In this paper, we present a new application of the Exp–function method to carry out the integration of nonlinear evolution equations in terms of multi–wave and rational solutions. To elucidate the solution procedure, we analytically investigate the Sharma–Tasso–Olver equation and the fifth–order Korteweg de Vries equation. Unlike Hirota’s method, our procedure does not require the bilinear formalism of the equations studied.

Keywords: Exp–function method; Multi–wave solution; Rational solution; Sharma–Tasso–Olver equation; Fifth–order Korteweg de Vries equation.

1. Introduction

Nonlinear evolution equations (NEEs) have been of fundamental importance in the study of applied physical and mathematical sciences. They are crucial for obtaining an understanding of the physical sciences, as well as the biological and social sciences. Thus, exactly solving NEEs has gained increasing importance. Nowadays, some modern analytic methods are available for tackling NEEs. To make mention of some; tanh function method [1], Adomian decomposition method [2], variational iteration method [3], first integral method [4], Exp–function method [5], homotopy perturbation method [6], (G'/G)–expansion method [7], multiple–exp function method [8] and so forth.

It is an important fact that one should be aware of the limitations of these methods and there is no guarantee that any of these techniques will succeed for a specific nonlinear problem. Among the others, the Exp–function method has received more attention and consequently it has been adapted, extended, and generalized to various kinds of nonlinear problems; for example, NEEs with variable coefficients [9], multi–dimensional equations [10–12], differential–difference equations [13, 14], coupled NEEs [15], stochastic equations [16], n–soliton solutions [17–19], rational solutions [20–22]. Hence, the Exp–function method provide a valuable addition to the wave theory.

On the other hand, traveling waves of NEEs may be coupled with different frequencies and different velocities. Multi–wave solutions are crucial in the sense
that they may sometimes be converted into a single soliton of very high energy that propagates over large domains of space without dispersion. Therefore, an extremely destructive wave may be produced. The tsunami is a good example for this kind of phenomena. As is well known, Hirota’s method [23] can be used to construct multi–wave solutions if the equation considered can be transformed into a bilinear form. However, the existence of the bilinear form cannot be guaranteed or it may not be known.

The main objective of this work is to show the applicability of the Exp–function method to two important equations of mathematical physics having distinct physical structures (namely, the Sharma–Tasso–Olver equation and the fifth–order Korteweg de Vries equation) for rational and travelling wave solutions with distinct velocities and distinct frequencies. The rest of this paper is organized as follows. In Section 2, we briefly describe our method. In Sections 3 and 4, we analyze our problems. In Section 5, we give some concluding remarks.

2. The Exp–function method

For a given NEE, say, in two variables $x$ and $t$,

$$P(u,u_{x},u_{t},u_{x},u_{tx},u_{tx},\ldots)=0,$$

where $P$ is a polynomial in its arguments, the Exp–function method is based on the assumption that its solutions can be expressed as

$$u(x,t) = \sum_{i=0}^{m} a_i \exp(i\xi) + \sum_{j=0}^{n} b_j \exp(j\xi), \quad \xi = kx + wt + \delta,$$

where $m$ and $n$ are positive integers to be determined; $a_i$, $b_j$, $k$ and $w$ are arbitrary constants to be specified; $\delta$ is the phase shift. Substituting the function (2) into Eq. (1) and balancing the highest–order terms, one can determine the constants $m$ and $n$.

To seek for $N$–wave solutions to Eq. (1), the function (2) can be generalized as

$$u(x,t) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} a_{ij} \exp(i_{1}\xi_{1} + i_{2}\xi_{2}), \quad \xi_{i} = k_{i}x + w_{i}t + \delta_{i}, \quad i=1,2,$$

which corresponds to the case $N=2$; and

$$u(x,t) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \sum_{l=0}^{m_3} a_{ijl} \exp(i_{1}\xi_{1} + i_{2}\xi_{2} + i_{3}\xi_{3}), \quad \xi_{i} = k_{i}x + w_{i}t + \delta_{i}, \quad i=1,2,3,$$

which corresponds to the case $N=3$; and so on.

To obtain a rational solution for Eq. (1), we modify the function (2) as
Rational and multi–wave solutions to nonlinear evolution equations [\ldots]–function method

\[ u(x,t) = \sum_{i=0}^{\infty} a_i \left( \mu_i \exp(\xi) + \mu_2 \xi \right)^i, \quad \xi = kx + wt + \delta, \quad \text{(5)} \]

where \( \mu_i \) and \( \mu_2 \) are two embedded constants. It is obvious that when \( \mu_1 = 1 \) and \( \mu_2 = 0 \), the function (5) turns out to be the function (2).

**Remark 1.** The positive integers \( m \) and \( n \) in the function (2) are determined by balancing the linear and nonlinear terms of highest order in Eq. (1). However, as far as we could verify through the research literature, performing this procedure takes a lot of effort and time consuming. Recently, Ali [24] proved that the balancing step in the Exp–function method is redundant. Thus, taking the function (2) into account directly and assigning arbitrary values to the constants \( m \) and \( n \) will make laborious calculations unnecessary.

### 3. The Sharma–Tasso–Olver equation

Let us consider the so–called Sharma–Tasso–Olver (STO) equation which reads

\[ u_t + 3aa^2 u_x + 3aa^2 u_{xx} + 3aa^2 u_{xxx} + au_{xxxx} = 0, \quad \text{(6)} \]

where \( \alpha \) is a nonzero constant, and \( u = u(x,t) \). We suppose that Eq. (6) admits a solution of the form

\[ u(x,t) = \frac{a_i \exp(\xi)}{1 + b_i \exp(\xi)}, \quad \xi = kx + wt + \delta, \quad \text{(7)} \]

which is embedded in (2). Substituting (7) into Eq. (6), we get the relation

\[ (1 + b_i \exp(\xi)) \sum_{n=1}^{\infty} C_n \exp(n\xi) = 0, \quad \text{(8)} \]

where

\[ C_1 = \alpha a_k^3 + a^w, \]
\[ C_2 = 6\alpha a_k^3 k^2 - 4\alpha a_k^3 k^3 + 2a_k b_1 w, \]
\[ C_3 = 3\alpha a_k^3 k - 3\alpha a_k^3 b_1 k^2 + \alpha a_k b_1^3 k^3 + a_k b_1^2 w. \]

Then, solving the system \( C_n = 0 \) \( (n = 1, 2, 3) \) simultaneously, we obtain the solution set

\[ w = -\alpha k^3, \quad a_i = k b_1, \quad \text{(9)} \]

which yields a one–wave solution to Eq. (6) as

\[ u_i(x,t) = \frac{b_k \exp(\xi k - \alpha k^3 t + \delta)}{1 + b_i \exp(\xi k - \alpha k^3 t + \delta)}, \quad \text{(10)} \]

where \( k, b, \) and \( \delta \) remain arbitrary.
3.1. Two–wave solutions

Assume that Eq. (6) admits a solution of the form

$$u(x,t) = \frac{a_{i0} \exp(\xi_i) + a_{0i} \exp(\xi_0) + a_{1i} \exp(\xi_1 + \xi_2) + a_{i1} \exp(\xi_i + \xi_2) + a_{10i} \exp(\xi_1 + \xi_3) + a_{0i1} \exp(\xi_0 + \xi_2) + a_{i01} \exp(\xi_i + \xi_3) + a_{i10} \exp(\xi_i + \xi_3) + a_{101} \exp(\xi_1 + \xi_3) + a_{110} \exp(\xi_1 + \xi_3)}{1 + b_{i0} \exp(\xi_i) + b_{0i} \exp(\xi_0) + b_{1i} \exp(\xi_1 + \xi_2) + b_{i1} \exp(\xi_i + \xi_2) + b_{10i} \exp(\xi_1 + \xi_3) + b_{i01} \exp(\xi_i + \xi_3) + b_{i10} \exp(\xi_i + \xi_3) + b_{101} \exp(\xi_1 + \xi_3)}, \quad (11)$$

It is clear that the function (11) is embedded in (3). Substituting (11) into Eq. (6), we obtain the relation

$$\left(1 + b_{i0} \exp(\xi_i) + b_{0i} \exp(\xi_0) + b_{1i} \exp(\xi_1 + \xi_2) + b_{i1} \exp(\xi_i + \xi_2) + b_{10i} \exp(\xi_1 + \xi_3) + b_{i01} \exp(\xi_i + \xi_3) + b_{i10} \exp(\xi_i + \xi_3) + b_{101} \exp(\xi_1 + \xi_3)\right)^n \sum_{i=0}^n \sum_{j=0}^n C_{ij} \exp(i\xi_i + j\xi_2) = 0, \quad (12)$$

where $C_{i0} = C_{0i} = C_{44} = 0$. Hence, solving the system $C_{ij} = 0 \ (0 \leq i, j \leq 4)$ simultaneously, we obtain the solution set

$$w_l = -\alpha k_l^1, \quad w_i = -\alpha k_i^2, \quad b_i = 0, \quad a_{i0} = k_i h_{0i}, \quad a_{0i} = k_i h_{i0}, \quad a_{i1} = k_i h_{i1}, \quad a_{10i} = k_i h_{i0}, \quad (13)$$

which gives a two–wave solution to Eq. (6) as

$$u_2(x,t) = \frac{k_i h_{i0} \exp(k_i x - \alpha k_i^1 t + \delta_i) + k_i h_{0i} \exp(k_i x - \alpha k_i^1 t + \delta_i)}{1 + b_{i0} \exp(k_i x - \alpha k_i^1 t + \delta_i) + b_{0i} \exp(k_i x - \alpha k_i^1 t + \delta_i) + b_{i1} \exp(k_i x - \alpha k_i^1 t + \delta_i)}, \quad (14)$$

where $b_{i0}$, $b_{0i}$, $k_1$, $k_2$, $\delta_1$, and $\delta_2$ remain arbitrary.

3.2. Three–wave solutions

Assume that Eq. (6) admits a solution of the form

$$u(x,t) = \frac{a_{i00} \exp(\xi_i) + a_{i01} \exp(\xi_0) + a_{i10} \exp(\xi_1) + a_{i011} \exp(\xi_0) + a_{i02} \exp(\xi_0) + a_{i11} \exp(\xi_1 + \xi_2) + a_{10i} \exp(\xi_1 + \xi_2) + a_{110} \exp(\xi_1 + \xi_3) + a_{111} \exp(\xi_1 + \xi_3) + a_{101} \exp(\xi_1 + \xi_3)}{1 + b_{i00} \exp(\xi_i) + b_{i01} \exp(\xi_0) + b_{i10} \exp(\xi_1) + b_{i11} \exp(\xi_1 + \xi_2) + b_{i011} \exp(\xi_0) + b_{i02} \exp(\xi_0) + b_{i111} \exp(\xi_1 + \xi_3) + b_{i101} \exp(\xi_1 + \xi_3) + b_{i110} \exp(\xi_1 + \xi_3)}, \quad (15)$$

where $\xi_i = k_i x + w_it + \delta_i \quad (l = 1, 2, 3)$.

Obviously, the function (15) is embedded in (4). After substituting (15) into Eq. (6) and making similar manipulations, we get the solution set of the resultant algebraic system as

$$w_i = -\alpha k_i^1, \quad w_i = -\alpha k_i^2, \quad w_i = -\alpha k_i^3, \quad a_{100} = k_i h_{0i}, \quad a_{010} = k_i h_{0i}, \quad a_{001} = k_i h_{0i}, \quad b_{101} = 0, \quad b_{010} = 0, \quad a_{011} = 0, \quad a_{101} = 0, \quad a_{110} = 0, \quad a_{111} = 0, \quad (16)$$

which leads to a three–wave solution to Eq. (6) as

$$u_3(x,t) = \frac{k_{i00} \exp(k_i x - \alpha k_i^1 t + \delta_i) + k_{i01} \exp(k_i x - \alpha k_i^1 t + \delta_i) + k_{i10} \exp(k_i x - \alpha k_i^1 t + \delta_i) + k_{i11} \exp(k_i x - \alpha k_i^1 t + \delta_i)}{1 + b_{i00} \exp(k_i x - \alpha k_i^1 t + \delta_i) + b_{i01} \exp(k_i x - \alpha k_i^1 t + \delta_i) + b_{i10} \exp(k_i x - \alpha k_i^1 t + \delta_i) + b_{i11} \exp(k_i x - \alpha k_i^1 t + \delta_i)}, \quad (17)$$

where $b_{001}$, $b_{010}$, $b_{100}$, $k_1$, $k_2$, $k_3$, $\delta_1$, $\delta_2$, and $\delta_3$ remain arbitrary.
3.3. Rational solutions

Suppose that Eq. (6) admits a solution of the form

\[
\begin{aligned}
    u(x,t) &= \frac{a_1 \left( \mu_1 \exp(\xi) + \mu_2 \xi \right) + a_0 + a_{-1} \left( \mu_1 \exp(\xi) + \mu_2 \xi \right)^{-1}}{b_1 \left( \mu_1 \exp(\xi) + \mu_2 \xi \right) + b_0 + b_{-1} \left( \mu_1 \exp(\xi) + \mu_2 \xi \right)^{-1}}, \\
    \xi &= kx + \omega t.
\end{aligned}
\]  

(18)

By the same procedure, we obtain the solution set of the resultant algebraic system as

\[
\begin{aligned}
    a_0 &= \frac{a_1 b_0}{b_1} + k b_1, \\
    a_{-1} &= \frac{k b_0}{2 b_1} + \frac{1}{2} \sqrt{b_1^2 - 4 b_0 b_1}, \\
    w &= -\frac{3 k a_1^2}{b_1^2}, \\
    \mu_1 &= 0, \\
    \mu_2 &= 1,
\end{aligned}
\]

(19)

which leads to a rational solution to Eq. (6) as

\[
\begin{aligned}
    u(x,t) &= \frac{a_1 \left( k x - \frac{3 k a_1^2}{b_1^2} t \right)^2 + \left( \frac{a_1 b_0}{b_1} + k b_1 \right) \left( k x - \frac{3 k a_1^2}{b_1^2} t \right) + \frac{k b_0}{2 b_1} + \frac{1}{2} k \sqrt{b_1^2 - 4 b_0 b_1}}{b_1 \left( k x - \frac{3 k a_1^2}{b_1^2} t \right)^2 + b_0 \left( k x - \frac{3 k a_1^2}{b_1^2} t \right) + b_{-1}},
\end{aligned}
\]  

(20)

where \( a_1, b_{-1}, b_0, b_1, \) and \( k \) remain arbitrary.

4. The fifth–order Korteweg de Vries equation

Let us consider the so–called fifth–order Korteweg de Vries equation which reads

\[
u_{xxx} + 10 u u_{xxx} + 30 u^2 u_x + 20 u u_{xx} + u_{xxxx} = 0.
\]

(21)

First, we assume that Eq. (21) admits a solution in the form

\[
\begin{aligned}
    u(x,t) &= a_1 \exp(\xi) \left( 1 + b_1 \exp(\xi) \right)^2, \\
    \xi &= k x + \omega t + \delta,
\end{aligned}
\]

(22)

which is embedded in (2). Substituting (22) into Eq. (21) and solving the resultant algebraic system for the unknowns \( a_1, b_1, k, \) and \( w, \) we obtain the solution set

\[
w = -k^5, \\
a_1 = 2 b_1 k^2,
\]

(23)

which leads a one–wave solution to Eq. (6) as

\[
\begin{aligned}
    u(x,t) &= \frac{2 b_1 k^2 \exp(k x - k^5 t + \delta)}{\left( 1 + b_1 \exp(k x - k^5 t + \delta) \right)^2},
\end{aligned}
\]

(24)

where \( k, b_1, \) and \( \delta \) remain arbitrary.

4.1. Two–wave solutions

Second, we suppose that Eq. (21) admits a solution of the form
where ξ_l = k_l x + w_l t + δ_l, l = 1, 2.

One can see that the function (25) is embedded in (3). Substituting (25) into Eq. (21) and solving the resultant algebraic system for the unknowns a_10, a_01, a_{11}, a_{21}, b_10, b_01, b_11, k_1, k_2, w_1, and w_2, we get the solution set

\[
a_{10} = 2 h_0 k_1^2, a_{01} = 2 h_0 k_2^2, a_{11} = 4 h_0 b_10 (k_1 - k_2)^2, a_{21} = 2 h_0 b_01 (k_1 - k_2)^2 / (k_1 + k_2)^2, \\
a_{12} = 2 h_0 k_1 (k_1 - k_2)^2 / (k_1 + k_2)^2, b_{11} = b_01 b_{10} (k_1 - k_2)^2 / (k_1 + k_2)^2, w_1 = -k_1^2, w_2 = -k_2^2,
\]

which provides a two–wave solution to Eq. (6) as

\[
u(x,t) = \frac{a_{10} \exp(\xi_1) + a_{01} \exp(\xi_2) + a_{11} \exp(\xi_1 + \xi_2) + a_{21} \exp(2\xi_1 + \xi_2) + a_{12} \exp(\xi_1 + 2\xi_2)}{1 + b_{10} \exp(\xi_1) + b_{01} \exp(\xi_2) + b_{11} \exp(\xi_1 + \xi_2)}, \quad (27)
\]

where \( \xi_1 = k_1 x - k_1^2 t + \delta_1, \xi_2 = k_2 x - k_2^2 t + \delta_2, \) and \( b_{10}, b_{01}, b_11, k_1, k_2, \delta_1, \delta_2 \) remain arbitrary.

4.2. Three–wave solutions

Third, we assume that Eq. (21) admits a solution of the form

\[
u(x,t) = \frac{v_1(\xi_1, \xi_2, \xi_3)}{v_2(\xi_1, \xi_2, \xi_3)}, \quad (28)
\]

where \( \xi_l = k_l x + w_l t + \delta_l, l = 1, 2, 3, \) and

\[
v_1(\xi_1, \xi_2, \xi_3) = a_{100} \exp(\xi_1) + a_{010} \exp(\xi_2) + a_{001} \exp(\xi_3) + a_{101} \exp(\xi_1 + \xi_2) + a_{110} \exp(\xi_1 + \xi_3) + a_{011} \exp(\xi_2 + \xi_3) + a_{200} \exp(2\xi_1) + a_{102} \exp(\xi_1 + 2\xi_2) + a_{012} \exp(\xi_1 + 2\xi_3) + a_{201} \exp(2\xi_2) + a_{210} \exp(2\xi_3) + a_{111} \exp(\xi_1 + \xi_2 + \xi_3) + a_{120} \exp(2\xi_1 + \xi_2) + a_{101} \exp(2\xi_1 + \xi_3) + a_{012} \exp(2\xi_2 + \xi_3) + a_{201} \exp(2\xi_1 + 2\xi_2) + a_{210} \exp(2\xi_1 + 2\xi_3) + a_{112} \exp(2\xi_1 + 2\xi_3) + a_{211} \exp(2\xi_2 + 2\xi_3) + a_{220} \exp(2\xi_2 + 2\xi_3)
\]

\[
v_2(\xi_1, \xi_2, \xi_3) = \left(1 + b_{100} \exp(\xi_1) + b_{010} \exp(\xi_2) + b_{001} \exp(\xi_3) + b_{101} \exp(\xi_1 + \xi_2) + b_{110} \exp(\xi_1 + \xi_3) + b_{011} \exp(\xi_2 + \xi_3) + b_{200} \exp(2\xi_1) + b_{102} \exp(\xi_1 + 2\xi_2) + b_{012} \exp(\xi_1 + 2\xi_3) + b_{201} \exp(2\xi_2) + b_{210} \exp(2\xi_3) + b_{111} \exp(\xi_1 + \xi_2 + \xi_3) + b_{120} \exp(2\xi_1 + \xi_2) + b_{101} \exp(2\xi_1 + \xi_3) + b_{012} \exp(2\xi_2 + \xi_3) + b_{201} \exp(2\xi_1 + 2\xi_2) + b_{210} \exp(2\xi_1 + 2\xi_3) + b_{112} \exp(2\xi_1 + 2\xi_3) + b_{211} \exp(2\xi_2 + 2\xi_3) + b_{220} \exp(2\xi_2 + 2\xi_3)\right)^2.
\]
It is obvious that the function (28) is embedded in (4). After substituting (28) into Eq. (21) and proceeding as before, we get the solution set of the resultant algebraic system as

\[ w_1 = -k_1^s, \quad w_2 = -k_2^s, \quad w_3 = -k_3^s, \quad a_{100} = 2b_{100}k_1^2, \quad a_{010} = 2b_{010}k_2^2, \quad a_{001} = 2b_{001}k_3^2, \]

(29)

\[ a_{011} = 4b_{000}h_{010}(k_2 - k_3)^2, \quad a_{101} = 4b_{000}h_{100}(k_1 - k_3)^2, \quad a_{110} = 4b_{000}h_{100}(k_1 - k_2)^2, \]

(30)

\[ a_{002} = \frac{2b_{00}^2b_{100}^2(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad a_{012} = \frac{2b_{00}^2b_{100}^2(k_2 - k_3)^2}{(k_2 + k_3)^2}, \quad b_{011} = \frac{b_{001}h_{100}^2(k_2 - k_3)^2}{(k_2 + k_3)^2}, \]

(31)

\[ a_{021} = \frac{2b_{00}^2b_{010}^2(k_2 - k_3)^2}{(k_2 + k_3)^2}, \quad b_{101} = \frac{b_{001}h_{010}^2(k_1 - k_3)^2}{(k_1 + k_3)^2}, \quad a_{201} = \frac{2b_{00}^2b_{100}^2(k_1 - k_3)^2}{(k_1 + k_3)^2}, \]

(32)

\[ a_{120} = \frac{b_{00}h_{010}^2(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad a_{141} = \frac{4b_{00}^2b_{010}^2b_{100}^2(k_1 - k_2)^2(k_2 - k_3)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(33)

\[ b_{110} = \frac{b_{00}h_{100}^2(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad a_{221} = \frac{2b_{00}^2b_{010}^2b_{100}^2(k_2 - k_3)^2(k_1 - k_3)^2(k_2 - k_3)^2k_1^2}{(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(34)

\[ a_{210} = \frac{b_{00}h_{010}^2(k_1 - k_2)^2k_2^2}{(k_1 + k_2)^2}, \quad a_{122} = \frac{2b_{00}^2b_{010}^2b_{100}^2k_2^2(k_1 - k_3)^2(k_2 - k_3)^2k_1^2}{(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(35)

\[ a_{141} = \frac{4b_{00}^2b_{010}^2b_{100}^2(k_1 - k_2)^2(k_2 - k_3)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_3)^2(k_2 + k_3)^2}, \quad b_{111} = \frac{b_{00}h_{010}^2h_{100}^2(k_1 - k_2)^2(k_2 - k_3)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(36)

\[ a_{111} = \frac{8b_{00}h_{010}h_{100}^2(k_2^2 + k_3^2)(k_2 - k_3)^2 + k_1^2(k_2^2 + k_3^2) - 2k_1^2(k_2^2 + k_3^2) + k_1^2(k_2^2 + k_3^2)}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(37)

\[ a_{212} = \frac{2b_{00}^2b_{010}^2b_{100}^2(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2(k_2 + k_3)^2}, \]

(38)

\[ a_{211} = \frac{4b_{00}h_{010}^2b_{100}^2(k_1 - k_2)^2(k_1 - k_3)^2(k_2 - k_3)^2}{(k_1 + k_2)^2(k_1 + k_3)^2}, \]

(39)
Finally, employing the determined coefficients (29)–(39) to (28), we derive a three–wave solution to Eq. (21), where $b_{01}$, $b_{001}$, $k_1$, $k_2$, $k_3$, $\delta_1$, $\delta_2$, and $\delta_3$ remain arbitrary.

### 4.3. Rational solutions

For a rational solution, we suppose that Eq. (21) admits the function (18) as a solution. Then, following the same procedure, we obtain the solution set of the resultant algebraic system as

\[
\begin{align*}
\mu_0 &= -a_1 b_1^3 - 8 k_2 b_1^3, \\
\mu_1 &= -a_1 b_1^3, \\
\mu_2 &= b_1^3, \\
\delta_1 &= b_1^3 / 4 b_1, \\
\delta_2 &= b_1^3 / 4 b_1, \\
\delta_3 &= b_1^3 / 4 b_1,
\end{align*}
\]

which leads to a rational solution to Eq. (21) as

\[
u(x,t) = \frac{4 a_1 b_1^3 \left(kx - (30ka_1^2 / b_1^3) t + \delta_1 \right)^3 + 4 a_1 b_1^3 \left(kx - (30ka_1^2 / b_1^3) t + \delta_2 \right)^3 + 4 b_1 b_1^3 \left(kx - (30ka_1^2 / b_1^3) t + \delta_3 \right)^3 + b_1^3 b_1}{4 b_1^3 \left(kx - (30ka_1^2 / b_1^3) t + \delta_1 \right)^3 + 4 b_1 b_1^3 \left(kx - (30ka_1^2 / b_1^3) t + \delta_2 \right)^3 + b_1^3 b_1},
\]

where $a_1$, $b_1$, $k_1$, and $\delta$ remain arbitrary.

**Remark 2.** The existence of $N$–wave solutions often implies the integrability of the equation considered. The determination of three–wave solutions for our equations confirms the fact that $N(\geq 4)$–wave solutions exist and can be obtained in a parallel manner. However, we observed that the computation becomes tedious and much more complicated.

### 5. Conclusions

Seeking exact and explicit solutions with multi–velocities and multi–frequencies for NEEs is an important and active research area in the applied mathematical and physical sciences. In this study, we implemented the Exp–function method to two completely integrable NEEs for explicitly constructing one–wave, two–wave, and three–wave solutions, as well as rational solutions. We successfully obtained such solutions involving more arbitrary parameters. Our results indicate that the Exp–function method can be used as a simplified form of the Hirota’s bilinear method. We conclude that the Exp–function method posses powerful features that make it practical for the determination of multi–wave solutions for a wide class of NEEs; this will be our future task.

**REFERENCES**
