FIXED POINT RESULTS AND (α, β)-TRIANGULAR ADMISSIBILITY IN THE FRAME OF COMPLETE EXTENDED b-METRIC SPACES AND APPLICATION

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We establish the notion of τ-generalized contraction for a pair of mappings S_1 and S_2 over a set Z, where $\tau : Z^2 \to [1, +\infty)$ is a function. We appoint our new notion to formulate and prove many common fixed point results in the setting of generalized b-metric spaces. examples are provided to analyze our results. Also, we set up applications to show the importance of our results. Our results are modification for many exciting results in the literature.

Keywords: Extended b-metric spaces, b-metric spaces, fixed and common fixed point theorems, α-admissibility, (α, β)-triangular admissibility, τ-generalized contraction.

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1. Introduction

The notion of metric spaces is consider to be one of significant notions in the society of sciences since this notion can be used to guarantee a unique solution of such problems in engineering, physics, mathematics etc. Due to the importance of the notion of the metric spaces, the mathematicians extended this notion to many new notions such as partial metric spaces, b-metric spaces, G-metric spaces, extended b-metric spaces and others.

The constructing of new contractive conditions on such metric spaces play an important way to generalized Banach contraction theorem [11]. For some generalization of Banach contraction theorem, see [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 21, 25, 27, 28, 35, 40]. Samet et al. [31] established the concept of α-admissibility and employed this important notions to create new fixed point theorems. Karapinar [23] initiated the study of fixed point theorems through the notion of triangular α-admissibility. Hussain et al. [17] utilized the notion of $\alpha - \psi$-contractions to derive many fixed point theorems. In 2013, Abdeljawad [1] extended the notion α-admissibility to a pair of self mappings. While, Shatanawi [36] introduced the notion of (α, β)-admissibility for pair of self mappings. For more study in admissibility contractive conditions, see [6, 29, 30, 36, 37].

Definition 1.1. *On a set Z, let S be a self mapping and $\alpha : Z^2 \to [0, +\infty)$ be a function. Then*

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(1) *S is called α-admissible* [31] *if $\forall\ z_1, z_2 \in Z$ with $1 \leq \alpha(z_1, z_2)$ it holds $1 \leq \alpha(Sz_1, Sz_2)$.*
(2) *S is called triangular α-admissible* [23] *if S is α-admissible and $\forall\ z_1, z_2, z_3 \in Z$ with $1 \leq \alpha(z_1, z_2)$ and $1 \leq \alpha(z_2, z_3)$ imply $1 \leq \alpha(z_1, z_3)$.*

Definition 1.2. *Let S_1, S_2 be two self mappings on Z and $\alpha, \beta : Z^2 \to [0, +\infty)$ be two functions. Then:*

(1) *The pair (S_1, S_2) is called α-admissible* [1] *if $\forall\ z_1, z_2 \in Z$ with $1 \leq \alpha(z_1, z_2)$ implies $1 \leq \alpha(S_1z_1, S_2z_2)$ and $1 \leq \alpha(S_2z_1, S_1z_2)$.*
(2) *The pair (S_1, S_2) is called (α, β)-admissible* [36] *if $\forall\ z_1, z_2 \in Z$ and $\beta(z_1, z_2) \leq \alpha(z_1, z_2)$ imply $\beta(S_1z_1, S_2z_2) \leq \alpha(S_1z_1, S_2z_2)$ and $\beta(S_2z_1, S_1z_2) \leq \alpha(S_2z_1, S_1z_2)$.*

Definition 1.3. [29] *Let S_1 and S_2 be two self mappings on Z and $\alpha, \beta : Z^2 \to [0, \infty)$ be two functions. Then the pair (S_1, S_2) is said to be (α, β)-triangular admissible if*

(i) *(S_1, S_2) is a pair of (α, β)-admissible;*
(ii) *$\forall\ z_1, z_2, z_3 \in Z$ with $\beta(z_1, z_2) \leq \alpha(z_1, z_2)$ and $\beta(z_2, z_3) \leq \alpha(z_2, z_3)$ implies $\beta(z_1, z_3) \leq \alpha(z_1, z_3)$.*

The notion of extended b-metric spaces was set up by Kamran et al.[20] as follows:

Definition 1.4. [20] *On the set $Z \neq \phi$, we consider the function $\tau : Z^2 \to [1, +\infty)$. The mapping $d_\tau : Z^2 \to [0, +\infty)$ is said to be an extended b-metric space if the following conditions hold:*

(EM1) $d_\tau(z', z) = 0 \Leftrightarrow z' = z$,
(EM2) $d_\tau(z', z) = d_\tau(z, z')$,
(EM3) $d_\tau(z', z) \leq \tau(z', z)\big[d_\tau(z', z'') + d_\tau(z'', z)\big]\ \forall\ z'', z', z \in Z$.

The pair (Z, d_τ) is called an extended b-metric space.

Remark 1.1. *If $\tau(z_1, z_2) = s \geq 1$ in (Z, d_τ), then (Z, d_τ) b-metric space.*

Definition 1.5. [20] *On the set Z, consider an extended b-metric space (Z, d_τ) and a sequence (z_r) in Z. Then:*

(1) *(z_r) converges to some element $z \in Z$ if*

$$\lim_{r \to +\infty} d_\tau(z_r, z) = 0.$$

(2) *(z_r) is Cauchy if*

$$\lim_{r,s \to +\infty} d_\tau(z_r, z_s) = 0.$$

For more results and theorems see, [10, 22, 26, 24, 38, 39].

2. Main results

From now on, we let Z to be a nonempty set. If $S_1, S_2 : Z \to Z$ are two self mappings, we denote by $C_{(S_1,S_2)}$ the set of all common fixed points for S_1 and S_2 and by F_S the set of all fixed points for S. In the rest of this paper, $\tau : Z^2 \to [1, +\infty)$ denotes a function, (Z, d_τ) denotes to an extended b-metric space and $(Z, \tilde{d_\tau})$ is a b-metric space with constant $s^* \geq 1$ unless otherwise are stated.
Now, we furnish our main definition followed by our main result:

Definition 2.1. *On Z, we let $S_1, S_2 : Z \to Z$ be two mappings. The pair (S_1, S_2) is called τ-generalized contraction if there exist $0 \leq \lambda < 1$ and $\delta > 0$ such that $\forall z_1, z_2 \in Z$, we have*

$$d_\tau(S_1z_1, S_2z_2) \leq \lambda^2 \tau(z_1, z_2) M(z_1, z_2), \tag{2.1}$$

where $M(z_1, z_2) = \max\left\{d_\tau(z_1, z_2), d_\tau(z_1, S_1z_1), d_\tau(z_2, S_2z_2), \frac{d_\tau(z_1,S_1z_1)(d_\tau(z_2,S_2z_2)}{\delta+d_\tau(z_1,z_2)}\right\}.$

Example 2.1. *On $Z=[0,+\infty)$, we define the two self mappings $S_1,S_2:[0,+\infty)\to[0,+\infty)$ via $S_1z=\frac{z}{4}$ and $S_2z=kz$ where $k\in[0,\frac{1}{4})$. Also, define $\tau:[0,+\infty)^2\to[1,+\infty)$ via $\tau(z,z')=(1+\max\{z,z'\})$ and $d_\tau:[0,+\infty)^2\to[0,+\infty)$ by:*

$$d_\tau(z,z')=\begin{cases}0 & \text{if } z=z'\\ \max\{z,z'\} & \text{elsewhere.}\end{cases}$$

Then it is obviously that (Z,d_τ) is an extended b-metric space. Then the pair (S_1,S_2) is τ-generalized contraction with $\delta=1$ and $\lambda=\frac{1}{2}$.

Proof. For $z_1,z_2\in Z$, we consider the following cases:
Case (i): If $z_2=z_1$, then

$$d_\tau(S_1z_1,S_2z_2)=\max\left\{\frac{z_1}{4},kz_2\right\}=\frac{z_1}{4}$$

and

$$\begin{aligned}&\tfrac{1}{4}\tau(z_1,z_2)\max\left\{d_\tau(z_1,z_2),d_\tau(z_1,S_1z_1),d_\tau(z_2,S_2z_2),\tfrac{d_\tau(z_1,S_1z_1)(d_\tau(z_2,S_2z_2)}{1+d_\tau(z_1,z_2)}\right\}\\&=\tfrac{1}{4}(1+\max\{z_1,z_2\})z_1\\&\geq\tfrac{z_1}{4}.\end{aligned}$$

Therefore,

$$d_\tau(S_1z_1,S_2z_2)=\frac{z_1}{4}\leq\frac{1}{4}\tau(z_1,z_2)M(z_1,z_2).$$

Case (ii): If $z_2<z_1$, then the proof is similar to case (i).
Case (iii): If $z_2>z_1$, then we have the following sub-cases:
Sub-case (1): If $kz_2=\frac{z_1}{4}$, then $d_\tau(S_1z_1,S_2z_2)=0$.
Sub-case (2): If $kz_2<\frac{z_1}{4}$, then the proof is similar to case (i).
Sub-case (3): If $kz_2>\frac{z_1}{4}$, then

$$d_\tau(S_1z_1,S_2z_2)=kz_2\leq\frac{z_2}{4}$$

and

$$\begin{aligned}&\tfrac{1}{4}\tau(z_1,z_2)\max\left\{d_\tau(z_1,z_2),d_\tau(z_1,S_1z_1),d_\tau(z_2,S_2z_2),\tfrac{d_\tau(z_1,S_1z_1)(d_\tau(z_2,S_2z_2)}{1+d_\tau(z_1,z_2)}\right\}\\&=\tfrac{1}{4}(1+\max\{z_1,z_2\})z_2\\&\geq\tfrac{z_2}{4}.\end{aligned}$$

Consequently, the pair (S_1,S_2) is τ-generalized contraction. □

Definition 2.2. *On Z, we let $S_1,S_2:Z\to Z$ be two mappings. The sequence (z_r) in Z is called an (S_1,S_2)-sequence with starting point $z'\in Z$ if $z_{2r+1}=S_1z_{2r}$ and $z_{2r+2}=S_2z_{2r+1}$ for all $r=0,1,2,\ldots$, where $z'=z_0$.*

Theorem 2.1. *On Z, we consider the two mappings S_1,S_2 and the two functions $\alpha,\beta:Z^2\to[0,+\infty)$. Suppose the following conditions hold:*

1. *(Z,d_τ) is complete,*
2. *there are $0\leq\lambda<1$ and $\delta>0$ such that the pair (S_1,S_2) is τ- generalized contraction,*
3. *for each $z_1,z_2\in Z$, $\tau(z_1,z_2)\leq\frac{1}{\lambda}$,*
4. *$\exists\, z_0\in Z$ with $\beta(S_1z_0,S_2(S_1z_0))\leq\alpha(S_1z_0,S_2(S_1z_0))$ and $\beta(S_2(S_1z_0),S_1z_0)\leq\alpha(S_2(S_1z_0),S_1z_0)$,*
5. *the pair (S_1,S_2) is (α,β)-triangular admissible.*

If S_1 or S_2 is a continuous function, then $C_{(S_1,S_2)}$ consists of only one element.

Proof. Begin with condition (4) and construct the (S_1, S_2)-sequence with starting point z_0. In view of $z_1 = S_1 z_0$ and $z_2 = S_2 z_1 = S_2(S_1 z_0)$, we obtain that

$$\beta(z_1, z_2) \leq \alpha(z_1, z_2) \text{ and } \beta(z_2, z_1) \leq \alpha(z_2, z_1).$$

The triangular admissibility of (S_1, S_2) ensures that

$$\beta(S_2 z_1, S_1 z_2) \leq \alpha(S_2 z_1, S_1 z_2),$$

and

$$\beta(S_1 z_2, S_2 z_1) \leq \alpha(S_1 z_2, S_2 z_1).$$

Again, in view of $z_3 = S_1 z_2$, we obtain that

$$\beta(z_2, z_3) \leq \alpha(z_2, z_3) \text{ and } \beta(z_3, z_2) \leq \alpha(z_3, z_2).$$

Inductively, we realize that the sequence (z_r) satisfies

$$\beta(z_r, z_{r+1}) \leq \alpha(z_r, z_{r+1}), \tag{\mathbf{2}.2}$$

and

$$\beta(z_{r+1}, z_r) \leq \alpha(z_{r+1}, z_r). \tag{\mathbf{2}.3}$$

For positive integers t, r with $t > r$, $\exists j \in \mathbb{N}$ such that $t = r + j$. The Equations (**2**.2) and (**2**.3) in credit to (α, β)-triangular admissibility of the pair (S_1, S_2) imply that:

$$\beta(z_r, z_t) = \beta(z_r, z_{t+j}) \leq \alpha(z_r, z_{t+j}) = \alpha(z_r, z_t), \tag{\mathbf{2}.4}$$

and

$$\beta(z_t, z_r) \leq \alpha(z_t, z_r). \tag{\mathbf{2}.5}$$

Now, if r is even, then $r = 2i$ for some $i \in \mathbb{N}$. So

$$\begin{aligned} d_\tau(z_{2i+2}, z_{2i+1}) &= d_\tau(S_2 z_{2i+1}, S_1 z_{2i}) \\ &\leq \lambda^2 \tau(z_{2i+1}, z_{2i}) \max \Big\{ d_\tau(z_{2i+1}, z_{2i}), d_\tau(z_{2i}, S_1 z_{2i}), \\ &\qquad d_\tau(z_{2i+1}, S_2 z_{2i+1}), \tfrac{d_\tau(z_{2i}, S_1 z_{2i}) d_\tau(z_{2i+1}, S_2 z_{2i+1})}{\delta + d_\tau(z_{2i+1}, z_{2i})} \Big\} \\ &= \lambda^2 \tau(z_{2i+1}, z_{2i}) \max \Big\{ d_\tau(z_{2i+1}, z_{2i}), d_\tau(z_{2i}, z_{2i+1}), \\ &\qquad d_\tau(z_{2i+1}, z_{2i+2}), \tfrac{d_\tau(z_{2i}, z_{2i+1}) d_\tau(z_{2i+1}, z_{2i+2})}{\delta + d_\tau(z_{2i+1}, z_{2i+2})} \Big\} \\ &= \lambda^2 \tau(z_{2i+1}, z_{2i}) \max \big\{ d_\tau(z_{2i+1}, z_{2i}), d_\tau(z_{2i+1}, z_{2i+2}) \big\} \\ &< \max \big\{ d_\tau(z_{2i+1}, z_{2i}), d_\tau(z_{2i+1}, z_{2i+2}) \big\}. \end{aligned}$$

So,

$$d_\tau(z_{2i+1}, z_{2i+2}) \leq \lambda^2 \tau(z_{2i+1}, z_{2i+2}) d_\tau(z_{2i+1}, z_{2i}). \tag{\mathbf{2}.6}$$

Also, if r is odd, then $r = 2i + 1$ for some $i \in \mathbb{N}$, then

$$\begin{aligned} d_\tau(z_{2i+3}, z_{2i+2}) &= d_\tau(S_1 z_{2i+2}, S_2 z_{2i+1}) \\ &\leq \lambda^2 \tau(z_{2i+2}, z_{2i+1}) \max \Big\{ d_\tau(z_{2i+2}, z_{2i+1}), d_\tau(z_{2i+2}, z_{2i+3}), \\ &\qquad d_\tau(z_{2i+1}, z_{2i+2}), \tfrac{d_\tau(z_{2i+2}, z_{2i+3}) d_\tau(z_{2i+1}, z_{2i+2})}{\delta + d_\tau(z_{2i+2}, z_{2i+1})} \Big\} \\ &= \lambda^2 \tau(z_{2i+2}, z_{2i+1}) \max \big\{ d_\tau(z_{2i+2}, z_{2i+1}), d_\tau(z_{2i+2}, z_{2i+3}) \big\} \\ &< \max \big\{ d_\tau(z_{2i+2}, z_{2i+1}), d_\tau(z_{2i+2}, z_{2i+3}) \big\}. \end{aligned}$$

So,

$$d_\tau(z_{2i+3}, z_{2i+2}) \leq \lambda^2 \tau(z_{2i+2}, z_{2i+1}) d_\tau(z_{2i+2}, z_{2i+1}). \tag{\mathbf{2}.7}$$

From (**2**.6) and (**2**.7), we get that

$$\begin{aligned} d_\tau(z_{r+1}, z_r) &\leq \lambda^2 \tau(z_r, z_{r-1}) d_\tau(z_r, z_{r-1}) \\ &\leq \lambda^4 \tau(z_r, z_{r-1}) \tau(z_{r-1}, z_{r-2}) d_\tau(z_{r-1}, z_{r-2}) \\ &\leq \lambda^{2r} \prod_{i=1}^{r} \tau(z_i, z_{i-1}) d_\tau(z_1, z_0). \end{aligned} \quad (\mathbf{2}.8)$$

Claim: (z_r) is a Cauchy sequence in Z. To prove our claim, it is enough to prove that the sequence (z_{2r}) is Cauchy. Given $r, t \in \mathbb{N}$. Assume that $t > r$. Then $(EM3)$ implies that

$$\begin{aligned} d_\tau(z_{2r}, z_{2t}) &\leq \tau(z_{2r}, z_{2t}) \big[d_\tau(z_{2r}, z_{2r+1}) + d_\tau(z_{2r+1}, z_{2t}) \big] \\ &\leq \tau(z_{2r}, z_{2t}) d_\tau(z_{2r}, z_{2r+1}) \\ &\quad + \big[\tau(z_{2r}, z_{2t}) \tau(z_{2r+1}, z_{2t}) \big] \big[d_\tau(z_{2r+1}, z_{2r+2}) + d_\tau(z_{2r+2}, z_{2t}) \big] \\ &\vdots \\ &\leq \sum_{j=2r+1}^{2t-1} \prod_{i=2r}^{j-1} \big[\tau(z_i, z_{2t}) d_\tau(z_j, z_{j+1}) \big]. \end{aligned} \quad (\mathbf{2}.9)$$

Employing (**2**.8) in (**2**.9), we get that

$$d_\tau(z_r, z_t) \leq \sum_{j=2r+1}^{2t-1} \prod_{i=2r}^{j-1} \tau(z_i, z_{2t}) \prod_{l=r}^{j} \lambda^{2j} \tau(z_{l-1}, z_l) d_\tau(z_0, z_1) \quad (\mathbf{2}.10)$$

Now, let $c_j = \prod_{i=2r}^{j-1} \tau(z_i, z_{2t}) \prod_{l=r}^{j} \lambda^{2j} \tau(z_{l-1}, z_l) d_\tau(z_0, z_1)$. Then

$$\limsup_{j \to \infty} \frac{c_{j+1}}{c_j} = \limsup_{j,t \to \infty} \big[\lambda^2 \tau(z_{j+1}, z_t) \tau(z_j, z_{j+1}) \big] < 1. \quad (\mathbf{2}.11)$$

So

$$\sum_{j=2r+1}^{+\infty} \prod_{i=2r}^{j-1} \tau(z_i, z_{2t}) \prod_{l=r}^{j} \lambda^{2j} \tau(z_{l-1}, z_l) d_\tau(z_0, z_1) < \infty. \quad (\mathbf{2}.12)$$

Hence

$$\lim_{r,t \to +\infty} d_\tau(z_r, z_t) = 0.$$

Thus (z_r) is a Cauchy sequence in (Z, d_τ). So the completeness of (Z, d_τ) ensures that $\exists\ \omega^* \in Z$ such that $z_r \to \omega^*$. Without lose of generality, we may assume that S_1 is continuous function. Then $z_{2r+1} = S_1 z_{2r} \to S_1 \omega^*$. The uniqueness of limit informs us that $S_1 \omega^* = \omega^*$.

Now,

$$\begin{aligned} d_\tau(\omega^*, S_2\omega^*) &= d_\tau(S_1\omega^*, S_2\omega^*) \\ &\leq \lambda^2 \tau(\omega^*, \omega^*) \max \Big\{ d_\tau(\omega^*, \omega^*), d_\tau(\omega^*, S_1\omega^*), d_\tau(\omega^*, S_2\omega^*), \\ &\qquad \frac{d_\tau(\omega^*, S_1\omega^*)(d_\tau(\omega^*, S_2\omega^*))}{\delta + d_\tau(\omega^*, \omega^*)} \Big\} \\ &= \lambda^2 \tau(\omega^*, \omega^*) d_\tau(\omega^*, S_2\omega^*) \\ &< d_\tau(\omega^*, S_2\omega^*). \end{aligned}$$

Hence, $\{\omega^*\} \subseteq C_{(S_1,S_2)}$.

Now, assume $\exists\ z_* \in C_{(S_1,S_2)}$; i.e, $S_1 z_* = S_2 z_* = z_*$. Then, we have

$$\begin{aligned} d_\tau(\omega^*, z_*) &= d_\tau(S_1\omega^*, S_2 z_*) \\ &\le \lambda^2 \tau(\omega^*, z_*) \max\Big\{ d_\tau(\omega^*, z_*), d_\tau(\omega^*, S_1\omega^*), d_\tau(z_*, S_2 z_*), \\ &\qquad \tfrac{d_\tau(\omega^*, S_1\omega^*) d_\tau(z_*, S_2 z_*)}{\delta + d_\tau(\omega^*, z_*)} \Big\} \\ &< d_\tau(\omega^*, z_*). \end{aligned}$$

So, $z_* = \omega^*$. Consequently, $C_{(S_1,S_2)} = \{\omega^*\}$. □

Corollary 2.1. *On Z, we consider the two mappings S_1, S_2 and the two functions $\alpha, \beta : Z^2 \to [0, +\infty)$. Suppose (Z, d_τ) is complete and the pair (S_1, S_2) is (α, β)-triangular admissible. Also, assume that there exist $\gamma_1, \gamma_2 \in [0, 1]$ with $\gamma_1 + \gamma_2 \le 1$ such that $\forall\ z_1, z_2 \in Z$, we have*

$$d_\tau(S_1 z_1, S_2 z_2) \le \tau(z_1, z_2)[\gamma_1^2 d_\tau(z_1, S_1 z_1) + \gamma_2^2 d_\tau(z_2, S_2 z_2)].$$

Furthermore, suppose these properties hold true

1. *for each $z_1, z_2 \in Z$, $\tau(z_1, z_2) \le \frac{1}{\gamma_1+\gamma_2}$,*
2. *$\exists z_0 \in Z$ with $\beta(S_1 z_0, S_2(S_1 z_0)) \le \alpha(S_1 z_0, S_2(S_1 z_0))$ and $\beta(s_2(S_1 z_0), S_1 z_0) \le \alpha(S_2(S_1 z_0), S_1 z_0)$.*

If S_1 or S_2 is a continuous function, then $C_{(S_1,S_2)}$ consists of only one element.

Proof. In advantage of

$$\begin{aligned} d_\tau(S_1 z_1, S_2 z_2) &\le \tau(z_1, z_2)\big[\gamma_1^2 d_\tau(z_1, S_1 z_1) + \gamma_2^2 d_\tau(z_2, S_2 z_2)\big] \\ &\le (\gamma_1^2 + \gamma_2^2)\tau(z_1, z_2) \max\big\{d_\tau(z_1, S_1 z_1), d_\tau(z_2, S_2 z_2)\big\} \\ &\le (\gamma_1 + \gamma_2)^2 \tau(z_1, z_2) \max\big\{d_\tau(z_1, S_1 z_1), d_\tau(z_2, S_2 z_2)\big\} \\ &\le (\gamma_1 + \gamma_2)^2 \tau(z_1, z_2) \max\Big\{ d_\tau(z_1, z_2), d_\tau(z_1, S_1 z_1), \\ &\qquad d_\tau(z_2, S_2 z_2), \tfrac{d_\tau(z_1, S_1 z_1)(d_\tau(z_2, S_2 z_2))}{\delta + d_\tau(z_1, z_2)} \Big\}. \end{aligned}$$

So the pair (S_1, S_2) is τ-generalized contraction. So, we catch the result from Theorem **2**.1. □

Corollary 2.2. *On Z, we consider the two mappings S_1, S_2 and the two functions $\alpha, \beta : Z^2 \to [0, +\infty)$. Suppose that $(Z, \tilde{d}_\tau)$ is complete and the pair (S_1, S_2) is (α, β)-triangular admissible. Also, assume there exist $\lambda \in [0, 1)$ with $s^* \le \frac{1}{\lambda}$ and $\delta > 0$ such that $\forall\ z_1, z_2 \in Z$, we have*

$$d_\tau(S_1 z_1, S_2 z_2) \le \lambda^2 s^* M(z_1, z_2),$$

where,

$$M(z_1, z_2) = \max\left\{ d_\tau(z_1, z_2), d_\tau(z_1, S_1 z_1), d_\tau(z_2, S_2 z_2), \frac{d_\tau(z_1, S_1 z_1) d_\tau(z_2, S_2 z_2)}{\delta + d_\tau(z_1, z_2)} \right\}.$$

Furthermore, suppose that there exists $z_0 \in Z$ with $\beta(S_1 z_0, S_2(S_1 z_0)) \le \alpha(S_1 z_0, S_2(S_1 z_0))$ and $\beta(S_2(S_1 z_0), S_1 z_0) \le \alpha(S_2(S_1 z_0), S_1 z_0)$. If S_1 or S_2 is continuous, then $C_{(S_1,S_2)}$ consists of only one element.

Proof. The result can be caught from Theorem **2**.1. □

Corollary 2.3. *On Z, we consider the two mappings S_1, S_2 and the two functions $\alpha, \beta : Z^2 \to [0, +\infty)$. Suppose that $(Z, \tilde{d}_\tau)$ is complete and the pair (S_1, S_2) is (α, β)-triangular admissible. Also, assume there exist $\gamma_1, \gamma_2 \in [0, 1]$ with $\gamma_1 + \gamma_2 < 1$ and $s^* \le \frac{1}{\gamma_1+\gamma_2}$ such that $\forall\ z_1, z_2 \in Z$, we have*

$$d_\tau(S_1 z_1, S_2 z_2) \le s^*\big[\gamma_1^2 d_\tau(z_1, S_1 z_1) + \gamma_2^2 d_\tau(z_2, S_2 z_2)\big].$$

Furthermore, suppose $\exists z_0 \in Z$ *with* $\beta(S_1z_0, S_2(S_1z_0)) \leq \alpha(S_1z_0, S_2(S_1z_0))$ *and* $\beta(S_2(S_1z_0), S_1z_0) \leq \alpha(S_2(S_1z_0), S_1z_0)$. *If* S_1 *or* S_2 *is continuous, then* $C_{(S_1,S_2)}$ *consists of only one element.*

Theorem 2.2. *On* Z*, we consider the mapping* $S : Z \to Z$*. Assume there exist* $\lambda \in [0,1)$ *and* $\delta > 0$ *such that*

$$d_\tau(Sz_1, Sz_2) \leq \lambda^2 \tau(z_1,z_2) M(z_1,z_2),$$

where, $M(z_1,z_2) = \max\left\{d_\tau(z_1,z_2), d_\tau(z_1,Sz_1), d_\tau(z_2,Sz_2), \frac{d_\tau(z_1,Sz_1)d_\tau(z_2,Sz_2)}{\delta + d_\tau(z_1,z_2)}\right\}$. *Furthermore, suppose the following conditions:*

(1) (Z, d_τ) *is complete,*
(2) *for each* $z_1, z_2 \in Z$, $\tau(z_1,z_2) \leq \frac{1}{\lambda}$.

If S *is a continuous function, then* F_S *consists of only one element.*

Proof. Given $z_0 \in Z$. We construct an (S,S)-sequence in Z with starting point z_0 by putting $z_{r+1} = Sz_r = S^{r+1}z_0$. To show that (z_r) is a Cauchy sequence, given $r, t \in \mathbb{N}$ with $r < t$. By using (EM3), we get that:

$$\begin{aligned} d_\tau(z_r, z_t) &\leq \tau(z_r,z_t)\big[d_\tau(z_r,z_{r+1}) + d_\tau(z_{r+1},z_t)\big] \\ &\vdots \\ &\leq \sum_{j=r}^{t-1}\prod_{i=r}^{j}\big[\tau(z_i,z_t)d_\tau(z_j,z_{j+1})\big]. \end{aligned} \qquad (\mathbf{2}.13)$$

Now,

$$\begin{aligned} d_\tau(z_{r+1}, z_r) &\leq \lambda^2 \tau(z_r, z_{r-1}) d_\tau(z_r, z_{r-1}) \\ &\leq \lambda^{2r}\textstyle\prod_{i=1}^{r}\tau(z_i,z_{i-1})d_\tau(z_1,z_0). \end{aligned} \qquad (\mathbf{2}.14)$$

Utilizing Equations (**2**.13) and (**2**.14), one can prove that (z_r) is a Cauchy sequence. The completeness of (Z, d_τ) insures that $\exists\, \beta_* \in Z$ such that $z_r \to \beta_*$. The continuity of S implies that $z_{r+1} = Sz_r \to S\beta_*$. So $\{\beta_*\} \subseteq F_S$. Now, assume $\exists\, z_* \in Z$ such that $z_* \in F_S$. Then

$$\begin{aligned} d_\tau(\beta_*, z_*) &= d_\tau(S\beta_*, Sz_*) \\ &\leq \lambda^2\tau(\beta_*,z_*)\max\left\{d_\tau(\beta_*,z_*)d_\tau(\beta_*,S\beta_*), d_\tau(z_*,Sz_*), \tfrac{d_\tau(\beta_*,S\beta_*)d_\tau(z_*,Sz_*)}{\delta+d_\tau(\beta_*,z_*)}\right\} \\ &= \lambda^2\tau(\beta_*,z_*)d_\tau(\beta_*,z_*). \end{aligned} \qquad (\mathbf{2}.15)$$

Hence, we get $\beta_* = z_*$, and so, $F_S = \{\beta_*\}$. □

Corollary 2.4. *On* Z*, we consider the self mapping* S*. Suppose* $(Z, \tilde{d_\tau})$ *is complete. Also, assume there exist* $\lambda \in [0,1)$ *with* $s^* < \frac{1}{\lambda}$ *and* $\delta > 0$ *such that* $\forall\ z_1, z_2 \in Z$*, we have:*

$$d_\tau(Sz_1,Sz_2) \leq \lambda^2 s^* \max\left\{d_\tau(z_1,z_2), d_\tau(z_1,Sz_1), d_\tau(z_2,Sz_2), \frac{d_\tau(z_1,Sz_1)(d_\tau(z_2,Sz_2)}{\delta + d_\tau(z_1,z_2)}\right\}.$$

If S *is continuous, then* S *has a unique fixed point in* Z.

Next, we introduce some examples to illustrate our results.

Example 2.2. *Let* $Z = [0,1]$ *and let* $K : Z \times Z \to [1,2]$ *be defined by* $K(x,y) = \frac{1+\max\{z_1,z_2\}}{1+\min\{z_1,z_2\}}$. *Let* $\tau : Z \times Z \to [1,+\infty)$ *and* $d_\tau : Z \times Z \to [0,+\infty)$ *be defined by* $\tau(z_1,z_2) = 2K(x,y)$ *and* $d_\tau(z_1,z_2) = \begin{cases} 0 & , z_1 = z_2 \\ (z_1+z_2)^2 & , z_1 \neq z_2 \end{cases}$.

Also, let $\alpha, \beta : Z \times Z \to [0,+\infty)$ *be defined by* $\alpha(z_1,z_2) = e^{z_1+z_2}$ *and*

$\beta(z_1, z_2) = e^{z_1+z_2} - 1$. *Let* $S_1, S_2 : Z \to Z$ *be defined by* $S_1(z) = \frac{z}{\sqrt{8}}$, *and* $S_2(z) = \frac{1}{\sqrt{8}} \ln(1+z)$. *Then, we have the following:*

(1) (Z, d_τ) *is complete,*
(2) *the pair* (S_1, S_2) *is* τ*- generalized contraction with* $\lambda = \frac{1}{4}$,
(3) *for each* $z_1, z_2 \in Z$, $\tau(z_1, z_2) \leq 4 = \frac{1}{\lambda}$,
(4) $\exists\, z_0 \in Z$ *with* $\beta(S_1 z_0, S_2(S_1 z_0)) \leq \alpha(S_1 z_0, S_2(S_1 z_0))$ *and* $\beta(S_2(S_1 z_0), S_1 z_0) \leq \alpha(S_2(S_1 z_0), S_1 z_0)$,
(5) S_1 *is a continuous function,*
(6) *the pair* (S_1, S_2) *is* (α, β)*-triangular admissible.*

Proof. The proofs of (1), (3), (4), (5) and (6) are obvious. So, we just show (2). Let $z_1, z_2 \in [0, 1]$. If $z_1 = z_2$, then it is trivial. Now, let $z_1 \neq z_2$. Then,

$$\begin{aligned} d_\tau(S_1 z_1, S_2 z_2) &= \left(\frac{z_1}{\sqrt{8}} + \frac{1}{\sqrt{8}} \ln(1 + z_2) \right)^2 \\ &\leq \frac{1}{8} (z_1 + z_2)^2 \\ &\leq \frac{1}{16} \tau(z_1, z_2) d_\tau(z_1, z_2). \end{aligned}$$

Hence, by Theorem **2**.1, $C_{(S_1, S_2)}$ consists of only one element. □

Example 2.3. *On* $Z = [0, 1]$,*let* $K : Z \times Z \to [1, 8]$ *be defined by* $K(x, y) = \frac{1+7\max\{x,y\}}{1+\min\{x,y\}}$. *Let* $d_\tau : Z \times Z \to [0, +\infty)$ *and* $\tau : Z \times Z \to [1, +\infty)$ *be defined by* $d_\tau(x, y) = (x - y)^2$ *and* $\tau(x, y) = 2K(x, y)$. *Let* $S : Z \to Z$ *be defined by* $S(x) = \frac{2-\frac{X}{2}}{5\sqrt{2}(2-x^2)}$. *Then, we have the following:*

(1) (d_τ, Z) *is a complete extended b-metric space,*
(2) S *satisfies condition* **2**.*1, with* $\lambda = \frac{1}{16}$.

Proof. First, observe that for each $x, y \in Z$, $\tau(x, y) \leq 16 = \frac{1}{\lambda}$.
We just show (2). Let $x, y \in [0, 1]$. Then

$$\begin{aligned} d_\tau(Sx, Sy) &= \left(\frac{2 - \frac{x}{2}}{5\sqrt{2}(2 - x^2)} - \frac{2 - \frac{y}{2}}{5\sqrt{2}(2 - y^2)} \right)^2 \\ &= \frac{2}{25(2 - x^2)^2(2 - y^2)^2} \left(x + y - \frac{1}{4}xy - \frac{1}{2} \right)^2 (x - y)^2 \\ &\leq \frac{2}{25(2 - x^2)^2(2 - y^2)^2} \left(\frac{5}{4} \right)^2 (x - y)^2 \\ &\leq \frac{1}{128} (x - y)^2 \\ &\leq \lambda^2 \tau(x, y) d_\tau(x, y). \end{aligned}$$

Hence, by Theorem **2**.2, F_S consists of only one element. □

3. Applications

To show the novelty of our work, we employ our results to prove the existence and uniqueness of solution for some nonlinear equations in the unit interval.

Theorem 3.1. *For integer k with $k \geq 2$, the equation*

$$x^{k+1} + x^k + 1 = Ax \text{ where } A \geq 3k+1$$

has a unique solution in the unit interval $I = [0,1]$.

Proof. Define $\tau : I^2 \to [0,+\infty)$ via $\tau(z_1,z_2) = 1 + \frac{3}{7}\max\{z_1,z_2\}$ and $d_\tau : I^2 \to [0,+\infty)$ via $d_\tau(z_1,z_2) = |z_1 - z_2|$. Then it is obviously that d_τ is a complete b-metric space.
Note that, our problem owns a unique solution in I iff the following self mapping S on I

$$S(z) = \frac{1+z^k}{A-z^k}$$

owns a unique fixed point. Now, we show that for all $z_1, z_2 \in Z$, we have

$$d_\tau(Sz_1,Sz_2) \leq \lambda^2\tau(z_1,z_2)d_\tau(z_1,z_2) \text{ with } \lambda = \frac{7}{10}.$$

First it is clear that for each $z_1, z_2 \in Z$, $\tau(z_1,z_2) \leq \frac{10}{7} = \frac{1}{\lambda}$.
Now,

$$\begin{aligned}
d_\tau(Sz_1,Sz_2) &= \left|\frac{1+z_1^k}{A-z_1^k} - \frac{1+z_2^k}{A-z_2^k}\right| \\
&= \left|\frac{(1+z_1^k)(A-z_2^k)-(1+z_2^k)(A-z_1^k)}{(A-z_1^k)(A-z_2^k)}\right| \\
&= \left(\frac{(A-1)}{(A-z_1^k)(A-z_2^k)}\right)\left|z_1^k - z_2^k\right| \\
&= \left(\frac{(A-1)}{(A-z_1^k)(A-z_2^k)}\right)\left[z_1^{k-1} + z_2z_1^{k-2} + \cdots + z_1z_2^{k-2} + z_2^{k-1}\right]\left|z_1 - z_2\right| \\
&\leq \frac{(A-1)(k)}{(A-1)^2}\left|z_1 - z_2\right| \\
&= \frac{(k)}{(A-1)}\left|z_1 - z_2\right| \\
&\leq \frac{1}{3}\left|z_1 - z_2\right| \\
&\leq \left(\frac{7}{10}\right)^2\left|z_1 - z_2\right| \\
&\leq \left(\frac{7}{10}\right)^2\left[1 + \frac{3}{7}\max\{z_1,z_2\}\right]\left|z_1 - z_2\right| \\
&= \lambda^2\tau(z_1,z_2)d_\tau(z_1,z_2) \\
&\leq \lambda^2\tau(z_1,z_2)\max\left\{d_\tau(z_1,z_2), d_\tau(z_1,Sz_1), d_\tau(z_2,Sz_2), \frac{d_\tau(z_1,Sz_1)(d_\tau(z_2,Sz_2)}{\delta + d_\tau(z_1,z_2)}\right\}.
\end{aligned}$$

Hence, S meets expectations of Theorem **2**.2, and so, F_S consists of only one element. □

Theorem 3.2. *For any integer $m \geq 1$, the equation*

$$\sum_{i=0}^{m} x^i = Bx \text{ where } B \geq 2m(m+1),$$

has a unique solution in the unit interval $I = [0,1]$.

Proof. Let $K : I^2 \to [1, \frac{3}{2}]$ be defined by $K(z_1, z_2) = \frac{1+2z_1z_2}{1+z_1z_2}$. Define $\tau : I^2 \to [0, +\infty)$ via $\tau(z_1, z_2) = K(z_1, z_2)$ and $d_\tau : I^2 \to [0, +\infty)$ via $d_\tau(z_1, z_2) = \frac{1}{2}K(z_1, z_2)(z_1 - z_2)^2$. Then it is obviously that d_τ is a complete extended b-metric space.
Note that, our problem owns a unique solution in I iff the following self mapping s on I

$$S(z) = \frac{1}{B}\sum_{i=0}^{m} z^i$$

owns a unique fixed point. Now, we show that for all $z_1, z_2 \in Z$, we have

$$d_\tau(sz_1, sz_2) \le \lambda^2 \tau(z_1, z_2) d_\tau(z_1, z_2) \text{ with } \lambda = \frac{2}{3}.$$

Now,

$$\begin{aligned}
d_\tau(Sz_1, Sz_2) &= \frac{1}{2}K(Sz_1, Sz_2)\left(\frac{1}{B}\sum_{i=0}^{m}(z_1^i - z_2^i)\right)^2 \\
&\le \frac{3}{4B^2}\left(\sum_{i=1}^{m}(z_1^i - z_2^i)\right)^2 \\
&\le \frac{3}{4B^2}(z_1 - z_2)^2(1 + 2 + \cdots + m)^2 \\
&= \frac{3m^2(m+1)^2}{16B^2}(z_1 - z_2)^2 \\
&\le \frac{3m^2(m+1)^2}{8B^2}\tau(z_1, z_2) d_\tau(z_1 - z_2) \\
&\le \lambda^2 \tau(z_1, z_2) d_\tau(z_1, z_2) \\
&\le \lambda^2 \tau(z_1, z_2) \max\left\{d_\tau(z_1, z_2), d_\tau(z_1, Sz_1), d_\tau(z_2, Sz_2), \frac{d_\tau(z_1, Sz_1)(d_\tau(z_2, Sz_2)}{\delta + d_\tau(z_1, z_2)}\right\}.
\end{aligned}$$

Hence, S meets expectations of Theorem **2**.2, and so, F_S consists of only one element. □

4. Conclusions

In this study, we introduced and studied τ-generalized contraction for a pair of mappings S_1 and S_2 over a non empty set Z endowed with an extended b-metric. Based on this a new contraction, some exciting fixed and common fixed point results were obtained. Our results are modifications and improvements for many existing results in the literature. Finally, we show the novelty of our work by setting up some examples and applications.

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