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$\mathcal{I}$-STATISTICAL CONVERGENCE IN PROBABILISTIC NORMED SPACES

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In this paper, we introduce a new type of summability notion, namely, $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence for double sequences in probabilistic normed space, which is a natural generalization of the notion of natural density, statistical convergence and lacunary statistical convergence using the notion of ideals of the set of positive integers $\mathbb{N}$. In this context, we investigate their relationship, and make some observations about these classes using the tools of probabilistic normed space.

Keywords: Double sequence, Ideal, t-norm, probabilistic normed space, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence.
MSC2000: 40G99

1. Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [4] and other authors independently as follows: A real number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. Our aim is to propose some new variants of statistical convergence for double sequences in probabilistic normed space. We put special emphasis on certain summability methods in probabilistic normed space, in a sense extending original ideas of Das et al. [3] and Savaş and Das [14]. In particular, we obtain a new type of summability notion, namely, $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence for double sequences in probabilistic normed space, which is a natural generalization of the notion of natural density, statistical convergence and lacunary statistical convergence using the notion of ideals of the set of positive integers $\mathbb{N}$.

Motivated by a result of Salat [13] and Fridy [5] about statistically convergent sequences, the authors of [7] also defined so called $\mathcal{I}$-convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals of $\mathbb{N}$ can be found in [2, 11, 16, 17] where many important references can be found.

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In [6], the relation between lacunary statistical convergence and statistical convergence was established, among other things. Recently, in [9], Mohiuddine and Savaş extended the idea of lacunary statistically convergent double sequences with respect to the probabilistic normed space. More results on this convergence can be found in [8].

As for the theory of probabilistic normed (PN) spaces, it was first introduced by Śerstnev [19] by means of a definition that was closely modelled on the theory of normed spaces. PN spaces are the vector spaces in which the norms of the vectors are uncertain due to randomness. In a PN space, the norms of the vectors are represented by probability distribution functions instead of non-negative real numbers. If x is an element of a PN space, then its norm is denoted by F_x, and the value $F_x(t)$ is interpreted as the probability that the norm of x is smaller than t. Quite recently, this subject was studied by various authors, see [1, 15]. We refer to [18] for more details.

In 1900, Pringsheim [12] introduced the notion of convergence of double sequences as follows: A double sequence $x = (x_{jk})$ is said to converge to the limit L in Pringsheim's sense if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. This concept was extended to $\mathbf{I}$-convergence of double sequences by B.C. Tripathy in [20] as follows: Let $\mathbf{I}_2$ be an ideal of $\mathbf{P}(\mathbb{N} \times \mathbb{N})$. Then a double sequence (x_{jk}) is said to be $\mathbf{I}$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - L| \geq \varepsilon\} \in \mathbf{I}_2.$$

Recently in [16] we used ideals to introduce the concept of $\mathbf{I}$-statistical convergence which naturally extends the notion of statistical convergence and studied some basic properties of this more general convergence.

As a natural consequence, in this note, we continue our investigation of $\mathbf{I}$-statistical convergence and introduce the notion of $\mathbf{I}$-statistical convergence and $\mathbf{I}_2$-lacunary statistical convergence for double sequences with respect to the probabilistic norm $\mathbf{F}$ in line of [3]. We mainly try to establish the relation between these two summability notions.

2. $\mathbf{I}$-Statistical convergence and $\mathbf{I}$-Lacunary statistical convergence on PN space

In this section we deal with the ideal statistical convergence and ideal lacunary statistical convergence on the probabilistic normed spaces.

The authors of [3] and [14] defined so called $\mathbf{I}$-statistical convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences.

Definition 1. *A sequence* $x=\{x_k\}_{k\in\mathbb{N}}$ *is said to be* $\mathbf{I}$*-statistically convergent to* L *or* $S(\mathbf{I})$*-convergent to* L *if, for each* $\varepsilon>0$ *and* $\delta>0$,

$$\left\{n\in\mathbb{N}\ :\ \frac{1}{n}\left|\{k\le n\ :\ \|x_k-L\|\ge\varepsilon\}\right|\ge\delta\right\}\in\mathbf{I}$$

or equivalently if for each $\varepsilon>0$

$$\delta_{\mathbf{I}}(A(\varepsilon))=\mathbf{I}\text{-}\lim\delta_n(A(\varepsilon))=0,$$

where $A(\varepsilon)=\{k\le n\ :\ \|x_k-L\|\ge\varepsilon\}$ *and* $\delta_n(A(\varepsilon))=\frac{|A(\varepsilon)|}{n}$.

In this case we write $x_k\to L(S(\mathbf{I}))$. The class of all $\mathbf{I}$-statistically convergent sequences will be denoted simply by $S(\mathbf{I})$. Let $\mathbf{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathbf{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathbf{I}$-statistically convergent is the statistical convergence.

By a double lacunary sequence $\theta_{rs}=\{(k_r,l_s)\}$, we mean there exists two lacunary sequences $\theta_r=(k_r)$ and $\theta_s=(l_s)$. Let $h_r=k_r-k_{r-1}$, $q_r=\frac{k_r}{k_{r-1}}$, $I_r=(k_{r-1},k_r]$, $\overline{h}_s=l_s-l_{s-1}$, $\overline{q}_s=\frac{l_s}{l_{s-1}}$, $\overline{I}_s=(l_{s-1},l_s]$, $k_{rs}=k_rl_s$, $h_{rs}=h_r\overline{h}_s$, $q_{rs}=q_r\overline{q}_s$ and the interval determined by θ_{rs} is denoted by

$$I_{rs}=\{(i,j)\ :\ k_{r-1}<i\le k_r,\ l_{s-1}<j\le l_s\}.$$

We now ready to obtain our main definitions and results.

Definition 2. *Let* $(X,\mathbf{F},*)$ *be a PN space. Then, a sequence* $x=\{x_{ij}\}$ *is said to be* $\mathbf{I}$*-statistically convergent to* $L\in X$ *with respect to the probabilistic norm* $\mathbf{F}$ *if, for each* $\varepsilon\in(0,1)$, $t>0$ *and* $\delta>0$,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{mn}\left|\{i\le m,j\le n\ :\ \mathbf{F}(x_{ij}-L;t)\le 1-\varepsilon\}\right|\ge\delta\right\}\in\mathbf{I}_2.$$

In this case we write $x_{ij}\xrightarrow{\mathbf{F}}L(S^{PN}(\mathbf{I}_2))$.

Definition 3. *Let* $(X,\mathbf{F},*)$ *be a PN space. and* θ_{rs} *be a double lacunary sequence. A sequence* $x=\{x_{ij}\}$ *is said to be* $\mathbf{I}$*-lacunary statistically convergent to* $L\in X$ *with respect to the probabilistic norm* $\mathbf{F}$ *if, for any* $\varepsilon\in(0,1)$, $t>0$ *and* $\delta>0$,

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{h_{rs}}\left|\{(i,j)\in I_{rs}\ :\ \mathbf{F}(x_{ij}-L;t)\le 1-\varepsilon\}\right|\ge\delta\right\}\in\mathbf{I}_2.$$

In this case, we write $x_{ij} \xrightarrow{\mathsf{F}} L\left(S_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right)$. *The class of all* $\mathtt{I}$*-lacunary statistically convergent double sequences will be denoted by* $S_{\theta_{rs}}^{PN}(\mathtt{I}_2)$.

In the following, we investigate the relationship between $\mathtt{I}$-statistical and $\mathtt{I}$-lacunary statistical convergence with respect to the probabilistic norm F. However, to prove Theorem 2 which describes the above mentioned relation we will need the following result, which gives an alternative characterization of $\mathtt{I}$-lacunary statistical convergence of bounded real double sequences similar to the characterization of lacunary statistical convergence given in [6].

Definition 4. *Let* $(X, \mathsf{F}, *)$ *be a PN space and* θ_{rs} *be a double lacunary sequence. Then* $x = \{x_{ij}\}$ *is said to be* $N_{\theta_{rs}}^{PN}(\mathtt{I}_2)$*-convergent to* $L \in X$ *with respect to the probabilistic norm* F *if, for any* $\varepsilon \in (0,1)$, $t > 0$ *and* $\delta > 0$,

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} \ : \ \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} \mathsf{F}(x_{ij} - L, t) \leq 1 - \varepsilon \right\} \in \mathtt{I}_2.$$

This convergence is denoted by $x_{ij} \to L\left(N_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right)$, *and the class of such sequences will be denoted simply by* $N_{\theta_{rs}}^{PN}(\mathtt{I}_2)$.

Theorem 1. *Let* $(X, \mathsf{F}, *)$ *be a PN space and* $\theta_{rs} = \{(k_r, l_s)\}$ *be a double lacunary sequence. Then*

(i) (a) $x_{ij} \to L\left(N_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right) \Rightarrow x_{ij} \to L\left(S_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right)$, *and*

(b) $N_{\theta_{rs}}^{PN}(\mathtt{I}_2)$ *is a proper subset of* $S_{\theta_{rs}}^{PN}(\mathtt{I}_2)$.

(ii) $x_{ij} \in l_\infty^2$ *and* $x_{ij} \to L\left(S_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right) \Rightarrow x_{ij} \to L\left(N_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right)$,

(iii) $S_{\theta_{rs}}^{PN}(\mathtt{I}_2) \cap l_\infty^2 = N_{\theta_{rs}}^{PN}(\mathtt{I}_2) \cap l_\infty^2$.

Proof. (i) (a) If $\varepsilon \in (0,1)$ and $x_{ij} \to L\left(N_{\theta_{rs}}^{PN}(\mathtt{I}_2)\right)$, we can write

$$\sum_{(i,j) \in I_{rs}} \mathsf{F}(x_{ij} - L, t) \geq \sum_{(i,j) \in I_{rs},\, \mathsf{F}(x_{ij} - L, t) \leq 1-\varepsilon} \mathsf{F}(x_{ij} - L, t)$$

$$\geq \varepsilon \left| \{(i,j) \in I_{rs} \ : \ \mathsf{F}(x_{ij} - L, t) \leq 1 - \varepsilon\} \right|$$

and so

$$\frac{1}{\varepsilon h_{rs}} \sum_{(i,j) \in I_{rs}} \mathsf{F}(x_{ij} - L, t) \geq \frac{1}{h_{rs}} \left| \{(i,j) \in I_{rs} \ : \ \mathsf{F}(x_{ij} - L, t) \leq 1 - \varepsilon\} \right|.$$

Then, for any $\delta > 0$ and $t > 0$

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs} \; : \; \mathsf{F}\left(x_{ij}-L,t\right)\leq 1-\varepsilon\right\}\right|\geq\delta\right\}$$

$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathsf{F}\left(x_{ij}-L,t\right)\leq(1-\varepsilon)\delta\right\}\in\mathcal{I}_2.$$

This proves the result.

(b) In order to establish that the inclusion $N_{\theta_{rs}}^{PN}(\mathcal{I}_2)\subseteq S_{\theta_{rs}}^{PN}(\mathcal{I}_2)$ is proper, let θ_{rs} be given, and define x_{ij} to be $1,2,...,\left[\sqrt{h_{rs}}\right]$ for the first $\left[\sqrt{h_{rs}}\right]$ integers in I_{rs} and $x_{ij}=0$ otherwise, for all $r,s=1,2,...$. Then, for any $\varepsilon\in(0,1)$ and $t>0$

$$\frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs} \; : \; \mathsf{F}\left(x_{ij}-0,t\right)\leq 1-\varepsilon\right\}\right|\leq\frac{\left[\sqrt{h_{rs}}\right]}{h_{rs}},$$

and for any $\delta>0$ we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs} \; : \; \mathsf{F}\left(x_{ij}-0,t\right)\leq 1-\varepsilon\right\}\right|\geq\delta\right\}$$

$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{\left[\sqrt{h_{rs}}\right]}{h_{rs}}\geq\delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to $\mathcal{I}_2$, it follows that $x_{ij}\rightarrow 0\left(S_{\theta_{rs}}^{PN}(\mathcal{I}_2)\right)$.

On the other hand,

$$\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathsf{F}\left(x_{ij}-0,t\right)=\frac{1}{h_{rs}}.\frac{\left[\sqrt{h_{rs}}\right]\left(\left[\sqrt{h_{rs}}\right]+1\right)}{2}.$$

Then

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathsf{F}\left(x_{ij}-0,t\right)\leq 1-\frac{1}{4}\right\}$$

$$=\left\{(r,s)\in\mathbb{N}\times\mathbb{N} \; : \; \frac{\left[\sqrt{h_{rs}}\right]\left(\left[\sqrt{h_{rs}}\right]+1\right)}{h_{rs}}\geq\frac{1}{2}\right\}$$

which belongs to $F(\mathcal{I})$, since $\mathcal{I}$ is admissible. So $x_{ij}\not\rightarrow 0\left(N_{\theta_{rs}}^{PN}(\mathcal{I}_2)\right)$.

(ii) Suppose that $x_{ij}\rightarrow L\left(S_{\theta_{rs}}^{PN}(\mathcal{I}_2)\right)$ and $x\in l_\infty^2$. Then there exists an $M>0$ such that $\mathsf{F}\left(x_{ij}-L,t\right)\geq 1-M \;\; \forall(i,j)\in\mathbb{N}\times\mathbb{N}$. Given $\varepsilon\in(0,1)$, we have

$$\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathsf{F}(x_{ij}-L,t)=\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs},\,\mathsf{F}(x_{ij}-L,t)\le 1-\frac{\varepsilon}{2}}\mathsf{F}(x_{ij}-L,t)$$
$$+\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs},\,\mathsf{F}(x_{ij}-L,t)>1-\frac{\varepsilon}{2}}\mathsf{F}(x_{ij}-L,t)$$
$$\le\frac{M}{h_{rs}}\left|\left\{(i,j)\in I_{rs}\ :\ \mathsf{F}(x_{ij}-L,t)\right\}\right|+\frac{\varepsilon}{2}.$$

Consequently, we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathsf{F}(x_{ij}-L,t)\le 1-\varepsilon\right\}$$
$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs}\ :\ \mathsf{F}(x_{ij}-L,t)\le 1-\frac{\varepsilon}{2}\right\}\right|\ge\frac{\varepsilon}{2M}\right\}\in\mathbf{I}_2.$$

This proves the result.

(iii) Follows from (i) and (ii).

Theorem 2. *For any lacunary sequence θ_{rs}, $\mathbf{I}$-statistical convergence with respect to the probabilistic norm F implies $\mathbf{I}$-lacunary statistical convergence with respect to the probabilistic norm F if and only if $\liminf_r q_r>1$ and $\liminf_s \overline{q}_s>1$. If $\liminf_r q_r=1$ and $\liminf_s \overline{q}_s=1$, then there exists a bounded double sequence $x=\{x_{ij}\}$ which is $\mathbf{I}$-statistically convergent but not $\mathbf{I}$-lacunary statistically convergent.*

Proof. Suppose first that $\liminf_r q_r>1$ and $\liminf_s \overline{q}_s>1$. Then there exists $\alpha>0$ such that $q_r\ge 1+\alpha$ and $\overline{q}_s\ge 1+\alpha$ for sufficiently large r,s, which implies that

$$\frac{h_{rs}}{k_r l_s}\ge\frac{\alpha}{\alpha+1}.$$

Since $x_{ij}\to L(S^{PN}(\mathbf{I}_2))$, for every $\varepsilon\in(0,1)$, $t>0$, and for sufficiently large r and s, we have

$$\frac{1}{k_r l_s}\left|\left\{i\le k_r,\,j\le l_s\ :\ \mathsf{F}(x_{ij}-L,t)\le 1-\varepsilon\right\}\right|$$
$$\ge\frac{\alpha}{\alpha+1}\frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs}\ :\ \mathsf{F}(x_{ij}-L,t)\le 1-\varepsilon\right\}\right|.$$

Then, for any $\delta>0$, we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{h_{rs}}\left|\left\{(i,j)\in I_{rs}\ :\ \mathbf{F}\left(x_{ij}-L,t\right)\le 1-\varepsilon\right\}\right|\ge\delta\right\}$$

$$\subseteq\left\{(r,s)\in\mathbb{N}\times\mathbb{N}\ :\ \frac{1}{k_r}\left|\left\{i\le k_r, j\le l_s\ :\ \mathbf{F}\left(x_{ij}-L,t\right)\le 1-\varepsilon\right\}\right|\ge\frac{\delta\alpha}{(1+\alpha)}\right\}\in\mathbf{I}_2.$$

This proves the sufficiency.

Conversely, suppose that $\liminf_r q_r = 1$ and $\liminf_s \overline{q}_s = 1$. Proceeding as in [1, p. 510], we can select a subsequence $\{k_{r_p}\}$ and $\{l_{s_q}\}$ such that

$$\frac{k_{r_p} l_{s_q}}{k_{r_p-1} l_{s_q-1}} < 1+\frac{1}{pq} \text{ and } \frac{k_{r_p-1} l_{s_q-1}}{k_{r_{p-1}} l_{s_{q-1}}} > pq, \text{ where } r_p \ge r_{p-1}+2,\ s_q \ge s_{q-1}+2.$$

Define a sequence $x=\{x_{ij}\}$ by

$$x=\{x_{ij}\}=\begin{cases}1, & \text{if } i\in I_{r_p} \text{ and } j\in I_{s_q}\\ 0, & \text{otherwise.}\end{cases}$$

Then, for any real L,

$$\frac{1}{h_{rs}}\sum_{(i,j)\in I_{r_p}\times I_{s_q}}\mathbf{F}\left(x_{ij}-L,t\right)=\mathbf{F}(1-L,t) \text{ for } i,j=1,2,...$$

and

$$\frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}\mathbf{F}\left(x_{ij}-L,t\right)=\mathbf{F}(L,t) \text{ for } r\ne r_j \text{ and } s\ne s_j$$

Then it is quite clear that x does not belong to $N_{\theta_{rs}}^{PN}(\mathbf{I}_2)$. Since x is bounded, Theorem 2 (iii) implies that $x \not\to L\left(S_{\theta_{rs}}^{PN}(\mathbf{I}_2)\right)$.

Next, let $k_{r_j-1}\le n\le k_{r_{j+1}-1}$. Then, from Theorem 2.1 in [4], we can write

$$\frac{\varepsilon}{mn}\left|\left\{j\le m, j\le n\ :\ \mathbf{F}\left(x_{ij}-L,t\right)\le 1-\varepsilon\right\}\right|$$

$$\le\frac{1}{mn}\sum_{i,j=1}^{m,n}\mathbf{F}\left(x_{ij},t\right)\le 1-\varepsilon\le\frac{k_{r_p-1}l_{s_q-1}+h_{r_p}\overline{h}_{s_q}}{k_{r_p-1}l_{s_q-1}}\le\frac{1}{pq}+\frac{1}{pq}=\frac{2}{pq}.$$

Hence $\{x_{ij}\}$ is $\mathbf{I}$-statistically convergent with respect to the probabilistic norm $\mathbf{F}$ for any admissible ideal $\mathbf{I}$.

It is known that [6] lacunary statistical convergence with respect to the probabilistic norm $\mathbf{F}$ implies statistical convergence with respect to the probabilistic norm $\mathbf{F}$ if and only if $\lim_r \sup q_r<\infty$ and $\lim_s \sup \overline{q}_s<\infty$ (i.e., when $\mathbf{I}=\mathbf{I}_{fin}$ is the ideal of finite subsets of $\mathbb{N}$). However, for arbitrary admissible ideal $\mathbf{I}$, this is not clear, and we leave it as an open problem.

Problem 1. *When does $\mathcal{I}$-lacunary statistical convergence with respect to the probabilistic norm F imply $\mathcal{I}$-statistical convergence with respect to the probabilistic norm F?*

Now, before stating an important result for the relationship between the concepts $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence with respect to the probabilistic norm F, we introduce the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence, by using the condition (AP).

Definition 5. *([10]) An admissible ideal $\mathcal{I}_2 \subset P(\mathbb{N}\times\mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}_2$ there are sets $B_n \subset \mathbb{N}$, $n\in\mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{I}_2$.*

It was observed in [7, 10] that, for a sequence $\{x_{jk}\}$, $\mathcal{I}$-convergence is equivalent to $\mathcal{I}^*$-convergence iff the ideal $\mathcal{I}$ satisfies the condition (AP).

Theorem 3. *Let $\mathcal{I}$ be an admissible ideal satisfying condition (AP), and let $\theta_{rs} \in F(\mathcal{I})$. If $x_{ij} \in S^{PN}(\mathcal{I}_2) \cap S^{PN}_{\theta_{rs}}(\mathcal{I}_2)$, then $S^{PN}(\mathcal{I}_2)\text{-}\lim x = S^{PN}_{\theta_{rs}}(\mathcal{I}_2)\text{-}\lim x$.*

Proof. Suppose that $S^{PN}(\mathcal{I}_2)\text{-}\lim x = L$, $S^{PN}_{\theta_{rs}}(\mathcal{I}_2)\text{-}\lim x = L'$, and $L \neq L'$. Let $0<\varepsilon<\frac{1}{2}\left|L-L'\right|$. Since $\mathcal{I}$ satisfies the condition (AP), there exists $M \in F(\mathcal{I})$ (i.e., $\mathbb{N}\setminus M \in \mathcal{I}$) such that

$$\lim_{r,s\to\infty} \frac{1}{m_r n_s}\left|\left\{i \le m_r, j \le n_s \ :\ \mathsf{F}(x_{ij}-L,t) \le 1-\varepsilon\right\}\right| = 0,$$

where $M = \{(m_r, n_s)\ :\ r,s = 1,2,\ldots\}$. Let

$$A = \left\{i \le m_r, j \le n_s \ :\ \mathsf{F}(x_{ij}-L,t) \le 1-\varepsilon\right\}$$

and

$$B = \left\{i \le m_r, j \le n_s \ :\ \mathsf{F}(x_{ij}-L',t) \le 1-\varepsilon\right\}.$$

Then $m_r n_s = |A \cup B| \le |A| + |B|$. This implies that

$$1 \le \frac{|A|}{m_r n_s} + \frac{|B|}{m_r n_s}.$$

Since $\frac{|B|}{m_r n_s} \le 1$ and $\lim\limits_{r,s\to\infty} \frac{|A|}{m_r n_s} = 0$, so we must have $\lim\limits_{r,s\to\infty} \frac{|B|}{m_r n_s} = 1$.

Let $M^{*}=\left\{\left(k_{\alpha_t},l_{\beta_{t'}}\right) : t,t'=1,2,...\right\}\cap\theta_{rs}\in F(\mathbf{I})$. Then the $\left(k_{\alpha_t},l_{\beta_{t'}}\right)$ th term of the statistical limit expression $\frac{1}{m_r n_s}\left\{i\le m_r, j\le n_s : \mathbf{F}\left(x_{ij}-L',t\right)\le 1-\varepsilon\right\}$ is

$$\frac{1}{k_{\alpha_t}l_{\beta_{t'}}}\left|\left\{(i,j)\in\bigcup_{r,s=1}^{\alpha_t,\beta_{t'}} I_{rs} : \mathbf{F}\left(x_{ij}-L',t\right)\le 1-\varepsilon\right\}\right|=\frac{1}{\sum_{r,s=1}^{\alpha_t,\beta_{t'}} h_{rs}}\sum_{r,s=1}^{\alpha_t,\beta_{t'}} t_{rs}h_{rs}, \tag{2.1}$$

where $t_{rs}=h_{rs}^{-1}\left|\left\{(i,j)\in I_{rs} : \mathbf{F}\left(x_{ij}-L',t\right)\le 1-\varepsilon\right\}\right|\xrightarrow{\mathbf{I}}0$ because $x_{ij}\to L'\left(S_{\theta_{rs}}^{PN}(\mathbf{I}_2)\right)$. Since θ_{rs} is a double lacunary sequence, (2.1) is a regular weighted mean transform of t_{rs}'s, and therefore it is also $\mathbf{I}$-convergent to zero as $t,t'\to\infty$, and so it has a subsequence which is convergent to zero since $\mathbf{I}$ satisfies condition (AP). But since this is a subsequence of

$$\left\{\frac{1}{mn}\left|\left\{1\le i\le m, 1\le j\le n : \mathbf{F}\left(x_{ij}-L',t\right)\le 1-\varepsilon\right\}\right|\right\}_{(m,n)\in M},$$

we infer that $\left\{\frac{1}{mn}\left|\left\{1\le i\le m, 1\le j\le n : \mathbf{F}\left(x_{ij}-L',t\right)\le 1-\varepsilon\right\}\right|\right\}_{(m,n)\in M}$ is not convergent to 1, which is a contradiction. This completes the proof of the theorem.

It can be checked as in the case of statistically and lacunary statistically convergent double sequences that both $S^{PN}(\mathbf{I}_2)$ and $S_{\theta_{rs}}^{PN}(\mathbf{I}_2)$ are linear subspaces of the space of real sequences.

Theorem 4. *Let* $(X,\mathbf{F},*)$ *be a PN space.* $S_{\theta_{rs}}^{PN}(\mathbf{I}_2)\cap l_\infty^2$ *is a closed subset of* l_∞^2 *where* l_∞^2 *stands for the space of all bounded double sequences of probabilistic norm* $\mathbf{F}$.

Proof. The proof readily follows from Theorem 1 of [16]. This theorem establishes the topological character of the space $S_{\theta_{rs}}^{PN}(\mathbf{I}_2)$.

3. Conclusions

In this paper, we introduce a new type of summability notion, namely, $\mathbf{I}$-statistical convergence and $\mathbf{I}$-lacunary statistical convergence for double sequences in probabilistic normed space, which is a natural generalization of the notion of natural density, statistical convergence and lacunary statistical convergence using the notion of ideals of the set of positive integers $\mathbb{N}$. In this context, we investigate their relationship, and make some observations about these classes using the tools of probabilistic normed space.

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