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FUZZY APPROXIMATE OF DERIVATIONS

Reza SAADATI[1], Choonkil PARK[2], Jung Rye LEE*[3]

Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy derivations on fuzzy Banach algebras associated with the Cauchy-Jensen functional equation and the Cauchy-Jensen functional inequality.

Keywords: fuzzy Banach algebra, fixed point, Hyers-Ulam stability, fuzzy derivation, Cauchy-Jensen functional equation, Cauchy-Jensen functional inequality.

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1. Introduction and preliminaries

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [2, 15, 23, 25, 28, 45]. Following Cheng and Mordeson [8], Bag and Samanta [2] Saadati and Vaezpour [40] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [24] and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29, 40] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

Definition 1.1. [2, 28, 29, 30, 40] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

Definition 1.2. [2, 28, 29, 30, 40] (1) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converges if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

[1]Professor, Department of Mathematics, Iran University of Science and Technology, Tehran, Iran, e-mail: rsaadati@eml.cc

[2]Professor, Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea

[3]Professor, Department of Mathematics, Daejin University, Kyeonggi 487-711, Republic of Korea

(2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

Definition 1.3. [34] Let X be an algebra and (X, N) a fuzzy normed space.

(1) The fuzzy normed space (X, N) is called a *fuzzy normed algebra* if

$$N(xy, st) \geq N(x, s) \cdot N(y, t)$$

for all $x, y \in X$ and all positive real numbers s and t.

(2) A complete fuzzy normed algebra is called a *fuzzy Banach algebra.*

Example 1.1. [34] Let $(X, \|\cdot\|)$ be a normed algebra. Let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & t > 0, x \in X \\ 0 & t \leq 0, x \in X. \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on X and $(X, N(x, t))$ is a fuzzy normed algebra.

Definition 1.4. Let (X, N) be a fuzzy normed algebra. Then an $\mathbb{R}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in X$.

The stability problem of functional equations was originated from a question of Ulam [43] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [38] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [38] has provided a lot of influence in the development of what we call the *Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [42] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [11] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9, 12, 20, 22, 41]).

Gilányi [17] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [39]. Fechner [14] and Gilányi [18] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + f(2z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{2}$$

and proved the Hyers-Ulam stability of the functional inequality (1.2) in Banach spaces.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 7, 27, 31, 32, 37, 44, 46]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional equation in fuzzy Banach algebras by using the fixed point method. In Section 3, we prove the Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional inequality in fuzzy Banach algebras by using the fixed point method.

Throughout this paper, assume that (X, N) is a fuzzy Banach algebra.

2. Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional equation in fuzzy Banach algebras

Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional equation in fuzzy Banach algebras.

Theorem 2.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < \frac{1}{2}$ with*

$$\varphi(x,y,z) \le \frac{L}{2}\varphi(2x,2y,2z)$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying

$$N\left(2f\left(\frac{rx+ry}{2}+rz\right) - rf(x) - rf(y) - 2rf(z), t\right) \ge \frac{t}{t+\varphi(x,y,z)}, \quad (3)$$

$$N\left(f(xy) - f(x)y - xf(y), t\right) \ge \frac{t}{t+\varphi(x,y,0)} \quad (4)$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Then $D(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a fuzzy derivation $D : X \to X$ such that

$$N\left(f(x) - D(x), t\right) \ge \frac{(1-L)t}{(1-L)t+\varphi(x,0,0)} \quad (5)$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $r = 1$ and $y = z = 0$ in (2.1), we get

$$N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \ge \frac{t}{t+\varphi(x,0,0)} \quad (6)$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t+\varphi(x,0,0)}, \ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N\left(g(x) - h(x), \varepsilon t\right) \ge \frac{t}{t+\varphi(x,0,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2},0,0\right)} \ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,0,0)} = \frac{t}{t+\varphi(x,0,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le L d(g,h)$$

for all $g, h \in S$.

On the other hand, (2.4) implies that $d(f, Jf) \le 1$.

By Theorem 1.6, there exists a mapping $D : X \to X$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{2}\right) = \frac{1}{2}D(x) \tag{7}$$

for all $x \in X$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that D is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - D(x), \mu t) \ge \frac{t}{t + \varphi(x, 0, 0)}$$

for all $x \in X$;

(2) $d(J^n f, D) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = D(x)$$

for all $x \in X$;

(3) $d(f, D) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, D) \le \frac{1}{1-L}.$$

This implies that the inequality (2.3) holds.

By (2.1),

$$N\left(2^{k+1} f\left(\frac{rx + ry}{2^{k+1}} + \frac{rz}{2^k}\right) - 2^k r f\left(\frac{x}{2^k}\right) - 2^k r f\left(\frac{y}{2^k}\right) - 2^{k+1} r f\left(\frac{z}{2^k}\right), 2^k t\right) \ge \frac{t}{t + \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right)}$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. So

$$N\left(2^{k+1} f\left(\frac{rx + ry}{2^{k+1}} + \frac{rz}{2^k}\right) - 2^k r f\left(\frac{x}{2^k}\right) - 2^k r f\left(\frac{y}{2^k}\right) - 2^{k+1} r f\left(\frac{z}{2^k}\right), t\right) \ge \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\varphi(x, y, z)}$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Since $\lim_{k\to\infty} \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}\varphi(x,y,z)} = 1$ for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$,

$$N\left(2D\left(\frac{rx + ry}{2} + rz\right) - rD(x) - rD(y) - 2rD(z), t\right) = 1$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Thus $2D\left(\frac{rx+ry}{2} + rz\right) - rD(x) - rD(y) - 2rD(z) = 0$. So the mapping $D : X \to X$ is additive and $\mathbb{R}$-linear.

By (2.2),

$$N\left(4^k f\left(\frac{xy}{4^k}\right) - 2^k f\left(\frac{x}{2^k}\right) \cdot y - x \cdot 2^k f\left(\frac{y}{2^k}\right), 4^k t\right) \ge \frac{t}{t + \varphi\left(\frac{x}{2^k}, \frac{y}{2^k}, 0\right)}$$

for all $x, y \in X$ and all $t > 0$. So

$$N\left(4^k f\left(\frac{xy}{4^k}\right) - 2^k f\left(\frac{x}{2^k}\right) \cdot y - x \cdot 2^k f\left(\frac{y}{2^k}\right), t\right) \ge \frac{\frac{t}{4^k}}{\frac{t}{4^k} + \frac{L^k}{2^k}\varphi(x, y, 0)}$$

for all $x, y \in X$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{\frac{t}{4^k}}{\frac{t}{4^k}+\frac{L^k}{2^k}\varphi(x,y,0)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(D(xy) - D(x)y - xD(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus $D(xy) - D(x)y - xD(y) = 0$. So the mapping $D : X \to X$ is a fuzzy derivation, as desired. □

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.1) and (2.2). Then $D(x) := N\text{-}\lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right)$ exists for each $x \in X$ and defines a fuzzy derivation $D : X \to X$ such that

$$N\left(f(x) - D(x), t\right) \ge \frac{(1-L)t}{(1-L)t + L\varphi(x, 0, 0)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \ge \frac{t}{t + \varphi(2x, 0, 0)} \ge \frac{t}{t + 2L\varphi(x, 0, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le L$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < \frac{1}{2}$ with*

$$\varphi(x, y, z) \le \frac{L}{2}\varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.2) and

$$N\left(f\left(\frac{rx + ry}{2} + rz\right) - \frac{r}{2}f(x) - \frac{r}{2}f(y) - rf(z), t\right) \ \ge \ \frac{t}{t + \varphi\left(x, y, z\right)} \qquad (8)$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Then $D(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a fuzzy derivation $D : X \to X$ such that

$$N\left(f(x) - D(x), t\right) \ge \frac{(2-2L)t}{(2-2L)t + L\varphi\left(x, x, x\right)}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $r = 1$ and $y = z = x$ in (2.6), we get

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi\left(x, x, x\right)} \tag{9}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to X\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi\left(x, x, x\right)},\ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.7) that

$$\begin{aligned} N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) &\geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)} \\ &\geq \frac{t}{t + \frac{L}{2}\varphi\left(x, x, x\right)} \end{aligned}$$

hence

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi\left(x, x, x\right)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.2) *and* (2.6)*. Then $D(x) := N\text{-}\lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right)$ exists for each $x \in X$ and defines a fuzzy derivation $D : X \to X$ such that*

$$N\left(f(x) - D(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g\left(2x\right)$$

for all $x \in X$.

It follows from (2.7) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right)\geq\frac{t}{t+\varphi(x,x,x)}$$

for all $x\in X$ and all $t>0$. So $d(f,Jf)\leq\frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. **Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional inequality in fuzzy Banach algebras**

We need the following lemma to prove the main results.

Lemma 3.1. [33, 36] *Let (X,N') and (Y,N) be fuzzy normed vector spaces. Let $f:X\to Y$ be a mapping such that*

$$N(f(x)+f(y)+2f(z),t)\geq N\left(2f\left(\frac{x+y}{2}+z\right),\frac{2t}{3}\right)$$

for all $x,y,z\in X$ and all $t>0$. Then f is Cauchy additive, i.e., $f(x+y)=f(x)+f(y)$ for all $x,y\in X$.

Using the fixed point method, we prove the Hyers-Ulam stability of fuzzy derivations associated with the Cauchy-Jensen functional inequality in fuzzy Banach algebras.

Theorem 3.1. *Let $\varphi:X^3\to[0,\infty)$ be a function such that there exists an $L<\frac{1}{2}$ with*

$$\varphi(x,y,z)\leq\frac{L}{2}\varphi\left(2x,2y,2z\right)$$

for all $x,y,z\in X$. Let $f:X\to X$ be an odd mapping satisfying

$$N\left(rf(x)+rf(y)+f(2rz),t\right)\geq\min\left\{N\left(2f\left(\frac{rx+ry}{2}+rz\right),\frac{2t}{3}\right),\frac{t}{t+\varphi(x,y,z)}\right\}\quad(10)$$

$$N\left(f(xy)-f(x)y-xf(y),t\right)\geq\frac{t}{t+\varphi\left(x,y,0\right)}\quad(11)$$

for all $x,y,z\in X$, all $t>0$ and all $r\in\mathbb{R}$. Then $D(x):=N\text{-}\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines a fuzzy derivation $D:X\to X$ such that

$$N\left(f(x)-D(x),t\right)\geq\frac{(2-2L)t}{(2-2L)t+L\varphi(x,x,-x)}\quad(12)$$

for all $x\in X$ and all $t>0$.

Proof. Letting $r=1$ and $y=x=-z$ in (3.1), we get

$$N\left(2f(x)-f\left(2x\right),t\right)\geq\frac{t}{t+\varphi(x,x,-x)}\quad(13)$$

for all $x\in X$ and all $r\in\mathbb{R}$.

Consider the set

$$S:=\{g:X\to X\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+:N(g(x)-h(x),\mu t)\geq\frac{t}{t+\varphi(x,x,-x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.4) that

$$\begin{aligned} N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) &\geq \frac{t}{t + \varphi(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2})} \\ &\geq \frac{t}{t + \frac{L}{2}\varphi(x, x, -x)} \end{aligned}$$

hence

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x, -x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.6, there exists a mapping $D : X \to X$ satisfying the following:

(1) D is a fixed point of J, i.e.,

$$D\left(\frac{x}{2}\right) = \frac{1}{2}D(x) \tag{14}$$

for all $x \in X$. The mapping D is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that D is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - D(x), \mu t) \geq \frac{t}{t + \varphi(x, x, -x)}$$

for all $x \in X$;

(2) $d(J^n f, D) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = D(x)$$

for all $x \in X$. Since $f : X \to X$ is odd, the above equality implies that, $D : X \to X$ is an odd mapping.

(3) $d(f, D) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, D) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (3.3) holds.

By (3.1),

$$\begin{aligned} &N\left(2^n\left(rf\left(\frac{x}{2^n}\right) + rf\left(\frac{y}{2^n}\right) + f\left(\frac{rz}{2^{n-1}}\right)\right), 2^n t\right) \\ &\quad\geq \min\left\{N\left(2^{n+1}f\left(\frac{rx + ry}{2^{n+1}} + \frac{rz}{2^n}\right), \frac{2^{n+1}}{3}t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y, z \in X$, all $t > 0$, all $r \in \mathbb{R}$ and all $n \in \mathbb{N}$. So

$$N\left(2^n\left(rf\left(\frac{x}{2^n}\right)+rf\left(\frac{y}{2^n}\right)+f\left(\frac{rz}{2^{n-1}}\right)\right),t\right)$$
$$\geq \min\left\{N\left(2^{n+1}f\left(\frac{rx+ry}{2^{n+1}}+\frac{rz}{2^n}\right),\frac{2t}{3}\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi\left(x,y,z\right)}\right\}$$

for all $x, y, z \in X$, all $t > 0$, all $r \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y,z)} = 1$ for all $x, y, z \in X$ and all $t > 0$,

$$N\left(rD(x)+rD(y)+D(2rz),t\right) \geq N\left(2D\left(\frac{rx+ry}{2}+rz\right),\frac{2t}{3}\right) \tag{15}$$

for all $x, y, z \in X$, all $t > 0$ and all $r \in \mathbb{R}$. Let $r = 1$ in (3.6). By Lemma 3.1, the mapping $D : X \to X$ is Cauchy additive. Letting $y = x$ and $z = -x$ in (3.6), we get $2rD(x) - 2D(rx) = 0$ for all $x \in X$ and all $r \in \mathbb{R}$. So the mapping $D : X \to X$ is $\mathbb{R}$-linear.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \leq 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \to X$ be an odd mapping satisfying (3.1) *and* (3.2) *Then $D(x) := N\text{-}\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)$ exists for each $x \in X$ and defines a fuzzy derivation $D : X \to X$ such that*

$$N\left(f(x)-D(x),t\right) \geq \frac{(2-2L)t}{(2-2L)t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.4) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right) \geq \frac{t}{t+\varphi(x,x,-x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.2. □

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