indicates the evolution term, while the last term indicates the dispersion term. The third term is the nonlinear term. Khalique in [18] obtained exact wave solutions of Eq.(1) using Lie symmetry method and simplest equation approach and Aminikhah et al. [19] proposed the functional variable method to solve this equation. The special case where $n=2$, the BBMP equation with power law nonlinearity is called the modified Benjamin-Bona-Mahony equation [20].

The rest of the paper is organized as follows. Details of the unified method, improved Riccati sub-equation and modified Kudryashov methods have been presented in the next Section. The obtained solutions of BBMP equations with power law nonlinearity using these methods are presented in Section 3. Conclusionsare presented in Section 4.

2. Methods

In this Section we express the first step of the unified, improved Riccati sub-equation and modified Kudryashov methods for finding traveling wave solutions of NLEEs.

Consider the nonlinear evolution equation (NLEE)

$$F\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x_{1}},\frac{\partial u}{\partial x_{2}},\dots\right)=0, \tag{2}$$

where F is a polynomial in u and its partial derivatives.

Using the wave transformation

$$u(x,t)=U(\xi),\quad \xi=x_{1}+x_{2}+\dots+x_{q}-\lambda t, \tag{3}$$

where λ is a constant, Eq. (2) is arranged as the following nonlinear ordinary differential equation

$$G(U,U',U'',...)=0. \tag{4}$$

2.1. The description of unified method

Herein, we find the traveling wave solutions (in polynomial function or rational function forms) for the NLEE given by Eq. (4) via the unified method [21,22,23,24,25]. The outline of this method is presented as follow.

(I) Polynomial solutions

To obtain the solutions of Eq. (4) in polynomial function forms, we assume that

$$\begin{aligned} &U = U(\xi) = \sum_{i=0}^{n} a_i \Gamma^i(\xi), \\ &(\Gamma'(\xi))^p = \sum_{j=0}^{pm} b_j \Gamma^i(\xi), \quad \xi = -\lambda t + \sum_{k=1}^{q} x_k, \quad p = 1,2, \end{aligned} \tag{5}$$

where $\Gamma'(\xi) = \frac{d}{d\xi}(\Gamma(\xi)), t, x_k, 1 \le k \le q$ are the arguments of u in Eq. (2), a_i and b_j are constants. The unified method provides the balance principle technique to evaluate the relation between the two parameters n and m and satisfies the consistency condition between the arbitrary functions in the solutions given by Eq. (5) (for details see [24, 25, 26, 27, 28]).

It worth mentioning that, the unified method solves Eq. (4) to elementary solutions or elliptic solutions when $p = 1$ or $p = 2$ respectively.

(II) Rational solutions

To get these solutions, we suppose that

$$\begin{aligned} &U = U(\xi) = \sum_{i=0}^{n} a_i \Gamma^i(\xi) / \sum_{i=0}^{k} r_i \Gamma^i(\xi), \quad n \ge k, \\ &(\Gamma'(\xi))^p = \sum_{j=0}^{pm} b_j \Gamma^i(\xi), \quad \xi = -\lambda t + \sum_{s=1}^{q} x_s, \quad p = 1,2, \end{aligned} \tag{6}$$

where $\Gamma'(\xi) = \frac{d}{d\xi}(\Gamma(\xi)), t, x_s, 1 \le s \le q$ are the arguments of u in Eq. (2), a_i, r_i, and b_j are constants. Similarly, The unified method provides the balance principle technique to evaluate the relation between the parameters n,k and m and satisfies the consistency condition between the arbitrary functions in the solutions given by Eq. (6) (for details see [24,25,26,27,28]). Furthermore, the values of p give different types for these solutions by the same criteria described in (I).

2.2. The description of improved Riccati sub-equation method

In this subsection we express the improved Riccati sub-equation which is proposed in [29,30] for finding exact solutions of NLEEs, that the essential steps of this method are described below.

Step 1. We suppose that Eq. (4) has the following solution

$$U(\xi)=\sum_{i=0}^{n}a_i(\omega+\psi(\xi))^i, \tag{7}$$

where ω and a_i are arbitrary, and $\psi(\xi)$ is a solution of the fractional Riccati equation that satisfies

$$\psi'(\xi)=\sigma+\psi^2(\xi), \tag{8}$$

Step 2. The positive integer n in Eq. (7) is determined with the balancing of the highest order derivatives and nonlinear terms in Eq. (4).

Step 3. Substituting (7) along with Eq. (8) into Eq. (4) and collecting the coefficients of $\psi(\xi)$ and setting the coefficients of $[\psi(\xi)]^i\ (i=0,1,2,\ldots,n)$ to be zero, we get a set of overdetermined nonlinear algebraic equations for $a_i\ (i=0,1,2,\ldots,n)$, ω and σ.

Step 4. Finally, assuming that ω,σ and $a_i\ (i=0,1,2,\ldots,n)$ can be obtained by solving the algebraic equations in step 3 and substituting these constants and the solutions of Eq. (8) into Eq. (7), we can finally obtain exact solutions of Eq. (4).

Step 5. The Riccati equation (8) admits the following exact solutions [31,32]

Type I. When $\sigma<0$,

$$\psi_1(\xi)=-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi),$$
$$\psi_2(\xi)=-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi),$$
$$\psi_3(\xi)=-\sqrt{-\sigma}\tanh(2\sqrt{-\sigma}\xi)\pm i\sqrt{-\sigma}\,\mathrm{sech}(2\sqrt{-\sigma}\xi),$$
$$\psi_4(\xi)=-\sqrt{-\sigma}\coth(2\sqrt{-\sigma}\xi)\pm \sqrt{-\sigma}\,\mathrm{csch}(2\sqrt{-\sigma}\xi),$$
$$\psi_5(\xi)=-\frac{1}{2}\left(\sqrt{-\sigma}\tanh(\frac{\sqrt{-\sigma}}{2}\xi)+\sqrt{-\sigma}\coth(\frac{\sqrt{-\sigma}}{2}\xi)\right),$$

Type II. When $\sigma>0$,

$$\psi_6(\xi)=\sqrt{\sigma}\tan(\sqrt{\sigma}\xi),$$
$$\psi_7(\xi)=-\sqrt{\sigma}\cot(\sqrt{\sigma}\xi),$$
$$\psi_8(\xi)=-\sqrt{\sigma}\tan(2\sqrt{\sigma}\xi)\pm\sqrt{\sigma}\sec(2\sqrt{\sigma}\xi),$$
$$\psi_9(\xi)=-\sqrt{\sigma}\cot(2\sqrt{\sigma}\xi)\pm\sqrt{\sigma}\csc(2\sqrt{\sigma}\xi),$$
$$\psi_{10}(\xi)=\frac{1}{2}\left(\sqrt{\sigma}\tan(\frac{\sqrt{\sigma}}{2}\xi)-\sqrt{\sigma}\cot(\frac{\sqrt{\sigma}}{2}\xi)\right),$$

Type III. When $\sigma=0$,

$$\psi_{11}(\xi)=-\frac{1}{\xi+d},\quad d=const.$$

2.3. The description of the modified Kudryashov method

A summary of the modified Kudryashov method [33,34] is given to extract new closed-form solutions to a nonlinear system of partial differential equations. The essential steps of modified Kudryashov method are described below.

Step 1. Lets assume that the solution $U(\xi)$ of the nonlinear Eq. (4) can be considered as

$$U(\xi)=a_0+\sum_{i=1}^{N}a_iQ^i(\xi), \tag{9}$$

where unknowns $a_i\,(i=0,1,\ldots N)$ are identified later,so that $a_N\neq 0$, N is a positive integer and $Q(\xi)$ satisfies the following new auxiliary equation

$$Q'(\xi)=\left(Q^2(\xi)-Q(\xi)\right)\ln(A),\quad A\neq 0,1, \tag{10}$$

where the solution of the Eq. (10) is

$$Q(\xi)=\frac{1}{1+dA^{\xi}} \tag{11}$$

Step 2. By inserting Eq. (9) along with Eq. (10) in Eq. (4) and equating the coefficient of each power of $Q(\xi)$ to zero, we get a system of algebraic equations in different parameters. The received system is then solved for finding some free parameters values. Finally, new closed-form solutions for the nonlinear Eq. (2) are produced.

3. The BBMP equation with power law nonlinearity

In this Section, we obtain several solutions of BBMP equations with power law nonlinearity using methods that we have presented in before Section.

Let

$$U(\xi)=u(x,t),\quad \xi=x-\lambda t, \tag{12}$$

from relation (12) and its derivatives we have

$$-\lambda U_{\xi}+aU_{\xi}+bU^nU_{\xi}-\lambda cU_{\xi\xi\xi}=0. \tag{13}$$

or

$$(a-\lambda)U+\frac{b}{n+1}U^{n+1}-\lambda cU_{\xi\xi}=0,\ a\neq\lambda. \tag{14}$$

By using the transformation

$$U=V^{\frac{1}{n}}, \tag{15}$$

Eq. (14) can be written as

$$-cn(n+1)\lambda VV_{\xi\xi}+c(n^2-1)\lambda V_{\xi}^2+n^2(1+n)(a-\lambda)V^2+bn^2V^3=0,\ n\neq 0, n\neq\pm 1. \tag{16}$$

3.1. The unified method

By considering the homogeneous balance between $VV_{\xi\xi}$ and V^3 in Eq. (16), we get $n=2(k-1), k=1,2,3,\ldots.$ Here, we confine ourselves to find these solutions when $k=2$ and $p=1$ or $p=2.$

3.1.1. The solitary wave solution

The unified method admits the solitary wave solution of Eq. (1) as

$$u_1(x,t)=\left(-\frac{ac(2+3n+n^2)R^2\operatorname{sech}^2(\frac{1}{2}R\,\xi)}{2b(n^2+cR^2)}\right)^{\frac{1}{n}}, \tag{17}$$

where $\xi=x-\frac{an^2}{n^2+cR^2)}t$ and $R=\sqrt{b_1^2-4b_0b_2}$.

3.1.2. The soliton wave solution

From Eq. (5) in Eq. (16), we get the soliton wave solution of Eq. (1) as

$$u_2(x,t)=\left(\frac{4ab_1^2b_2c(2+3n+n^2)e^{\frac{b_1\xi}{2\sqrt{b_2}}}}{b\left(1-2b_2e^{\frac{b_1\xi}{2\sqrt{b_2}}}\right)(b_1^2c+4b_2n^2)}\right)^{\frac{1}{n}},\quad b_2>0, \tag{18}$$

where $\xi=x-\frac{4ab_2n^2}{b_1^2c+4b_2n^2}t$.

3.1.3. The elliptic wave solution

Herewith we write the elliptic wave solution of (1) by using the unified method (something is missing here). This solution is given by (for details see the subsection 2.1)

$$u_3(x,t)=\left(-\frac{6ac\left(b_2+H+2b_4\Gamma^2(\xi)\right)}{b\left(-2+b_2c+3cH\right)}\right)^{\frac{1}{2}}, \tag{19}$$

where $\xi=x+\frac{2a}{-2+b_2c+3cH}t$ and $H=\sqrt{b_2^2-4b_0b_4}$.

The auxiliary function $\Gamma(\xi)$ in Eq. (19) is given by

$$\Gamma'(\xi)=\sqrt{b_4\Gamma^4(\xi)+b_2\Gamma^2+b_0},$$

which is classified into different types of elliptic functions according to the classification in [35].

Finally, we find the rational function solutions of Eq. (1) by a similar technique in subsection 3.1.4 that we did by using the unified method. These rational solutions (periodic type or soliton type) are introduced in the following subsection.

3.1.4. The rational solutions

Here, the rational solutions of Eq. (1) will be obtained in two different forms as follows.

Case 1:periodic type

$$u_4(x,t)=\left(\frac{3ab_2^2c(1-\sin(b_2\xi))}{b(2+b_2^2c)(1+\sin(b_2\xi))}\right)^{\frac{1}{2}}, \tag{20}$$

where $\xi = x-\frac{3bb_2^2}{2}t$.

Case 2:soliton type

$$u_5(x,t)=\left(\frac{\sqrt{3ab_2c}\left(e^{\sqrt{b_2}\,\xi}+R\right)}{b\sqrt{2-b_2c}\left(e^{\sqrt{b_2}\,\xi}-R\right)}\right), b_2>0, \tag{21}$$

where $\xi = x-\frac{2a}{2-b_2c}t$ and $R=\sqrt{b_1^2-4b_0b_2}$.

3.2. The improved Riccati sub-equation method

We assume that Eq. (16) has a solution in the form given below

$$V(\xi)=\sum_{i=0}^{n}a_i(\omega+\psi)^i, \tag{22}$$

where ω and a_i $(i=0,1,2,\ldots,n)$ are constants and $\psi(\xi)$ satisfies the Riccati Eq. (8). Balancing between $VV_{\xi\xi}$ and V^3 in Eq. (16) we can gain $n=2$. So, we have

$$V(\xi)=a_0+a_1(\omega+\psi)+a_2(\omega+\psi)^2. \tag{23}$$

By substituting Eq. (23) and its derivatives into Eq. (16) and collecting all terms with the same power of ψ together, the left-hand side of Eq. (16) is converted into another polynomial in ψ. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for the unknowns a_0, a_1, ω and λ .

On using Maple software package, we finally get a set of algebraic equations. Then solving the set of algebraic equations yields

$$\lambda=-\frac{n^2a}{-n^2+4c\sigma}, \qquad a_0=-\frac{ca\left(\sigma+\omega^2\right)\left(n^2+2+3n\right)}{b\left(-n^2+4c\sigma\right)},$$
$$a_1=4\frac{ca\left(n^2+2+3n\right)\omega}{b\left(-n^2+4c\sigma\right)}, \qquad a_2=-2\frac{ca\left(n^2+2+3n\right)}{b\left(-n^2+4c\sigma\right)}. \tag{24}$$

Using Eqs. (24), (23), (12), (15) and the solutions of (8), we can find the following travelling wave solutions of Eq. (1) as follows

Type I. When $\sigma < 0$,

$$u_6(x,y) = \left(L\left((\sigma+\omega^2)+4\omega(\omega-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi))-2(\omega-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi))^2\right)\right)^{\frac{1}{n}},$$

$$u_7(x,y) = \left(L\left((\sigma+\omega^2)+4\omega(\omega-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi))-2(\omega-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi))^2\right)\right)^{\frac{1}{n}},$$

$$u_8(x,y) = \Big(L\Big((\sigma+\omega^2)+4\omega(\omega-(\sqrt{-\sigma}\tanh(2\sqrt{-\sigma}\xi)\mp i\sqrt{-\sigma}\operatorname{sech}(2\sqrt{-\sigma}\xi))$$
$$-2(\omega-(\sqrt{-\sigma}\tanh(2\sqrt{-\sigma}\xi)\mp i\sqrt{-\sigma}\operatorname{sech}(2\sqrt{-\sigma}\xi)))^2\Big)\Big)^{\frac{1}{n}},$$

$$u_9(x,y) = \Big(L\Big((\sigma+\omega^2)+4\omega(\omega-(\sqrt{-\sigma}\coth(2\sqrt{-\sigma}\xi)\mp \sqrt{-\sigma}\operatorname{csch}(2\sqrt{-\sigma}\xi))$$
$$-2(\omega-(\sqrt{-\sigma}\coth(2\sqrt{-\sigma}\xi)\mp \sqrt{-\sigma}\operatorname{csch}(2\sqrt{-\sigma}\xi)))^2\Big)\Big)^{\frac{1}{n}},$$

$$u_7(x,y) = \left(L\left((\sigma+\omega^2)+4\omega(\omega-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi))-2(\omega-\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi))^2\right)\right)^{\frac{1}{n}},$$

$$u_{10}(x,y) = (L((\sigma+\omega^2)+4\omega(\omega-\frac{1}{2}(\sqrt{-\sigma}\tanh(\frac{\sqrt{-\sigma}}{2}\xi)+\sqrt{-\sigma}\coth(\frac{\sqrt{-\sigma}}{2}\xi)))$$

where $\xi = x + \dfrac{n^2 a}{-n^2+4c\sigma}t$ and $L = \dfrac{ca\left(n^2+2+3n\right)}{b\left(-n^2+4c\sigma\right)}$.

Type II. When $\sigma > 0$,

$$u_{11}(x,y) = \left(L\left((\sigma+\omega^2)+4\omega(\omega+\sqrt{\sigma}\tan(\sqrt{\sigma}\xi))-2(\omega+\sqrt{\sigma}\tan(\sqrt{\sigma}\xi))^2\right)\right)^{\frac{1}{n}},$$

$$u_{12}(x,y) = \left(L\left((\sigma+\omega^2)+4\omega(\omega-\sqrt{\sigma}\cot(\sqrt{\sigma}\xi))-2(\omega-\sqrt{\sigma}\cot(\sqrt{\sigma}\xi))^2\right)\right)^{\frac{1}{n}},$$

$$u_{13}(x,y) = \Big(L\Big((\sigma+\omega^2)+4\omega(\omega-(\sqrt{\sigma}\tan(2\sqrt{\sigma}\xi)\mp \sqrt{\sigma}\sec(2\sqrt{\sigma}\xi))$$
$$-2(\omega-(\sqrt{\sigma}\tan(2\sqrt{\sigma}\xi)\mp \sqrt{\sigma}\sec(2\sqrt{\sigma}\xi)))^2\Big)\Big)^{\frac{1}{n}},$$

$$u_{14}(x,y) = \Big(L\Big((\sigma+\omega^2)+4\omega(\omega-(\sqrt{\sigma}\cot(2\sqrt{\sigma}\xi)\mp \sqrt{\sigma}\csc(2\sqrt{\sigma}\xi))$$
$$-2(\omega-(\sqrt{\sigma}\cot(2\sqrt{\sigma}\xi)\mp \sqrt{\sigma}\csc(2\sqrt{\sigma}\xi)))^2\Big)\Big)^{\frac{1}{n}},$$

$$u_{15}(x,y) = (L((\sigma+\omega^2)+4\omega(\omega+\frac{1}{2}(\sqrt{\sigma}\tan(\frac{\sqrt{\sigma}}{2}\xi)-\sqrt{\sigma}\cot(\frac{\sqrt{\sigma}}{2}\xi)))$$

Type III. When $\sigma = 0$,

$$u_{16}(x,y) = (L((\sigma+\omega^2)+4\omega(\omega-\frac{1}{\xi+d})-2(\omega-\frac{1}{\xi+d})^2))^{\frac{1}{n}}, \quad d = const.$$

where $\xi = x + \dfrac{n^2 a}{4c\sigma-n^2}t$ and $L = \dfrac{ca\left(n^2+2+3n\right)}{b\left(4c\sigma-n^2\right)}$.

3.3. The modified Kudryashov method

Assume the formal solution of the Eq. (12) through the Eq. (9)

$$V(\xi) = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi). \tag{25}$$

We substitute Eq. (25) along with its first and second derivatives into Eq. (16) and collect all terms with the same order of $Q(\xi)$, we get a system of nonlinear algebraic equation. Solving this system, we acquire the following solution sets:

$$\lambda = \frac{n^2 a}{n^2 + c\left(Ln^2 A\right)}, \qquad a_0 = 0, \qquad a_1 = \frac{-2c\, a(n+1)(n+2)\left(Ln^2 A\right)}{b\left(n^2 + c\left(Ln^2 A\right)\right)},$$
$$a_2 = \frac{2c\, a(n+1)(n+2)\ln^2 A}{b\left(n^2 + c\ln^2 A\right)}. \tag{26}$$

Using Eqs. (26), (25), (12), (15) and auxiliary equation (11), we can find the following travelling wave solutions of Eq. (1) as follows

$$u_{17}(x,t) = \left(-\frac{2c\, a(n+1)(n+2)\left(Ln^2 A\right)}{b\left(n^2 + c\left(Ln^2 A\right)\right)\left(1 + d\left(\cosh\left(\left(x - \frac{n^2 a}{n^2 + c\left(Ln^2 A\right)}\right)LnA\right) + \sinh\left(\left(x - \frac{n^2 a}{n^2 + c\left(Ln^2 A\right)}\right)LnA\right)\right)\right)} + \right.$$
$$\left. \frac{2c\, a(n+1)(n+2)\left(Ln^2 A\right)}{b\left(n^2 + c\left(Ln^2 A\right)\right)\left(1 + d\left(\cosh\left(\left(x - \frac{n^2 a}{n^2 + c\left(Ln^2 A\right)}\right)LnA\right) + \sinh\left(\left(x - \frac{n^2 a}{n^2 + c\left(Ln^2 A\right)}\right)LnA\right)\right)\right)^2} \right)^{\frac{1}{n}}.$$

4. Conclusions

In this paper, the unified, improved Riccati sub-equation and modified Kudryashov methods were used to obtain the solutions to the BBMP equations with power law nonlinearity. These solutions include soliton wave solutions, solitary wave solutions, elliptic wave solutions, and periodic wave rational solutions, which may be helpful to better realize the mechanisms of the complex physical phenomena in different branches of engineering sciences, mathematical physics and other technical areas. Moreover, these methods can be applied to solve other NLEEs.

REFERENCES

[1] *Zhou, Q., Ekici, M., Mirzazadeh, M., Sonmezoglu, A.* (2017). The investigation of soliton solutions of the coupled sine-Gordon equation in nonlinear optics. Journal of Modern Optics, 1-6.

[2] *Ekici, M., Mirzazadeh, M., Sonmezoglu, A., Ullah, M. Z., Zhou, Q., Triki, H., Moshokoa S.P., Biswas, A.* (2017). Optical solitons with anti-cubic nonlinearity by extended trial equation method. Optik-International Journal for Light and Electron Optics, 136, 368-373.

[3] *Rezazadeh, H., Osman, M.S., Eslami, M., Ekici, M., Sonmezoglu, A., Asma, M., Othman, W.A.M., Wong, B.R., Mirzazadeh, M., Zhou, Q., and Biswas, A.* (2018). Mitigating Internet bottleneck with fractional temporal evolution of optical solitons having quadratic–cubic nonlinearity. Optik-International Journal for Light and Electron Opticsk, 164, 84-92.

[4] *Mirzazadeh, M., Yldrm, Y., Yaar, E., Triki, H., Zhou, Q., Moshokoa, S. P., .Belic, M.* (2018). Optical solitons and conservation law of Kundu-Eckhaus equation. Optik-International Journal for Light and Electron Optics, 154, 551-557.

[5] *Eslami, M., Khodadad, F. S., Nazari, F.,Rezazadeh, H.* (2017). The first integral method applied to the Bogoyavlenskii equations by means of conformable fractional derivative. Optical and Quantum Electronics, 49(12), 391.

[6] *Mehrdoust, F., & Mirzazadeh, M.* (2014). On Analytical Solution of the Black-Scholes Equation by the First Integral Method. University Polytechnic Bucharest Scientific Bulletin Series A, 76(4), 85-90.

[7] *Mirzazadeh, M., Eslami, M.* (2013). Exact multisoliton solutions of nonlinear Klein-Gordon equation in (1+2) dimensions. The European Physical Journal Plus, 128(11), 132.

[8] *Khodadad, F. S., Nazari, F., Eslami, M., Rezazadeh, H.* (2017). Soliton solutions of the conformable fractional Zakharov-Kuznetsov equation with dual-power law nonlinearity. Optical and Quantum Electronics, 49(11), 384.

[9] *Aminikhah, H., Sheikhani, A. R., & Rezazadeh, H.* (2016). Sub-equation method for the fractional regularized long-wave equations with conformable fractional derivatives. Scientia Iranica. Transaction B, Mechanical Engineering, 23(3), 1048.

[10] *Jawad, A. J. A. M., Petkovi, M. D., Biswas, A.* (2010). Modified simple equation method for nonlinear evolution equations. Applied Mathematics and Computation, 217(2), 869-877.

[11] *Bhrawy, A. H., Abdelkawy, M. A., Biswas, A.* (2013). Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi's elliptic function method. Communications in Nonlinear Science and Numerical Simulation, 18(4), 915-925.

[12] *Mirzazadeh, M., Biswas, A.* (2014). Optical solitons with spatio-temporal dispersion by first integral approach and functional variable method. Optik-International Journal for Light and Electron Optics, 125(19), 5467-5475.

[13] *Aminikhah, H., Sheikhani, A. H. R., Rezazadeh, H.* (2015). Travelling wave solutions of nonlinear systems of PDEs by using the functional variable method. Boletim da Sociedade Paranaense de Matematica, 34(2), 213-229.

[14] *Rezazadeh, H., Mirzazadeh, M., Mirhosseini-Alizamini, S. M., Neirameh, A., Eslami, M., & Zhou, Q.* (2018). Optical solitons of Lakshmanan-Porsezian-Daniel model with a couple of nonlinearities. Optik-International Journal for Light and Electron Optics,164, 414-423

[15] *Rezazadeh, H.* (2018). New solitons solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity. Optik International Journal for Light and Electron Optics, 167, 218-227.

[16] *Zheng, B.* (2010). Applicationof the (G'/G)-expansion method for (2+1)-dimensionalBoussinesq equations and Kadomtsev–Petviashvili equation. University Polytechnic Bucharest Scientific Bulletin Series A, 72(4), 175-184.

[17] *Triki, H., Chowdhury, A., & Biswas, A.* (2013). Solitary wave and shock wave solutions of the variants of Boussinesq equations. University Polytechnic Bucharest Scientific Bulletin Series A, 75(4).
[18] *Khalique, C. M.* (2013). Solutions and conservation laws of Benjamim-Bona-Mahony-Peregrine equation with power-law and dual power-law nonlinearities. Pramana, 80(3), 413-427.
[19] *Aminikhah, H., Pourreza Ziabary, B., Rezazadeh, H.* (2015). Exact traveling wave solutions of partial differential equations with power law nonlinearity. Nonlinear Engineering, 4(3), 181-188.
[20] *Benjamin, T. B., Bona, J. L., Mahony, J. J.* (1972). Model equations for long waves in nonlinear dispersive systems. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 272(1220), 47-78.
[21] *Abdel-Gawad, H. I., Tantawy, M., Osman, M. S.* (2016). Dynamic of DNA's possible impact on its damage. Mathematical Methods in the Applied Sciences, 39(2), 168-176.
[22] *Abdel-Gawad, H. I., Osman, M.* (2015). On shallow water waves in a medium with time-dependent dispersion and nonlinearity coefficients. Journal of advanced research, 6(4), 593-599.
[23] *Osman, M. S.* (2017). Analytical study of rational and double-soliton rational solutions governed by the KdV-Sawada-Kotera-Ramani equation with variable coefficients. Nonlinear Dynamics, 89(3), 2283-2289.
[24] *Osman, M. S., Abdel-Gawad, H. I.* (2015). Multi-wave solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equations with variable coefficients. The European Physical Journal Plus, 130(10), 215.
[25] *Osman, M. S.* (2017). Nonlinear interaction of solitary waves described by multi-rational wave solutions of the (2+1)-dimensional Kadomtsev-Petviashvili equation with variable coefficients. Nonlinear Dynamics, 87(2), 1209-1216.
[26] *Abdel-Gawad, H. I., Osman, M.* (2014). Exact solutions of the Korteweg-de Vries equation with space and time dependent coefficients by the extended unified method. Indian Journal of Pure and Applied Mathematics, 45(1), 1-12.
[27] *Abdel-Gawad, H. I., Osman, M. S.* (2013). On the variational approach for analyzing the stability of solutions of evolution equations. Kyungpook mathematical journal, 53(4), 661-680.
[28] *Osman, M.S.,Wazwaz, A.M.* (2018). An efficient algorithm to construct multi-soliton rational solutions ofthe(2+1)-dimensional KdV equation with variable coefficients. Applied Mathematics and Computation, 321, 282-289.
[29] *Islam, M. S., Khan, K., Akbar, M. A., Mastroberardino, A.* (2014). A note on improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations. Royal Society Open science, 1(2), 140038.
[30] *Islam, M. S., Khan, K., Akbar, M. A.* (2017). Application of the improved F-expansion method with Riccati equation to find the exact solution of the nonlinear evolution equations. Journal of the Egyptian Mathematical Society, 25(1), 13-18.
[31] *Yomba, E.* (2005). The Extended Fan Sub-Equation Method and its Application to the (2+1)-Dimensional Dispersive Long Wave and Whitham-Broer-Kaup Equations. Chinese Journal of Physics, 43(4), 789-805.
[32] *Zedan, H.* (2010). Applications of the New Compound Riccati Equations Rational Expansion Method and Fan's Subequation Method for the Davey-Stewartson Equations. Boundary Value Problems, 2010(1), 915721.
[33] *Hosseini, K., Mayeli, P., Ansari, R.* (2017). Modified Kudryashov method for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities. Optik-International Journal for Light and Electron Optics, 130, 737-742.

[34*] Biswas, A., Rezazadeh, H., Mirzazadeh, M., Eslami, M., Zhou, Q., Moshokoa, S. P., & Belic, M.* (2018). Optical Solitons Having Weak Non-local Nonlinearity by Two Integration Schemes. Optik-International Journal for Light and Electron Optics, 164, 380-384.

[35] *Zhang, L. H*. (2009). Travelling wave solutions for the generalized Zakharov-Kuznetsov equation with higher-order nonlinear terms. Applied Mathematics and Computation, 208(1), 144-155.