MODIFIED FRAME-BASED RICHARDSON ITERATIVE METHOD AND ITS CONVERGENCE ACCELERATION BY CHEBYSHEV POLYNOMIALS

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Recently, there have been some developments regarding frame-based Richardson iteration and corresponding convergence acceleration by using Chebyshev polynomials and their respective algorithms. To obtain better rate of convergence, we deal with Richardson iteration with another preconditioner yielding the second power of earlier convergence rate formed by the bounds of the given frame. Afterward, we conduct Chebyshev acceleration on modified Richardson iteration to obtain a convergence rate which is much smaller than both earlier Chebyshev iteration and the new version of Richardson iteration.

Keywords: Hilbert space, frame, operator equation, iterative method, modified Richardson iteration, convergence acceleration, Chebyshev polynomials.

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1. Introduction and preliminaries

In this paper, we establish some iterative schemes to solve the operator equation

$$Lu = f, \tag{1}$$

where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. These schemes give rise to some methods which are induced by *Chebyshev polynomials* and they are based on *frames*. Recently, there have been major developments in the field of frame-based numerical iterative methods for solving operator equation of the form (1) which give rise to extraordinary convergence rates formed by bounds of the given frame [7, 8, 9]. In [7], one can see the development of frame-based Richardson iteration and corresponding convergence acceleration by using Chebyshev polynomials and their respective algorithms. In this direction, we can still express other formulations and algorithms whose convergence rate is explicitly the second power of that of already associated approaches [7]. To understand the privilege of the case, we point out the advantages of the discussions in each section with respect to the similar earlier studies. These further productivities originates from more efficient preconditioning of (1). If this is the case, a number of numerical iterative approaches can be discussed.

Among all numerical iterative approaches for solving operator equation (1), stationary Richardson iterative method plays an important role in numerical linear algebra since long ago [1]. Along with other methods, it is also traditionally used for iteratively solving elliptic partial differential equations [1]. The abstract procedure will be discussed in the following subsection.

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1.1. Richardson Iteration

In general, the stationary Richardson iteration of the equation (1) is of the form

$$u_{k+1} = u_k + a\,(f - Lu_k)\,, \quad k = 0, 1, 2, \ldots \tag{2}$$

where u_0 is an initial guess and $a > 0$ is a parameter to be chosen appropriately. One could observe that better convergence could be obtained if a varied with k. By suitable choice of the parameter a in (2), it is possible to improve the rate of convergence of the iteration. Such process is called *convergence acceleration* [10].

Technically speaking, the way of obtaining this is with the aid of Chebyshev polynomials. These polynomials have the important minimum property that makes them useful for convergence acceleration. These polynomials are defined by

$$c_n(x) = \begin{cases} \cos(ncos^{-1}(x)), & |x| \le 1 \\ \cosh(n\cosh^{-1}(x)) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^{-n}\right], & |x| > 1 \end{cases}, \tag{3}$$

which satisfy the following recurrence relations

$$c_0(x) = 1, \quad c_1(x) = x, \quad c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x), \quad \forall n \ge 2. \tag{4}$$

In the first place, we state the following fact about these polynomials which will be used by us later on.

Lemma 1.1 ([4])**.** *Given any constants $a \le b \le 1$, set $P_n(x) = \frac{c_n\left(\frac{2x-a-b}{b-a}\right)}{c_n\left(\frac{2-a-b}{b-a}\right)}$ for $x \in [a,b]$, then*

$$\max_{a\le x\le b} \mid P_n(x) \mid \le \max_{a\le x\le b} \mid Q_n(x) \mid,$$

for all polynomials Q_n of degree n with the condition $Q_n(1) = 1$. Furthermore,

$$\max_{a\le x\le b} \mid P_n(x) \mid = \frac{1}{c_n\left(\frac{2-a-b}{b-a}\right)}.$$

Later in the third section, we express more applicable properties of these polynomials, using which, we illustrate how they serve as convergence acceleration. For this purpose, to take all the preliminary steps, we should deal with the basic notion of *frames*. The study of this concept is given below.

1.2. Frames

To date, frames have introduced themselves as a standard mathematical framework in applied mathematics, computer science, and engineering as a means to derive redundant, yet stable decompositions of a signal for analysis or transmission [3]. Beside these remarkable applications of frames, they can also be treated as a standard notion in numerical iterative schemes for solving operator equations of the type (1). For more details, we refer the reader to [2, 7, 8, 9].

In this section, following Casazza [3], we provide a brief review of the basics of frame theory upon which the subsequent sections are based. Let us first define the notion of the frame.

Definition 1.1. *A family of vectors $\{f_i\}_{i\in I}$ is a frame for the Hilbert space H if there are constants $0 < A \le B < \infty$ so that for all $f \in H$*

$$A \parallel f \parallel^2 \le \sum_{i\in I} \mid \langle f, f_i \rangle \mid^2 \le B \parallel f \parallel^2 .$$

A, B are called the lower (respectively, upper) frame bounds for the frame. If $A = B$ this is an A-tight frame.

If $\{f_i\}_{i\in I}$ is a frame for H with frame bounds A, B we define the **analysis operator** $T : H \to \ell_2(I)$ to be

$$T(f) = \sum_{i\in I} \langle f, f_i\rangle e_i, \ \ \forall f \in H,$$

where $\{e_i\}_{i\in I}$ is the natural orthonormal basis of $\ell_2(I)$. The adjoint of the analysis operator is the **synthesis operator** which is given by $T^*(e_i) = f_i$ or

$$T^*(\{c_i\}_i) = \sum_{i\in I} c_i f_i, \ \ \forall \{c_i\}_{i\in I} \in \ell_2(I).$$

To check this, for each $f \in H$ and $\{c_i\}_{i\in I}$, we see that

$$\begin{aligned}\langle T(f), \{c_i\}_i\rangle &= \left\langle \sum_{j\in I} \langle f, f_j\rangle e_j, \{c_i\}_i \right\rangle \\ &= \sum_{j\in I} \langle f, f_j\rangle \langle e_j, \{c_i\}_i\rangle \\ &= \sum_{j\in I} \langle f, f_j\rangle \bar{c}_j = \sum_{j\in I} \langle f, c_j f_j\rangle = \left\langle f, \sum_{j\in I} c_j f_j \right\rangle.\end{aligned}$$

The **frame operator** for the frame is $S = T^*T : H \to H$ given by

$$S(f) = T^*T(f) = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

A direct calculation now yields

$$\langle Sf, f\rangle = \sum_{i\in I} | \langle f, f_i\rangle |^2.$$

So the frame operator is a **positive**, **self-adjoint**, and **invertible** operator on H. Moreover,

$$AI \le S \le BI,$$

where I denotes the identity operator on H. Thus, the family $\{S^{-1}f_i\}_{i\in I}$ is also a frame for H called the **canonical dual frame**. In general, the following theorem holds true for all frames.

Theorem 1.1 ([5]). *Assume that $\{f_i\}_{i\in I}$ is a frame for H and that $L : H \to H$ is a bounded surjective operator. Then $\{Lf_i\}_{i\in I}$ is also a frame for H.*

We can **reconstruct** vectors in the space by the canonical dual frame as

$$f = SS^{-1}f = \sum_{i\in I} \langle S^{-1}f, f_i\rangle f_i = \sum_{i\in I} \langle f, S^{-1}f_i\rangle f_i.$$

The same reasoning yields $f = \sum_{i\in I} \langle f, f_i\rangle S^{-1}f_i$.

2. Modified Richardson Iterative Method

This section is devoted to the study of the abstract modified Richardson iterative method for solving the equation (1) by using a given frame $\{f_i\}_{i\in I}$ with frame bounds A, B. The important point to note here is that the convergence rate obtained in frame-based Richardson iterative method studied in [7] equals $\frac{B-A}{A+B}$, while in modified version it is squared as $\left(\frac{B-A}{A+B}\right)^2$, which shows more efficiency of this version. To begin with, we state the following lemma.

Lemma 2.1. *Let $\{f_i\}_{i\in I}$ be a frame for H with frame operator S. Suppose that A, B are the lower and upper bounds of the frame $\{Lf_i\}_{i\in I}$. Then*

$$\left\| I - \frac{4}{A+B}\left(I - \frac{1}{A+B}LSL\right)LSL \right\| \leq \left(\frac{B-A}{A+B}\right)^2. \tag{5}$$

Proof. First of all, we note that since L is self-adjoint, for each $f \in H$ the frame operator S' of $\{Lf_i\}_{i\in I}$ is obtained as follows.

$$S'f = \sum_{i\in I} \langle f, Lf_i\rangle Lf_i = \sum_{i\in I} \langle Lf, f_i\rangle Lf_i = L\left(\sum_{i\in I} \langle Lf, f_i\rangle f_i\right) = LSLf.$$

Hence, $S' = LSL$ and thus for all $v \in H$ we have

$$\begin{aligned}\left\langle \left(I - \frac{2}{A+B}LSL\right)v, v \right\rangle &= \|v\|^2 - \frac{2}{A+B}\langle LSLv, v\rangle \\ &= \|v\|^2 - \frac{2}{A+B}\langle S'v, v\rangle \\ &\leq \|v\|^2 - \frac{2A}{A+B}\|v\|^2 \\ &= \left(\frac{B-A}{A+B}\right)\|v\|^2.\end{aligned}$$

Similarly, we can obtain

$$-\left(\frac{B-A}{A+B}\right)\|v\|^2 \leq \left\langle \left(I - \frac{2}{A+B}LSL\right)v, v \right\rangle,$$

that yields altogether

$$\left\| I - \frac{2}{A+B}LSL \right\| \leq \left(\frac{B-A}{A+B}\right). \tag{6}$$

Therefore, by starting from left-hand side of (5) and applying (6), we see

$$\begin{aligned}\left\| I - \frac{4}{A+B}\left(I - \frac{1}{A+B}LSL\right)LSL \right\| &= \left\| I - \frac{4}{A+B}LSL + \frac{4}{(A+B)^2}(LSL)^2 \right\| \\ &= \left\| \left(I - \frac{2}{A+B}LSL\right)^2 \right\| \\ &\leq \left\| I - \frac{2}{A+B}LSL \right\|^2 \leq \left(\frac{B-A}{A+B}\right)^2,\end{aligned}$$

which completes the proof. □

Now, by using this lemma we can design the following iterative method based on Richardson iteration.

Theorem 2.1. *Let $\{f_i\}_{i\in I}$ be a frame for H with frame operator S and let A, B be the frame bounds of the frame $\{Lf_i\}_{i\in I}$. Then for any initial guess u_0 to the solution of (1), the sequence $\{u_k\}$ defined by*

$$u_k = u_{k-1} + \frac{4}{A+B}\left(I - \frac{1}{A+B}LSL\right)LS\left(f - Lu_{k-1}\right), \tag{7}$$

converges to the exact solution u of equation (1) with convergence rate $\left(\frac{B-A}{A+B}\right)^2$.

Proof. By the definition of u_k, it is obvious that

$$\begin{aligned} u-u_k &= u-u_{k-1}-\frac{4}{A+B}\left(I-\frac{1}{A+B}LSL\right)LS\,(f-Lu_{k-1}) \\ &= u-u_{k-1}-\left(\frac{4}{A+B}-\frac{4}{(A+B)^2}LSL\right)LS\,(f-Lu_{k-1}) \\ &= u-u_{k-1}-\left(\frac{4}{A+B}-\frac{4}{(A+B)^2}LSL\right)LSL\,(u-u_{k-1}) \\ &= \left(I-\frac{4}{A+B}LSL+\frac{4}{(A+B)^2}(LSL)^2\right)(u-u_{k-1}) \\ &= \left(I-\frac{2}{A+B}LSL\right)^2(u-u_{k-1}). \end{aligned}$$

Therefore

$$\|u-u_k\| = \left\|\left(I-\frac{2}{A+B}LSL\right)^2(u-u_{k-1})\right\| \leq \left\|I-\frac{2}{A+B}LSL\right\|^2\|u-u_{k-1}\|,$$

and the result follows from Lemma 2.1. □

In the sequel, we summarize this section with an algorithm which generates an approximate solution to equation (1) with prescribed accuracy and based on Richardson iteration of Theorem 2.1. For this, let $\{f_i\}_{i\in I}$ be a frame for H with frame operator S, let A, B be the frame bounds of $\{Lf_i\}_{i\in I}$ and let m be the lower bound of the linear operator L.

Algorithm1 $[L, m, \epsilon, A, B, S] \to u_\epsilon$

***(i)*:** Let $\alpha_0 = \frac{B-A}{A+B}$

***(ii)*:** $k := 0,\ u_k := 0$

***(iii)*:** $k := k+1$

(1) $u_k = u_{k-1} + \frac{4}{A+B}\left(I-\frac{1}{A+B}LSL\right)LS\,(f-Lu_{k-1})$

(2) $\alpha_k := (\alpha_0)^k\frac{\|f\|}{m}$

(iv)*:** If $\alpha_k \leq \epsilon$ stop and set $u_\epsilon := u_k$, if else Go to ***(iii).

3. Convergence Acceleration by Using Chebyshev Polynomials

By the setting of previous section, we conduct here the discussion of convergence acceleration. In fact, with the aid of Chebyshev polynomials the following convergence rate will be obtained $\frac{2\sigma^n}{1+\sigma^{2n}}$, where $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$.
To begin with the discussion, let $h_n = \sum_{k=1}^n a_{n_k}u_k$ be a polynomial such that $\sum_{k=1}^n a_{n_k} = 1$, where $\{u_k\}_{k\in\mathbb{N}}$ is as in Theorem 2.1. Note that if $u_1 = u_2 = \ldots = u_n = u$, then the condition $\sum_{k=1}^n a_{n_k} = 1$ implies $h_n = u$. Therefore, based on the proof of Theorem 2.1, we obtain

$$\begin{aligned} u-h_n = \sum_{k=1}^n a_{n_k}u - \sum_{k=1}^n a_{n_k}u_k &= \sum_{k=1}^n a_{n_k}(u-u_k) \\ &= \sum_{k=1}^n a_{n_k}\left(I-\frac{2}{A+B}LSL\right)^{2k}(u-u_0). \end{aligned}$$

By setting $R = \left(I-\frac{2}{A+B}LSL\right)^2$ and $Q_n(x) = \sum_{k=1}^n a_{n_k}x^k$, one sees

$$u-h_n = \sum_{k=1}^n a_{n_k}R^k(u-u_0) = Q_n(R)(u-u_0). \tag{8}$$

On the one hand, Lemma 2.1 implies that the spectrum of R is a subset of the interval $[-\alpha, \alpha]$ where $\alpha = \left(\frac{B-A}{B+A}\right)^2$. On the other hand, since L is an invertible and self-adjoint operator, and also S is positive definite, then LSL is a positive definite operator. Hence, in view of (8) the spectral theorem follows altogether

$$\|u - h_n\| \leq \|Q_n(R)\| \|u - u_0\| \leq \max_{|x|\leq\alpha} \mid Q_n(x) \mid \|u - u_0\|. \tag{9}$$

In order to minimize the error vector $\|u - h_n\|$, we have to find

$$\min_{Q_n \in \mathbb{Q}_n} \max_{|x|\leq\alpha} \mid Q_n(x) \mid, \tag{10}$$

where $\mathbb{Q}_n := \{Q(x) : \deg Q = n, Q(1) = 1\}$. By Lemma 1.1, this minimization problem can be solved in terms of the Chebyshev polynomials. To this end, we first set $a = -\alpha$ and $b = \alpha$ in Lemma 1.1, to obtain

$$P_n(x) = \frac{c_n\left(\frac{2x+\alpha-\alpha}{\alpha+\alpha}\right)}{c_n\left(\frac{2+\alpha-\alpha}{\alpha+\alpha}\right)} = \frac{c_n\left(\frac{x}{\alpha}\right)}{c_n\left(\frac{1}{\alpha}\right)}, \tag{11}$$

which solves (10).

Proposition 3.1. *The polynomial h_n satisfies the following recurrence relation.*

$$h_n = \beta_n \left[h_{n-1} - h_{n-2} + \frac{4}{A+B}\left(I - \frac{1}{A+B}LSL\right) LS\,(f - Lh_{n-1})\right] + h_{n-2},$$

where $\beta_n = \frac{\frac{2}{\alpha}c_{n-2}\left(\frac{1}{\alpha}\right)}{c_n\left(\frac{1}{\alpha}\right)}$.

Proof. Repeated combination of (11) with the relation (4), for $n \geq 2$, gives

$$\begin{aligned} c_n\left(\frac{1}{\alpha}\right) P_n(x) = c_n\left(\frac{x}{\alpha}\right) &= \frac{2x}{\alpha} c_{n-1}\left(\frac{x}{\alpha}\right) - c_{n-2}\left(\frac{x}{\alpha}\right) \\ &= \frac{2x}{\alpha} c_{n-1}\left(\frac{1}{\alpha}\right) P_{n-1}(x) - c_{n-2}\left(\frac{1}{\alpha}\right) P_{n-2}(x). \end{aligned}$$

Replacing x by R, and by applying the resulting operator identity on $(u - u_0)$, we get

$$\begin{aligned} c_n\left(\frac{1}{\alpha}\right) P_n(R)(u - u_0) =& \frac{2R}{\alpha} c_{n-1}\left(\frac{1}{\alpha}\right) P_{n-1}(R)(u - u_0) \\ & - c_{n-2}\left(\frac{1}{\alpha}\right) P_{n-2}(R)(u - u_0). \end{aligned}$$

Since $P_n(x)$ is the solution of minimization problem (10), by virtue of (8) one recovers above equation as

$$c_n\left(\frac{1}{\alpha}\right)(u - h_n) = \frac{2}{\alpha} c_{n-1}\left(\frac{1}{\alpha}\right) R(u - h_{n-1}) - c_{n-2}\left(\frac{1}{\alpha}\right)(u - h_{n-2}).$$

Writing $R = \left(I - \frac{2}{A+B}LSL\right)^2$, above relation induces

$$\begin{aligned} c_n\left(\frac{1}{\alpha}\right) u - c_n\left(\frac{1}{\alpha}\right) h_n = & \frac{2}{\alpha} c_{n-1}\left(\frac{1}{\alpha}\right)\left(I - \frac{2}{A+B}LSL\right)^2 (u - h_{n-1}) \\ & - c_{n-2}\left(\frac{1}{\alpha}\right)(u - h_{n-2}), \end{aligned}$$

or equivalently,

$$c_n\left(\frac{1}{\alpha}\right)u - c_n\left(\frac{1}{\alpha}\right)h_n = \frac{2}{\alpha}c_{n-1}\left(\frac{1}{\alpha}\right)u + \frac{2}{\alpha}c_{n-1}\left(\frac{1}{\alpha}\right)[-h_{n-1} - \left(\frac{4}{A+B}LSL - \frac{4}{(A+B)^2}(LSL)^2\right) (u-h_{n-1})] - c_{n-2}\left(\frac{1}{\alpha}\right)u + c_{n-2}\left(\frac{1}{\alpha}\right)h_{n-2}.$$

Repeated application of the relation (4), for $n \geq 2$, leads to

$$c_n\left(\frac{1}{\alpha}\right)h_n = \frac{2}{\alpha}c_{n-1}\left(\frac{1}{\alpha}\right)\left(h_{n-1} + \left(\frac{4}{A+B}LSL - \frac{4}{(A+B)^2}(LSL)^2\right) (u-h_{n-1})\right) - c_{n-2}\left(\frac{1}{\alpha}\right)h_{n-2}.$$

Therefore,

$$h_n = \frac{2}{\alpha}\frac{c_{n-1}(\frac{1}{\alpha})}{c_n(\frac{1}{\alpha})}\left[h_{n-1} + \left(\frac{4}{A+B}LSL - \frac{4}{(A+B)^2}(LSL)^2\right)(u-h_{n-1})\right] - \frac{c_{n-2}(\frac{1}{\alpha})}{c_n(\frac{1}{\alpha})}h_{n-2}. \tag{12}$$

On the other hand, by (4) we have $1-\beta_n = 1 - \frac{2}{\alpha}\frac{c_{n-1}(\frac{1}{\alpha})}{c_n(\frac{1}{\alpha})} = -\frac{c_{n-2}(\frac{1}{\alpha})}{c_n(\frac{1}{\alpha})}$. In this case, we can rewrite (12) as

$$h_n = \beta_n\left[h_{n-1} + \left(\frac{4}{A+B}LSL - \frac{4}{(A+B)^2}(LSL)^2\right)(u-h_{n-1})\right] + (1-\beta_n)h_{n-2},$$

or equivalently,

$$h_n = \beta_n\left[h_{n-1} - h_{n-2} + \left(\frac{4}{A+B}LSL - \frac{4}{(A+B)^2}(LSL)^2\right)(u-h_{n-1})\right] + h_{n-2},$$

which yields the following as we desired

$$h_n = \beta_n\left[h_{n-1} - h_{n-2} + \frac{4}{A+B}\left(I - \frac{1}{(A+B)^2}LSL\right)LS(f - Lh_{n-1})\right] + h_{n-2},$$

as we desired. ☐

To continue, we consider the following auxiliary lemma.

Lemma 3.1. *If* $\beta_n = \frac{\frac{2}{\alpha}c_{n-2}\left(\frac{1}{\alpha}\right)}{c_n\left(\frac{1}{\alpha}\right)}$, *then the following relation holds*

$$\beta_n = \left(1 - \frac{\alpha^2}{4}\beta_{n-1}\right)^{-1}.$$

Proof. By the recursive formula (12), we have

$$\beta_n = \left(\frac{\alpha}{2}\frac{c_n(\frac{1}{\alpha})}{c_{n-1}(\frac{1}{\alpha})}\right)^{-1} = \left(\frac{\alpha}{2}\frac{\frac{2}{\alpha}c_{n-1}(\frac{1}{\alpha}) - c_{n-2}(\frac{1}{\alpha})}{c_{n-1}(\frac{1}{\alpha})}\right)^{-1}$$
$$= \left(\frac{\alpha}{2}\frac{\frac{2}{\alpha}c_{n-1}(\frac{1}{\alpha})}{c_{n-1}(\frac{1}{\alpha})} - \frac{\alpha}{2}\frac{c_{n-2}(\frac{1}{\alpha})}{c_{n-1}(\frac{1}{\alpha})}\right)^{-1} = \left(1 - \frac{\alpha}{2}\frac{\alpha}{2}\frac{\frac{2}{\alpha}c_{n-2}(\frac{1}{\alpha})}{c_{n-1}(\frac{1}{\alpha})}\right)^{-1},$$

and, via the definition of β_{n-1}, this is precisely the assertion of the lemma. ☐

Therefore, based on above lemma, we can establish an algorithm for approximately solving operator equation (1). For this, let $\{f_i\}_{i\in I}$ be a frame for H with frame operator S and let A, B be the bounds of the frame $\{Lf_i\}_{i\in I}$.

Algorithm2 $[L, \epsilon, f, S, A, B, m] \to u_\epsilon$

***(i)*:** put $\alpha = \left(\frac{B-A}{B+A}\right)^2$, $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$,

set $h_0 = 0$, $h_1 = \frac{4}{A+B}\left(I - \frac{1}{(A+B)^2}\right) LSf$, $\beta_1 = 2$, $n = 1$

***(ii)*:** While $\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m} > \epsilon$, Do

***(1)*:** $n = n+1$

***(2)*:** $\beta_n = (1 - \frac{\alpha^2}{4}\beta_{n-1})^{-1}$

***(3)*:** $h_n = \beta_n\left(h_{n-1} - h_{n-2} + \frac{4}{A+B}\left(I - \frac{1}{(A+B)^2}LSL\right)LS(f - Lh_{n-1})\right) + h_{n-2}$

***(iii)*:** $u_\epsilon := h_n$.

3.1. Convergence Analysis

We verify, here, the convergence of the above algorithm. To this end, we consider the following theorem.

Theorem 3.1. *If u is the exact solution of the equation (1), then the approximate solution h_n satisfies*

$$\|u - h_n\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m},$$

where m is the lower bound of L. Also, the output u_ϵ of Algorithm2 satisfies $\|u - u_\epsilon\| < \epsilon$.

Proof. Combining Lemma 1.1 with relation (9), and by taking $u_0 = h_0 = 0$, we obtain

$$\|u - h_n\| \le \frac{1}{c_n\left(\frac{1}{\alpha}\right)}\|u - u_0\| = \frac{1}{c_n\left(\frac{1}{\alpha}\right)}\|u\| \le \frac{1}{c_n\left(\frac{1}{\alpha}\right)}\frac{\|f\|}{m}. \tag{13}$$

On the other hand, by (3) and by binomial expansion we have

$$\begin{aligned}
c_n\left(\frac{1}{\alpha}\right) &= c_n\left(\left(\frac{B+A}{B-A}\right)^2\right)\\
&= \frac{1}{2}\left[\left(\left(\frac{B+A}{B-A}\right)^2 + \sqrt{\left(\frac{B+A}{B-A}\right)^4 - 1}\right)^n\right.\\
&\qquad \left. + \left(\left(\frac{B+A}{B-A}\right)^2 + \sqrt{\left(\frac{B+A}{B-A}\right)^4 - 1}\right)^{-n}\right]\\
&= \frac{1}{2}\left[\left(\frac{(B+A)^2}{(B-A)^2} + \sqrt{\frac{8AB(A^2+B^2)}{(B-A)^4}}\right)^n\right.\\
&\qquad \left. + \left(\frac{(B+A)^2}{(B-A)^2} + \sqrt{\frac{8AB(A^2+B^2)}{(B-A)^4}}\right)^{-n}\right]\\
&= \frac{1}{2}\left[\left(\frac{1}{(B-A)^2}\left((B+A)^2 + \sqrt{8AB(A^2+B^2)}\right)\right)^n\right.\\
&\qquad \left. + \left(\frac{1}{(B-A)^2}\left((B+A)^2 + \sqrt{8AB(A^2+B^2)}\right)\right)^{-n}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\left(\frac{1}{B^2+A^2-2AB}\left(A^2+B^2+2AB+2\sqrt{2AB}\sqrt{A^2+B^2}\right)\right)^{n} + \left(\frac{1}{B^2+A^2-2AB}\left(A^2+B^2+2AB+2\sqrt{2AB}\sqrt{A^2+B^2}\right)\right)^{-n}\right]$$

$$= \frac{1}{2}\left[\left(\frac{A^2+B^2+2AB+2\sqrt{2AB}\sqrt{A^2+B^2}}{\left(\sqrt{B^2+A^2}-\sqrt{2AB}\right)\left(\sqrt{B^2+A^2}+\sqrt{2AB}\right)}\right)^{n} + \left(\frac{A^2+B^2+2AB+2\sqrt{2AB}\sqrt{A^2+B^2}}{\left(\sqrt{B^2+A^2}-\sqrt{2AB}\right)\left(\sqrt{B^2+A^2}+\sqrt{2AB}\right)}\right)^{-n}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\left(\sqrt{A^2+B^2}+\sqrt{2AB}\right)^2}{\left(\sqrt{B^2+A^2}-\sqrt{2AB}\right)\left(\sqrt{B^2+A^2}+\sqrt{2AB}\right)}\right)^{n} + \left(\frac{\left(\sqrt{A^2+B^2}+\sqrt{2AB}\right)^2}{\left(\sqrt{B^2+A^2}-\sqrt{2AB}\right)\left(\sqrt{B^2+A^2}+\sqrt{2AB}\right)}\right)^{-n}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sqrt{A^2+B^2}+\sqrt{2AB}}{\sqrt{A^2+B^2}-\sqrt{2AB}}\right)^{n} + \left(\frac{\sqrt{A^2+B^2}+\sqrt{2AB}}{\sqrt{A^2+B^2}-\sqrt{2AB}}\right)^{-n}\right]$$

$$= \frac{1}{2}\left(\frac{1}{\sigma^n}+\sigma^n\right) = \frac{1+\sigma^{2n}}{2\sigma^n},$$

where $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$. This equality together with (13) yields the last statement of the theorem $\|u-u_\epsilon\| < \epsilon$. □

4. **Conclusions**

We introduced and studied a new class of iterative methods derived by frame theory for solving operator equation $Lu = f$ where L is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. These methods give rise to two special convergence rates. The first one, deduced from modified Richardson iterative method, is as $\alpha = \left(\frac{B-A}{A+B}\right)^2$, where A, B are lower and upper bounds of given frame, and equals the second power of that of Richardson iteration introduced in [7]. The second one, derived by Chebyshev acceleration on modified Richardson iteration, has the following rate of convergence

$$\frac{2\sigma^n}{1+\sigma^{2n}},$$

where $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$. Note that, this quantity in associated Chebyshev acceleration type in [7] has the same formula as in (4), but for σ we observe that $\sigma = \frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}$. From this, we conclude here much faster convergence rate. Subsequently in this area, since for any $n > 1$ we have $\frac{2\sigma^n}{1+\sigma^{2n}} \le \alpha^n$, thus an accelerated rate of convergence in comparison with that of modified Richardson iterative method is achieved.

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