ON THE STABILITY OF HOMOMORPHISMS AND k-DERIVATIONS ON Γ-BANACH ALGEBRAS

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Let V be a Γ-Banach algebra over the complex field $\mathbb{C}$ and let $D: V \to V$ and $k: \Gamma \to \Gamma$ be two linear mappings. If $D(a\alpha b) = D(a)\alpha b + ak(\alpha)b + a\alpha D(b)$ for all $a, b \in V$, $\alpha \in \Gamma$, then D is called a k-derivation of V. In this paper, we prove the Hyers-Ulam-Rassias stability of algebra homomorphisms in Γ-Banach algebras with direct method. We also use the same method to study the stability and the superstability of k-derivations associated with the Cauchy functional equation and the mixed type additive and quadratic functional equation $f(x+3y) + f(x-3y) = f(x+y) + f(x-y) - 16f(y) + 8f(2y)$ in Γ-Banach algebras.

Keywords: Hyers-Ulam-Rassias stability, superstability, Γ-Banach algebra, k-derivation

MSC2010: 39B52, 39B82, 16W25, 16Y99

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [18] in 1940, concerning the stability of group homomorphisms: *Let $(G_1, *)$ be a group and $(G_2, \circ, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $f: G_1 \to G_2$ satisfies the inequality $d(f(x*y), f(x)\circ f(y)) < \delta$*, for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(f(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, under what conditions does there exist a homomorphism near an approximate homomorphism between a group and a metric group? When the answer is affirmative, we say that the homomorphisms from G_1 to G_2 are stable.

D.H. Hyers [8] gave a first affirmative answer to the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. In 1978, Th.M. Rassias [16] generalized the Hyers's stability theorem for linear mappings by considering an unbounded Cauchy difference. This phenomenon of stability proved by Th.M. Rassias [16] is called the *Hyers-Ulam-Rassias stability.* Since then several results concerning the Hyers-Ulam-Rassias stability of various functional equations with more general domains and ranges have been extensively investigated by a number of authors (see [6], [5], [7], [9], [13]).

Before giving our main results, we first present some preliminary definitions.

Let A be a real or complex algebra. A linear mapping $D: A \to A$ is said to be a *derivation* if $D(ab) = D(a)b + aD(b)$, for all $a, b \in A$. The stability of derivations between operator algebras was first obtained by P. Šemrl [17].

The concept of a Γ-ring was introduced by N. Nobusawa [14] and generalized by Barnes [1]. In recent years, many results of Γ-rings have been extended to Γ-algebras.

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Let V and Γ be two linear spaces over a field F. Then V is said to be a Γ-*algebra* (in the sense of Barnes [1]) over F if there exists a mapping $(a, \alpha, b) \mapsto a\alpha b$ of $V \times \Gamma \times V \to V$ satisfying the following conditions:

(i) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

(ii) $\lambda(a\alpha b) = (\lambda a)\alpha b = a(\lambda\alpha)b = a\alpha(\lambda b)$,

(iii) $a\alpha(b+c) = a\alpha b + a\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $(a+b)\alpha c = a\alpha c + b\alpha c$

for all $a, b, c \in V$, $\alpha, \beta \in \Gamma$, $\lambda \in F$. The Γ-algebra V is denoted by (Γ, V).

In addition, if there exists a mapping $(\alpha, a, \beta) \mapsto \alpha a\beta$ of $\Gamma \times V \times \Gamma \to \Gamma$ satisfying the following for all $a, b \in V$, $\alpha, \beta, \gamma \in \Gamma$ and $\lambda \in F$:

(iv) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$,

(v) $a\alpha b = 0$ for all $a, b \in V$ implies $\alpha = 0$,

then V is called a Γ_N-algebra and denoted by $(\Gamma, V)_N$.

If V and Γ are normed linear spaces over F, then Γ-algebra V is called a Γ-*normed algebra* if the condition

$$\|a\alpha b\| \leq \|a\| \cdot \|\alpha\| \cdot \|b\|$$

holds for all $a, b \in V$ and $\alpha \in \Gamma$.

Bhattacharya and Maity [2] gave the definition of a Γ-Banach algebra in their paper. A Γ-normed algebra V is called a Γ-*Banach algebra* if V is a Banach space. Any Banach algebra can be regarded as a Γ-Banach algebra by suitably choosing Γ. Similar definitions can be made for Γ_N-algebras.

Γ-Banach algebras are generalization of both the concepts of Banach algebras and Γ-rings. The set of all $m \times n$ rectangular matrices and the set of all bounded linear maps from an infinite dimensional normed linear space X into a Banach space Y are some examples of Γ-Banach algebras which are not general Banach algebras. Similarly an ordinary derivation can't be defined on Γ-Banach algebras since there is no natural way of introducing an algebraic multiplication into them. The notions of derivation of a Γ-ring has been introduced by F.J. Jing [11] in 1987, and later H. Kandamar [12] developed a new concept of derivation in Γ-rings known as k-derivation. We define a k-derivation in Γ-algebras as follows:

Let V be a Γ-algebra over a field F and let $d : V \to V$ and $k : \Gamma \to \Gamma$ be two linear mappings. If the condition

$$d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$$

holds for all $a, b \in V$ and $\alpha \in \Gamma$, then d is called a k-*derivation* of V. If $d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$ holds for all $a \in V$ and $\alpha \in \Gamma$, then d is called a *Jordan k-derivation* of V. It is clear that every k-derivation of a Γ-algebra V is a Jordan k-derivation of V. But, the converse is not true in general.

Let V_1 be a Γ_1-algebra and V_2 be a Γ_2-algebra over a same field F. An ordered pair (ψ, φ) of linear mappings $\psi : \Gamma_1 \to \Gamma_2$ and $\varphi : V_1 \to V_2$ is called an *algebra homomorphism* from (Γ_1, V_1) to (Γ_2, V_2) if the following condition holds:

$$\varphi(a\alpha b) = \varphi(a)\psi(\alpha)\varphi(b)$$

for all $a, b \in V_1$ and $\alpha \in \Gamma_1$.

In the proofs of our theorems, we shall use the following lemma which is proved in [15]:

Lemma 1.1. [15] *Let X and Y be linear spaces, $\mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}$. Then the mapping f is $\mathbb{C}$-linear.*

In this paper, using the direct method, we prove the Hyers-Ulam-Rassias stability and superstability of k-derivations associated with the Cauchy functional equation and the mixed type additive and quadratic functional equation

$$f(x+3y) + f(x-3y) = f(x+y) + f(x-y) - 16f(y) + 8f(2y)$$

in Γ-Banach algebras and Γ_N-Banach algebras, respectively.

2. Stability of homomorphisms and k-derivations

In this section, we first prove the Hyers-Ulam-Rassias stability of homomorphisms in Γ-Banach algebras, associated to the Cauchy functional equation.

Theorem 2.1. *Let V be a Γ-Banach algebra and V' be a Γ'-Banach algebra over the complex field $\mathbb{C}$. Suppose $f : V \to V'$ is a mapping with $f(0) = 0$ for which there exist a map $g : \Gamma \to \Gamma'$ with $g(0) = 0$ and functions $\phi_1 : V \times V \to [0, \infty)$, $\phi_2 : \Gamma \times \Gamma \to [0, \infty)$ such that*

$$\Phi(a, b) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\phi_1(2^n a, 2^n b) < \infty, \tag{1}$$

$$\Psi(\alpha, \beta) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\phi_2(2^n \alpha, 2^n \beta) < \infty, \tag{2}$$

$$\|f(\mu a + \mu b) - \mu f(a) - \mu f(b)\| \leq \phi_1(a, b), \tag{3}$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \phi_2(\alpha, \beta), \tag{4}$$

$$\|f(a\alpha b) - f(a)g(\alpha)f(b)\| \leq \phi_1(a, b), \mu \in \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}, a, b \in V, \alpha, \beta \in \Gamma. \tag{5}$$

for all Then there exists unique algebra homomorphism $(\psi, \varphi) : (\Gamma, V) \to (\Gamma', V')$ such that

$$\|g(\alpha) - \psi(\alpha)\| \leq \Psi(\alpha, \alpha)$$

and

$$\|f(a) - \varphi(a)\| \leq \Phi(a, a)$$

for all $a \in V, \alpha \in \Gamma$.

Proof. Putting $\mu = 1$ in (3), we have

$$\|f(a + b) - f(a) - f(b)\| \leq \phi_1(a, b) \qquad (a, b \in V). \tag{6}$$

Now we replace b by a in (6) to get

$$\|f(2a) - 2f(a)\| \leq \phi_1(a, a). \tag{7}$$

One can use the induction to show that

$$\Big\|\frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m}\Big\| \leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\phi_1(2^k a, 2^k a) \tag{8}$$

for all $a \in V$ and all non-negative integers m and n with $n > m$. It follows from (8) that the sequence $\left\{\frac{f(2^n a)}{2^n}\right\}$ is a Cauchy sequence for all $a \in V$. Since (Γ', V') is complete the sequence $\left\{\frac{f(2^n a)}{2^n}\right\}$ is convergent. Set

$$\varphi(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}.$$

Replacing a, b by $2^n a$, $2^n b$, respectively, in (3), we get

$$\|f(2^n(\mu a + \mu b)) - \mu f(2^n a) - \mu f(2^n b)\| \leq \phi_1(2^n a, 2^n b) \tag{9}$$

Now we divide the both sides of the above inequality by 2^n and we have

$$\|2^{-n} f(2^n(\mu a + \mu b)) - 2^{-n}\mu f(2^n a) - 2^{-n}\mu f(2^n b)\| \leq 2^{-n}\phi_1(2^n a, 2^n b). \tag{10}$$

Passing to the limit as $n \to \infty$ we obtain $\varphi(\mu a + \mu b) = \mu\varphi(a) + \mu\varphi(b)$ for all $a, b \in V$ and all $\mu \in \mathbb{T}$. By Lemma 1.1, we see that φ is $\mathbb{C}$-linear.

If we put $m = 0$ in the inequality (8) and take the limit as $n \to \infty$, then we get $\|f(a) - \varphi(a)\| \leq \Phi(a, a)$ for all $a \in A$. It is known that additive mapping φ satisfying (3) is unique [6].

Similarly it can be shown that there exists unique linear mapping ψ defined by $\psi(\alpha) := \lim_{n\to\infty} 2^{-n} g(2^n\alpha)$ by using (4).

Replacing a, b and α in (5) by $2^n a$, $2^n b$ and $2^n\alpha$, respectively, we have

$$\|f(8^n a\alpha b) - f(2^n a)g(2^n\alpha)f(2^n b)\| \leq \phi_1(2^n a, 2^n b).$$

If we divide both sides of the above inequality by 2^{3n}, then we obtain that

$$\|2^{-3n} f(2^{3n} a\alpha b) - 2^{-n} f(2^n a) 2^{-n} g(2^n\alpha) 2^{-n} f(2^n b)\| \leq 2^{-3n}\phi_1(2^n a, 2^n b)$$

for all $a, b \in V$, $\alpha \in \Gamma$. Thus we have

$$\varphi(a\alpha b) = \varphi(a)\psi(\alpha)\varphi(b).$$

Therefore, (ψ, φ) is an algebra homomorphism from (Γ, V) into (Γ', V'). □

Corollary 2.1. *Let V be a Γ-Banach algebra and V' be a Γ'-Banach algebra over the complex field $\mathbb{C}$. Suppose $f : V \to V'$ is a mapping with $f(0) = 0$ for which there exists a map $g : \Gamma \to \Gamma'$ with $g(0) = 0$ and there exist $\theta_1, \theta_2 \geq 0$ and $p, t \in [0, 1)$ such that*

$$\|f(\mu a + \mu b) - \mu f(a) - \mu f(b)\| \leq \theta_1(\|a\|^p + \|b\|^p),$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \theta_2(\|\alpha\|^t + \|\beta\|^t),$$

$$\|f(a\alpha b) - f(a)g(\alpha)f(b)\| \leq \theta_1(\|a\|^p + \|b\|^p), \mu \in \mathbb{T} = \{\mu \in \mathbb{C} \mid |\mu| = 1\}, a, b \in V, \alpha, \beta \in \Gamma.$$

Then there exists unique algebra homomorphism (ψ, φ) from (Γ, V) to (Γ', V') satisfying

$$\|g(\alpha) - \psi(\alpha)\| \leq \frac{2\theta_2}{2 - 2^t}\|\alpha\|^t$$

and

$$\|f(a) - \varphi(a)\| \leq \frac{2\theta_1}{2 - 2^p}\|a\|^p$$

for all $a \in V, \alpha \in \Gamma$.

Proof. Putting $\phi_1(a, b) := \theta_1(\|a\|^p + \|b\|^p)$ and $\phi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t + \|\beta\|^t)$ in Theorem 2.1, we get the desired result. □

Now, we prove the following theorem for k-derivations of Γ-Banach algebras.

Theorem 2.2. *Let V be a Γ-Banach algebra over the complex field $\mathbb{C}$. Suppose $f : V \to V$ is a mapping with $f(0) = 0$ and $g : \Gamma \to \Gamma$ is a mapping with $g(0) = 0$ for which there exist functions $\varphi_1 : V \times V \times V \times V \to [0, \infty)$ and $\varphi_2 : \Gamma \times \Gamma \to [0, \infty)$ such that*

$$\Phi_1(a, b, c, d) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi_1(2^n a, 2^n b, 2^n c, 2^n d) < \infty, \tag{11}$$

$$\Phi_2(\alpha, \beta) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi_2(2^n\alpha, 2^n\beta) < \infty, \tag{12}$$

$$\|f(\mu a + \mu b + c\alpha d) - \mu f(a) - \mu f(b) - f(c)\alpha d - cg(\alpha)d - c\alpha f(d)\| \leq \varphi_1(a, b, c, d) \tag{13}$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \varphi_2(\alpha, \beta) \tag{14}$$

for all $\mu \in \mathbb{T}$ *and all* $a, b, c, d \in V$, $\alpha, \beta \in \Gamma$. *Then there exists a unique linear map* k *from* Γ *to* Γ *satisfying* $\|g(\alpha) - k(\alpha)\| \leq \Phi_2(\alpha, \alpha)$, *and there exists a unique* k*-derivation* $D : V \to V$ *such that*

$$\|f(a) - D(a)\| \leq \Phi_1(a, a, 0, 0) \tag{15}$$

for all $a \in V$, $\alpha \in \Gamma$.

Proof. Put $\mu = 1$ and $c = d = 0$ in (13) to obtain

$$\|f(a + b) - f(a) - f(b)\| \leq \varphi_1(a, b, 0, 0). \tag{16}$$

Now replace b by a in (16) to get

$$\|f(2a) - 2f(a)\| \leq \varphi_1(a, a, 0, 0). \tag{17}$$

One can use induction to show that

$$\left\|\frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m}\right\| \leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\varphi_1(2^k a, 2^k a, 0, 0) \tag{18}$$

for all $a \in V$ and all $n > m \geq 0$. It follows from the convergence of series (11) that the sequence $\left\{\frac{f(2^n a)}{2^n}\right\}$ is Cauchy. So it is convergent, since (Γ, V) is complete. Set

$$D(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n} \qquad (a \in V). \tag{19}$$

Putting $c = d = 0$ and replacing a, b by $2^n a$, $2^n b$, respectively, in (13), we get

$$\|f(2^n(\mu a + \mu b)) - \mu f(2^n a) - \mu f(2^n b)\| \leq \varphi_1(2^n a, 2^n b, 0, 0).$$

Now we divide the both sides of the above inequality by 2^n and we have

$$\|2^{-n} f(2^n(\mu a + \mu b)) - 2^{-n}\mu f(2^n a) - 2^{-n}\mu f(2^n b)\| \leq 2^{-n}\varphi_1(2^n a, 2^n b, 0, 0). \tag{20}$$

Passing to the limit as $n \to \infty$ we obtain $D(\mu a + \mu b) = \mu D(a) + \mu D(b)$ for all $a, b \in V$ and all $\mu \in \mathbb{T}$. Therefore, by Lemma 1.1, we have that D is $\mathbb{C}$-linear.

It follows from (18) and (19) that

$$\|f(a) - D(a)\| \leq \Phi_1(a, a, 0, 0)$$

for all $a \in V$. Also it is known that additive mapping D satisfying (15) is unique (see [10]).

Similarly it can be shown that there exists unique linear mapping k defined by $k(\alpha) := \lim_{n\to\infty} 2^{-n} g(2^n\alpha)$ by using (14).

Putting $\mu = 1$, $a = b = 0$, and replacing c, d and α by $2^n c$, $2^n d$ and $2^n\alpha$, respectively, in (13), and divide both sides of the inequality by 2^{3n} we obtain

$$\begin{aligned} &\|2^{-3n} f(2^{3n}(c\alpha d)) - 2^{-n} f(2^n c)\alpha d - 2^{-n} c g(2^n\alpha) d - 2^{-n} c\alpha f(2^n d)\| \\ &\leq 2^{-3n}\varphi_1(0, 0, 2^n c, 2^n d) \end{aligned} \tag{21}$$

for all $c, d \in V$, $\alpha \in \Gamma$.

Then by using the convergence of series (11), we have

$$\begin{aligned} &\lim_{n\to\infty} \|2^{-3n} f(2^{3n} c\alpha d) - 2^{-n} f(2^n c)\alpha d - 2^{-n} c g(2^n\alpha) d - 2^{-n} c\alpha f(2^n d)\| \\ \leq\ &\lim_{n\to\infty} 2^{-3n}\varphi_1(0, 0, 2^n c, 2^n d) = 0. \end{aligned}$$

Thus we get

$$\begin{aligned} D(c\alpha d) &= \lim_{n\to\infty} \frac{f(2^{3n} c\alpha d)}{2^{3n}} \\ &= \lim_{n\to\infty} \frac{f(2^n c)\alpha d + c g(2^n\alpha) d + c\alpha f(2^n d)}{2^n} \\ &= D(c)\alpha d + c k(\alpha) d + c\alpha D(d), c, d \in V, \alpha \in \Gamma. \end{aligned}$$

Hence D is a k-derivation on (Γ, V). □

Corollary 2.2. *Let V be a Γ-Banach algebra over the complex field $\mathbb{C}$. Suppose $f : V \to V$ is a mapping with $f(0) = 0$ and $g : \Gamma \to \Gamma$ is a mapping with $g(0) = 0$ for which there exist $\theta_1, \theta_2 \geq 0$ and $p, t \in [0, 1)$ such that*

$$\begin{aligned} & \|f(\mu a + \mu b + c\alpha d) - \mu f(a) - \mu f(b) - f(c)\alpha d - cg(\alpha)d - c\alpha f(d)\| \\ \leq \; & \theta_1(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p), \end{aligned}$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \theta_2(\|\alpha\|^t + \|\beta\|^t), \mu \in \mathbb{T}, a, b, c, d \in V, \alpha, \beta \in \Gamma.$$

Then there exists unique linear mapping k from Γ to Γ satisfying $\|g(\alpha) - k(\alpha)\| \leq \frac{2\theta_2}{2 - 2^t}\|\alpha\|^t$, and there exists a unique k-derivation $D : V \to V$ such that

$$\|f(a) - D(a)\| \leq \frac{2\theta_1}{2 - 2^p}\|a\|^p$$

for all $a \in V$, $\alpha \in \Gamma$.

Proof. Put $\varphi_1(a, b, c, d) := \theta_1(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ and $\varphi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t + \|\beta\|^t)$ in Theorem 2.3. □

Moreover, we have the following result for the superstability of k-derivati- ons.

Corollary 2.3. *Suppose that V is a Γ-Banach algebra over the complex field $\mathbb{C}$. Let $p, q, r, s, t, l, \theta_1, \theta_2$ be non-negative real numbers with $0 < p + q + r + s \neq 1$, $0 < t + l \neq 1$ and let $f : V \to V$ and $g : \Gamma \to \Gamma$ be two mappings such that*

$$\|f(\mu a + \mu b + c\alpha d) - \mu f(a) - \mu f(b) - f(c)\alpha d - cg(\alpha)d - c\alpha f(d)\| \leq \theta_1(\|a\|^p\|b\|^q\|c\|^r\|d\|^s)$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \theta_2(\|\alpha\|^t\|\beta\|^l), \mu \in \mathbb{T}, a, b, c, d \in V, \alpha, \beta \in \Gamma. \tag{22}$$

Then f is a k-derivation on V, where $k : \Gamma \to \Gamma$ is a linear map.

Proof. Putting $a = b = c = d = 0$ and $\mu = 1$ in (22), we get $f(0) = 0$. Now, if we put $\beta = 0$, $\mu = 1$ in (22), then we have $g(0) = 0$. Again, putting $c = d = 0$, $a = b$ and $\mu = 1$ in (22), we conclude that $f(2a) = 2f(a)$ for all $a \in V$, and by induction we see that $f(a) = \frac{f(2^n a)}{2^n}$ for all $a \in V$ and $n \in \mathbb{N}$. Similarly, if we put $\alpha = \beta$ and $\mu = 1$ in (22), we have that $g(2\alpha) = 2g(\alpha)$ for all $\alpha \in \Gamma$, and using induction again we have that $g(\alpha) = \frac{g(2^n\alpha)}{2^n}$ for all $\alpha \in \Gamma$ and $n \in \mathbb{N}$. Therefore, we can obtain the desired result by Theorem 2.2 putting $\varphi_1(a, b, c, d) := \theta_1(\|a\|^p\|b\|^q\|c\|^r\|d\|^s)$ and $\varphi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t\|\beta\|^l)$. □

3. **Stability of k-derivations associated with the mixed type additive and quadratic functional equation**

In 2013, A. Bodaghi and S.O. Kim [3] obtained the general solution of the mixed type additive and quadratic functional equation

$$f(x + 3y) + f(x - 3y) = f(x + y) + f(x - y) - 16f(y) + 8f(2y), \tag{23}$$

and they investigated the Hyers-Ulam stability for these functional equations in non-Archimedean normed spaces.

Lemma 3.1. [3] *Let X and Y be linear spaces. If an odd mapping $f : X \to Y$ satisfies the functional equation (23), then f is additive.*

In this section, we consider the Hyers-Ulam-Rassias stability of k-derivations in Γ_N-Banach algebras with the functional equation (23).

Recall that a Γ_N-algebra V is called *prime* if, for any two elements a and b of V, $a\Gamma b = 0$ implies either $a = 0$ or $b = 0$. Also, V is said to be 2-*torsion free* if $2a = 0$ implies $a = 0$ for all $a \in V$.

Theorem 3.1. *Let V be a 2-torsion free prime Γ_N-Banach algebra over $\mathbb{C}$. Let $f : V \to V$ be an odd mapping, $g : \Gamma \to \Gamma$ be a map with $g(0) = 0$ and $\varphi_1, \phi : V \times V \to [0, \infty)$, $\varphi_2 : \Gamma \times \Gamma \to [0, \infty)$ satisfy*

$$\|f(a\alpha b + b\alpha a) - f(a)\alpha b - f(b)\alpha a - ag(\alpha)b - bg(\alpha)a - a\alpha f(b) - b\alpha f(a)\| \leq \phi(a, b), \quad (24)$$

$$\|f(\mu(a + 3b)) + f(\mu(a - 3b)) - \mu(f(a + b) + f(a - b) + 16f(b) - 8f(2b))\| \leq \varphi_1(a, b), \quad (25)$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \leq \varphi_2(\alpha, \beta) \quad (26)$$

for all $a, b \in V$, $\alpha, \beta \in \Gamma$ and $\mu \in \mathbb{T}$. Assume that

$$\Phi(a) := \sum_{n=0}^{\infty} 2^{-n}\varphi_1(0, 2^n a) < \infty, \quad (27)$$

$$\varphi(\alpha, \beta) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi_2(2^n\alpha, 2^n\beta) < \infty, \quad (28)$$

$$\liminf_{n\to\infty} 2^{-3n}\phi(2^n a, 2^n b) = 0, \quad (29)$$

$$\liminf_{n\to\infty} 2^{-n}\varphi_1(2^n a, 2^n b) = \liminf_{n\to\infty} 2^{-n}\varphi_1(2^n a, 0) = 0, a, b \in V, \alpha, \beta \in \Gamma. \quad (30)$$

Then there exists a unique linear map k from Γ to Γ satisfying $\|g(\alpha) - k(\alpha)\| \leq \varphi_2(\alpha, \alpha)$, and there exists a unique k-derivation $D : V \to V$ such that

$$\|f(a) - D(a)\| \leq \frac{1}{|2|^4}\ \Delta(a), a \in V, \quad (31)$$

where $\Delta(a) := \sup\{\varphi_1(0, 2^j a)/|2|^j \ \mid j \in \mathbb{N} \cup \{0\}\}$.

Proof. Putting $\mu = 1$ and $a = 0$ in (25), we have

$$\|2f(b) - f(2b)\| \leq \frac{1}{|2|^3}\ \varphi_1(0, b) \quad (32)$$

for all $b \in V$. Letting $b = 2^n a$ in (32) and then dividing by $|2|^{n+1}$, we get

$$\|\frac{1}{2^n}f(2^n a) - \frac{1}{2^{n+1}}f(2^{n+1}a)\| \leq \frac{1}{|2|^{n+4}}\ \varphi_1(0, 2^n a)$$

for all $a \in V$ and non-negative integer n. So we obtain

$$\|\frac{1}{2^n}f(2^n a) - \frac{1}{2^{n+j}}f(2^{n+j}a)\| \leq \frac{1}{|2|^3}\ \Big[\frac{\varphi_1(0, 2^{n+1}a)}{2^{n+1}} + \ldots + \frac{\varphi_1(0, 2^{n+j}a)}{2^{n+j}}\Big].$$

This implies that $\left\{\frac{f(2^n a)}{2^n}\right\}$ is a Cauchy sequence in V by (27). Hence, there exists a mapping D such that

$$D(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n} \qquad (a \in V). \quad (33)$$

One can use the inequality (26) to show that there exist unique linear mapping k defined by $k(\alpha) := \lim_{n\to\infty} \frac{g(2^n\alpha)}{2^n}$. On the other hand,

$$\begin{aligned}
&\|D(a\alpha b + b\alpha a) - D(a)\alpha b - D(b)\alpha a - ak(\alpha)b - bk(\alpha)a - a\alpha D(b) - b\alpha D(a)\| \\
\leq\ & \lim_{n\to\infty} \| \frac{1}{8^n}[f(8^n(a\alpha b + b\alpha a)) - 4^n f(2^n a)\alpha b - 4^n f(2^n b)\alpha a - 4^n a g(2^n\alpha)b \\
& -4^n b g(2^n\alpha)a - 4^n a\alpha f(2^n b) - 4^n b f(2^n a)]\| \\
\leq\ & \liminf_{n\to\infty} \frac{1}{8^n}\phi(2^n a, 2^n b) = 0.
\end{aligned}$$

Thus

$$D(a\alpha b + b\alpha a) = D(a)\alpha b + D(b)\alpha a + ak(\alpha)b + bk(\alpha)a + a\alpha D(b) + b\alpha D(a) \tag{34}$$

for all $a, b \in V$, $\alpha \in \Gamma$. It follows from the definition of D that

$$\begin{aligned}
&\|D(a+3b) + D(a-3b) - D(a+b) - D(a-b) - 16D(b) + 8D(2b)\| \\
\leq\ & \lim_{n\to\infty} \| \frac{1}{2^n}[f(2^n(a+3b)) + f(2^n(a-3b)) - f(2^n(a+b)) \\
& -f(2^n(a-b)) - 16f(2^n b) + 8f(2^n(2b))]\| \\
\leq\ & \liminf_{n\to\infty} \frac{1}{2^n}\varphi_1(2^n a, 2^n b) = 0
\end{aligned}$$

for all $a, b \in V$. Since D is an odd mapping, by Lemma 3.1 the mapping $D : V \to V$ is additive.

Letting $b = 0$ in (25), we have $\|2f(\mu a) - 2\mu f(a)\| \leq \varphi_1(a, 0)$ for all $a \in V$. Replacing a by $2^n a$, we get $\|f(2^n(\mu a)) - \mu f(2^n a)\| \leq \varphi_1(2^n a, 0)$.

Dividing the both sides of the above inequality by 2^n, we have $\|2^{-n}f(2^n(\mu a)) - 2^{-n}\mu f(2^n a)\| \leq 2^{-n}\varphi_1(2^n a, 0)$.

Passing to the limit as $n \to \infty$ we obtain $D(\mu a) = \mu D(a)$ for all $a \in V$ and all $\mu \in \mathbb{T}$. Therefore, by Lemma 1.1, D is $\mathbb{C}$-linear.

Since D is additive, we have

$$\begin{aligned}
2D(a\alpha a) &= D(a\alpha a) + D(a\alpha a) = D(a\alpha a + a\alpha a) \\
&= 2(D(a)\alpha a + ak(\alpha)a + a\alpha D(a)),
\end{aligned}$$

and it follows from that

$$2(D(a\alpha a) - D(a)\alpha a - ak(\alpha)a - a\alpha D(a)) = 0$$

for all $a \in V$, $\alpha \in \Gamma$. By 2-torsion freeness of V, we have that D is a Jordan k-derivation. Applying the result, which asserts that every Jordan k-derivation of a 2-torsion free prime Γ_N-ring is a k-derivation (see [4]), we see that D is a k-derivation.

For each $a \in V$ and nonnegative integers n, we have

$$\begin{aligned}
\left\|\frac{f(2^n a)}{2^n} - f(a)\right\| &= \left\|\sum_{j=0}^{k-1} \frac{f(2^{j+1}a)}{2^{j+1}} - \frac{f(2^j a)}{2^j}\right\| \\
&\leq \max\left\{\left\|\frac{f(2^{j+1}a)}{2^{j+1}} - \frac{f(2^j a)}{2^j}\right\| \mid 0 \leq j < k\right\} \\
&\leq \frac{1}{|2|^4}\max\left\{\frac{\varphi_1(0, 2^j a)}{|2|^j} \mid 0 \leq j < k\right\}.
\end{aligned} \tag{35}$$

Taking that n tends to approach infinity in (35) and applying (33), we can see that inequality (31) holds.

It remains to show that D is uniquely defined. Let $d : V \to V$ be another k-derivation satisfying $d(a\alpha b) = d(a)\alpha b + ak(\alpha)b + a\alpha d(b)$ and $\|f(a) - d(a)\| \le \frac{1}{|2|^4}\ \Delta(a)$.

Then for all $a \in V$, we have

$$\begin{aligned}\|D(a) - d(a)\| &= \frac{1}{2^n}\|D(2^n a) - d(2^n a)\| \le \frac{1}{2^n}(\|D(2^n a) - f(2^n a)\| + \|f(2^n a) - d(2^n a)\|)\\ &\le 2^{-n}\cdot\frac{1}{|2|^3}\ \Delta(2^n a) = \frac{1}{|2|^3}\sup\Big\{\frac{\varphi_1(0, 2^{k+n}a)}{|2|^{k+n}} \mid k \ge n,\ k \ge 0\Big\}.\end{aligned}$$

By letting $n \to \infty$ in the preceding inequality, we obtain $D(a) = d(a)$ for all $a \in V$. This completes the proof. □

Corollary 3.1. *Suppose that V is a 2-torsion free prime Γ_N-Banach algebra over $\mathbb{C}$. Let $f : V \to V$ be an odd mapping, $g : \Gamma \to \Gamma$ be a map with $g(0) = 0$ for which there exist nonnegative real numbers θ, θ_1, θ_2 and positive real numbers $p > 3$, $t < 1$ such that*

$$\begin{aligned}&\|f(a\alpha b + b\alpha a) - f(a)\alpha b - f(b)\alpha a - ag(\alpha)b - bg(\alpha)a - a\alpha f(b) - b\alpha f(a)\|\\ &\quad\le \theta(\|a\|^p + \|b\|^p),\end{aligned}$$

$$\|f(\mu(a+3b)) + f(\mu(a-3b)) - \mu(f(a+b) + f(a-b) + 16f(b) - 8f(2b))\| \le \theta_1(\|a\|^p + \|b\|^p),$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \le \theta_2(\|\alpha\|^t + \|\beta\|^t), a, b \in V, \alpha, \beta \in \Gamma, \mu \in \mathbb{T}$$

Then there exists a unique linear mapping k from Γ to Γ satisfying $\|g(\alpha) - k(\alpha)\| \le \frac{2\theta_2}{2 - 2^t}\|\alpha\|^t$, and there exists a unique k-derivation $D : V \to V$ such that

$$\|f(a) - D(a)\| \le \sup\Big\{\frac{\theta_1\|a\|^p}{|2|^{k(1-p)}} \mid k \in \mathbb{N}\cup\{0\}\Big\}, a \in V$$

Proof. Let $\phi : V\times V \to [0,\infty)$, $\varphi_1 : V\times V \to [0,\infty)$ and $\varphi_2 : \Gamma\times\Gamma \to [0,\infty)$ be functions such that $\phi(a,b) := \theta(\|a\|^p + \|b\|^p)$, $\varphi_1(a,b) := \theta_1(\|a\|^p + \|b\|^p)$ and $\varphi_2(\alpha,\beta) := \theta_2(\|\alpha\|^t + \|\beta\|^t)$ for all $a, b \in V$, $\alpha, \beta \in \Gamma$ and $\mu \in \mathbb{T}$. Then we get the required result by applying Theorem 3.1. □

The following result shows that under some conditions the superstability of k-derivations associated with the functional equation (23) is provided.

Corollary 3.2. *Suppose that V is a 2-torsion free prime Γ_N-Banach algebra over $\mathbb{C}$. Let $p, q, t, l, \theta, \theta_1, \theta_2$ be non-negative real numbers with $p+q > 3$, $0 < t+l \ne 1$ and let $f : V \to V$ be an odd mapping and $g : \Gamma \to \Gamma$ be a mapping such that*

$$\begin{aligned}&\|f(a\alpha b + b\alpha a) - f(a)\alpha b - f(b)\alpha a - ag(\alpha)b - bg(\alpha)a - a\alpha f(b) - b\alpha f(a)\|\\ &\quad\le \theta(\|a\|^p\|b\|^q),\end{aligned} \tag{36}$$

$$\begin{aligned}&\|f(\mu(a+3b)) + f(\mu(a-3b)) - \mu(f(a+b) + f(a-b) + 16f(b) - 8f(2b))\|\\ &\quad\le \theta_1(\|a\|^p\|b\|^q),\end{aligned} \tag{37}$$

$$\|g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta)\| \le \theta_2(\|\alpha\|^t\|\beta\|^l), a, b \in V, \alpha, \beta \in \Gamma, \mu \in \mathbb{T} \tag{38}$$

Then f is a k-derivation on V, where $k : \Gamma \to \Gamma$ is a linear map.

Proof. Putting $\beta = 0$, $\mu = 1$ in (38), we get $g(0) = 0$. If we put $a = 0$, $\mu = 1$ in (37), then we have $f(2b) = 2f(b)$ for all $b \in V$, and by induction we see that $f(b) = \dfrac{f(2^n b)}{2^n}$ for all $b \in V$ and $n \in \mathbb{N}$. Similarly, if we put $\beta = \alpha$ and $\mu = 1$ in (38), we have that $g(2\alpha) = 2g(\alpha)$ for all $\alpha \in \Gamma$, and by induction we conclude that $g(\alpha) = \dfrac{g(2^n \alpha)}{2^n}$ for all $\alpha \in \Gamma$ and $n \in \mathbb{N}$.

Now, if we put $\phi(a, b) := \theta(\|a\|^p \|b\|^q)$, $\varphi_1(a, b) := \theta_1(\|a\|^p \|b\|^q)$ and $\varphi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t \|\beta\|^l)$ for all $a, b \in V$, $\alpha, \beta \in \Gamma$ and $\mu \in \mathbb{T}$, then we get the desired result by using Theorem 3.1. □

Acknowledgement. The authors would like to thank the referees for the careful and detailed reading of the manuscript and valuable suggestions and comments.

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