HIGH ORDER COMPACT CRANK-NICOLSON DIFFERENCE SCHEME FOR A CLASS OF SPACE FRACTIONAL DIFFERENTIAL EQUATIONS

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In this paper, we present a high order compact Crank-Nicolson difference scheme for the initial boundary value problem of a class of space fractional differential equations, where the space fractional Riemann-Liouville derivative are approximated by a weighted and shifted Grünwald-Letnikov approximation formula with sixth order accuracy. This difference scheme is proved to be of unique solution, unconditionally stable, convergent with accuracy of second order and sixth order in temporal direction and space direction respectively. Numerical experiments are carried out to support the theoretical analytical results.

Keywords: Space fractional differential equation; Riemann-Liouville derivative; Compact difference scheme; Unconditionally stable.

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1. Introduction

Recently, research on the theory and applications of fractional differential equations (FDEs) has gained more and more attention by many researchers. Compared with integer-order differential equations, FDEs are better choices for describing some phenomena or processes with memory, hereditary and long-range interaction in diffusion, biology, relaxation vibrations, electrochemistry, finance, fluid mechanics and so on [1-6]. In the last few decades, a variety of models have been proposed by use of FDEs for the description of memory and hereditary properties of various materials and processes such as physical and biological processes. For the basic theory of fractional calculus, we refer the readers to the [7, 8].

In the research of FDEs, seeking solutions of FDEs are a hot topic, and have been paid much attention by many authors. However, in most cases, it is difficult to obtain exact solutions for FDEs due to the complexity of fractional operators and fractional calulus. Thus, it becomes very important to develop efficient and high accuracy numerical methods to obtain numerical

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solutions for FDEs. Among the existing numerical methods, the finite difference method is the most popularly used one easy to be fulfilled. So far many efficient finite difference schemes have been developed by many authors for solving a variety of time FDEs and space FDEs as well as space-time FDEs. In general, the Caputo fractional derivative is the most widely used one in time FDEs, and the main method for approximating the Caputo derivative is by L interpolation approximation formulas [9-16], while the Riemann-Liouville fractional derivative and the Riesz fractional derivative are usually used in space FDEs, and approximation formulas for them are usually constructed by use of the Grünwald-Letnikov (G-L) approximation method [17-23], which was initially proposed by Meerschaert and Tadjeran [24]. Besides, the Riesz fractional derivative can also be approximated by the fractional center difference approximation formula [25, 26]. On the other hand, due the global property of the fractional operator, computation and stored task become expensive for fractional finite difference schemes, especially for the computation of high dimensional problems. The alternating direction implicit method [27-29] is a valid approach to solve this problem, which reduce stored task to a large degree for computer, and reduce the high dimensional computation problem to several one dimensional computation problems improving computation efficiency.

In order to improve the accuracy of the finite difference scheme, compact techniques are usually used to develop compact difference schemes [30-32]. However, in the existing difference schemes for spatial FDEs, most of the approximation accuracy for the fractional derivatives are no more than fourth order. So motivated by the works above, we will construct an approximation formula with sixth accuracy for the Riemann-Liouville fractional derivatives, and then based on the approximation formula develop a compact Crank-Nicolson difference scheme for the initial boundary value problem of a class of space fractional differential equation, which is denoted as follows

$$\begin{cases} u_t(x,t) = k({}_0D_x^\alpha u(x,t) -{}_x D_L^\alpha u(x,t)) + f(x,t), \\ \qquad\qquad 1 < \alpha < 2,\ x \in [0,L],\ t \in [0,T], \\ u(x,0) = \varphi(x),\ x \in [0,L], \\ u(0,t) = u(L,t) = 0, \end{cases} \tag{1}$$

where the function u is smooth enough, $k > 0$ is a constant, and the fractional derivatives are defined in the sense of the left-side and right-side Riemann-Liouville derivatives as follows:

$$\begin{cases} {}_{-\infty}D_x^\alpha u(x,t) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^x (x-\sigma)^{n-1-\alpha}u(\sigma,t)d\sigma\right), \\ {}_xD_\infty^\alpha u(x,t) = (-1)^n\frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_x^\infty (\sigma-x)^{n-1-\alpha}u(\sigma,t)d\sigma\right), \end{cases} \tag{2}$$

where $n-1 \le \alpha < n,\ n \in \mathbb{N}$.

For the sake of convenience, we extend the definition domain of the function $u(x,t)$ to $\mathbb{R} \times [0,T]$, and satisfies $u(x,t) \equiv 0$ for $(x,t) \notin [0,L] \times [0,T]$. So under this extension we have

${}_{-\infty}D_x^\alpha u(x,t) ={}_0 D_x^\alpha u(x,t),\ {}_xD_\infty^\alpha u(x,t) ={}_x D_L^\alpha u(x,t)$.

The rest of this paper is organized as follows. In Section 2, we present some notations and preliminaries, and derive an approximation formula with sixth accuracy for the Riemann-Liouville fractional derivatives. In Section 3, we develop a compact Crank-Nicolson difference scheme for the problem (1). In Section 4, unique solvability, unconditionally stability and convergence for the Crank-Nicolson difference scheme are discussed. In Section 5, we carry out numerical experiments for checking the validity of the present difference scheme. In Section 6, some conclusions are given.

2. **Preliminaries**

Let $M,\ N$ be positive integers, and $h = \frac{L}{M}$ denotes the spatial step size, while $\tau = \frac{T}{N}$ denotes the temporal step size. Define $x_i = i * h(i \in \mathbb{Z}),\ t_n = n\tau(0 \le n \le N),\ \Omega_h = \{x_i | i \in \mathbb{Z}\},\ \Omega_\tau = \{t_n | 0 \le n \le N\}$, $(i,n) = (x_i, t^n)$, and then the domain $\mathbb{R} \times [0,T]$ is covered by $\Omega_h \times \Omega_\tau$. Let $U_i^n = u(x_i, t^n)$ and u_i^n denote the exact solution and numerical solution at the point (i,n) respectively. $U^n = (..., U_{-2}^n,\ U_{-1}^n,\ U_0^n,\ U_1^n,\ U_2^n, ...)^T,\ u^n = (..., u_{-2}^n,\ u_{-1}^n,\ u_0^n,\ u_1^n,\ u_2^n, ...)^T$.

Define the grid functions spaces $U_h = \{u | u = (..., u_{-2},\ u_{-1},\ u_0,\ u_1,\ u_2, ...)^T\}$ and $U_h^0 = \{u | u \in U_h,\ \lim\limits_{|i|\to\infty} u_i = 0,\ \lim\limits_{|i|\to\infty} \delta_x u_{i-\frac{1}{2}} = 0\}$, where $\delta_x u_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{h}$. For $u,\ v \in U_h^0$, define the inner product as $(u,v) = h \sum\limits_{i=-\infty}^{\infty} u_i v_i$, while define the discrete L_2 norm by $\|u\| = \sqrt{(u,u)} = (\sum\limits_{i=-\infty}^{\infty} h|u_i|^2)^{\frac{1}{2}}$.

For further use, denote

$$\delta_t u_i^{n-\frac{1}{2}} = \frac{u_i^n - u_i^{n-1}}{\tau},\ \delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2},\ u_i^{n-\frac{1}{2}} = \frac{u_i^n + u_i^{n-1}}{2}.$$

Lemma 1 [27]. Let $\alpha \in (1,2),\ u \in C^{n+3}(\mathbb{R})$ such that all derivatives of u up to order $n+3$ belong to $L_1(\mathbb{R})$. Define the left-side shifted Grünwald difference operator by

$$A_{h,p}^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x-(k-p)h),$$

where p is an integer, and $g_0^{(\alpha)} = 1,\ g_k^{(\alpha)} = (1 - \frac{\alpha+1}{k}) g_{k-1}^{(\alpha)},\ k = 1,2,...$. Then it holds that

$$A_{h,p}^\alpha u(x) ={}_{-\infty} D_x^\alpha u(x) + \sum_{l=1}^{n-1} c_l^{\alpha,p}\,{}_{-\infty}D_x^{\alpha+l} u(x) h^l + O(h^n) \tag{3}$$

uniformly for $x \in \mathbb{R}$, where $c_l^{\alpha,p},\ l = 1,2...$ are the coefficients of the power series expansion for the function $(\frac{1-e^{-z}}{z})^\alpha e^{pz}$.

Corollary 1. If we define the right-side shifted Grünwald difference operator by

$$B_{h,p}^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x+(k-p)h),$$

then we have

$$B_{h,p}^{\alpha}u(x) ={}_x D_{\infty}^{\alpha}u(x)+\sum_{l=1}^{n-1} c_l^{\alpha,p}{}_xD_{\infty}^{\alpha+l}u(x)h^l+O(h^n), \tag{4}$$

Lemma 2. Let $\alpha \in (1,2)$, $u \in C^9(\mathbb{R})$ such that all derivatives of u up to order 9 belong to $L_1(\mathbb{R})$, $c^{\alpha} = \sum_{p=-2}^{2} s_p c_4^{\alpha,p}$, where s_p, $p = 0, \pm 1$, ± 2 are constants, and $c_l^{\alpha,p}$ are defined as in (3). Define three operators Δ_1^{α}, Δ_2^{α}, A^{α} such that

$$\begin{cases} \Delta_1^{\alpha}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\lambda_k^{(\alpha)}u(x-(k-2)h), \\ \Delta_2^{\alpha}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\lambda_k^{(\alpha)}u(x+(k-2)h), \\ \mathcal{A}^{\alpha}u(x) = (1+c^{\alpha}h^4\delta_x^2\delta_x^2)u(x), \end{cases}$$

where

$$\begin{cases} \lambda_0^{(\alpha)} = s_2 g_0^{(\alpha)}, \\ \lambda_1^{(\alpha)} = s_2 g_1^{(\alpha)} + s_1 g_0^{(\alpha)}, \\ \lambda_2^{(\alpha)} = s_2 g_2^{(\alpha)} + s_1 g_1^{(\alpha)} + s_0 g_0^{(\alpha)}, \\ \lambda_3^{(\alpha)} = s_2 g_3^{(\alpha)} + s_1 g_2^{(\alpha)} + s_0 g_1^{(\alpha)} + s_{-1} g_0^{(\alpha)}, \\ \lambda_k^{(\alpha)} = s_2 g_k^{(\alpha)} + s_1 g_{k-1}^{(\alpha)} + s_0 g_{k-2}^{(\alpha)} + s_{-1} g_{k-3}^{(\alpha)} + s_{-2} g_{k-4}^{(\alpha)},\ k = 4,5,..., \end{cases}$$

If

$$\begin{cases} s_2 = -\frac{7}{144}\alpha - \frac{1}{96}\alpha^2 + \frac{1}{144}\alpha^3 - \frac{1}{30} + \frac{1}{480}\alpha^4, \\ s_1 = \frac{23}{72}\alpha + \frac{7}{48}\alpha^2 - \frac{1}{144}\alpha^3 + \frac{2}{15} - \frac{1}{120}\alpha^4, \\ s_0 = \frac{1}{24}\alpha - \frac{1}{4}\alpha^2 - \frac{1}{48}\alpha^3 + \frac{4}{5} + \frac{1}{80}\alpha^4, \\ s_{-1} = -\frac{25}{72}\alpha + \frac{5}{48}\alpha^2 + \frac{5}{144}\alpha^3 + \frac{2}{15} - \frac{1}{120}\alpha^4, \\ s_{-2} = \frac{5}{144}\alpha + \frac{1}{96}\alpha^2 - \frac{1}{72}\alpha^3 - \frac{1}{30} + \frac{1}{480}\alpha^4. \end{cases} \tag{5}$$

Then it holds that

$$\begin{cases} \Delta_1^{\alpha}u(x) = \mathcal{A}^{\alpha}[{}_{-\infty}D_x^{\alpha}u(x)] + O(h^6), \\ \Delta_2^{\alpha}u(x) = \mathcal{A}^{\alpha}[{}_xD_{\infty}^{\alpha}u(x)] + O(h^6), \end{cases} \tag{6}$$

Proof. From the definition of Δ_1^{α} we have in fact $\Delta_1^{\alpha}u(x) = \sum_{p=-2}^{2} s_p A_{h,p}^{\alpha}u(x)$. It follows from Lemma 1 that

$$\Delta_1^{\alpha}u(x) = \sum_{p=-2}^{2} s_p[{}_{-\infty}D_x^{\alpha}u(x)] + \sum_{l=1}^{n-1}\sum_{p=-2}^{2} s_p c_l^{\alpha,p}[{}_{-\infty}D_x^{\alpha+l}u(x)]h^l + O(h^6).$$

After setting $\sum_{p=-2}^{2} s_p = 1$ and the coefficients of $[{}_{-\infty}D_x^{\alpha+l}u(x)]h^i$, $i = 1,2,3,5$ be zero one can obtain (5), and furthermore,

$$\Delta_1^{\alpha}u(x) ={}_{-\infty} D_x^{\alpha}u(x) + ({}_{-\infty}D_x^{\alpha+4}u(x))c^{\alpha}h^4 + O(h^6)$$
$$={}_{-\infty} D_x^{\alpha}u(x) + c^{\alpha}h^4\delta_x^2\delta_x^2[{}_{-\infty}D_x^{\alpha}u(x)] + O(h^6) = \mathcal{A}^{\alpha}[{}_{-\infty}D_x^{\alpha}u(x)] + O(h^6),$$

where the center difference formula $\delta_x^2\delta_x^2[{}_{-\infty}D_x^{\alpha}u(x)]$ has been used for approximating ${}_{-\infty}D_x^{\alpha+4}u(x)$.

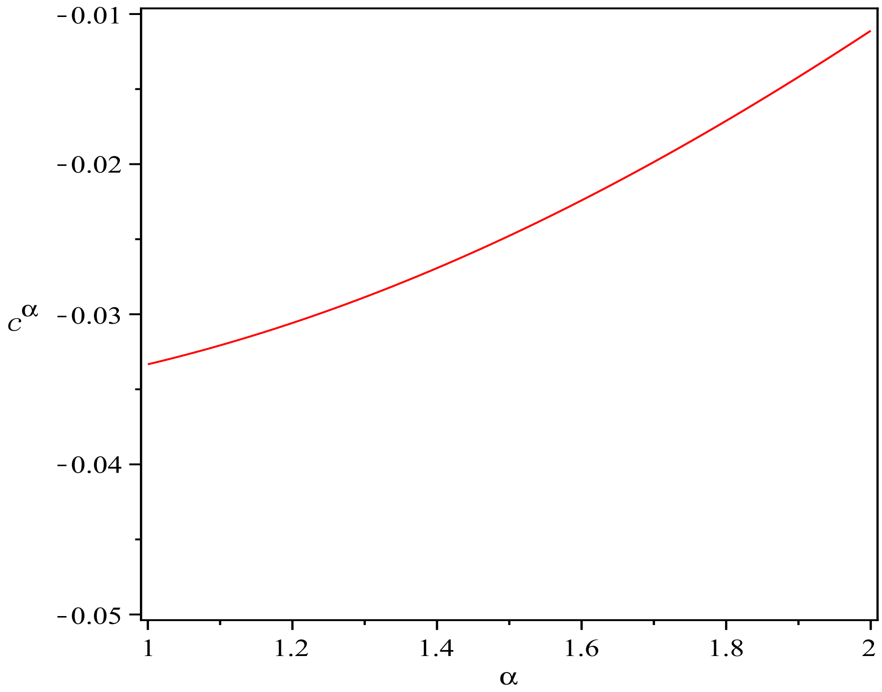

Fig. 1 The function curve of c^{α}

Similarly, one can prove that $\Delta_2^{\alpha}u(x) = \mathcal{A}^{\alpha}[{}_xD_{\infty}^{\alpha}u(x)] + O(h^6)$. The proof is complete.

Lemma 3. [33, Lemma 2.1.1]. Suppose $u \in U_h^0$. Then it holds that

$$\begin{cases} \frac{\sqrt{6}}{(b-a)}\|u\| \le \|\delta_x u\| \le \frac{2}{h}\|u\|, \\ \frac{6}{(b-a)^2}\|u\| \le \|\delta_x^2 u\| = \|\delta_x\delta_x u\| \le \frac{4}{h^2}\|u\|, \end{cases}$$

Lemma 4. For $u \in U_h^0$, we have

$$0.36\|u\|^2 \le (\mathcal{A}^{\alpha}u, u) \le \|u\|^2.$$

In fact, by use of the definition of $\mathcal{A}^{\alpha}$ and the discrete Green formula we have

$$(\mathcal{A}^{\alpha}u, u) = (u, u) + c^{\alpha}h^4(\delta_x^2\delta_x^2 u, u) = \|u\|^2 + c^{\alpha}h^4\|\delta_x^2 u\|^2.$$

According to the function curve of c^{α} shown in Fig. 1 one can see that $c^{\alpha} \in (-0.04, -0.01)$ for $\alpha \in (1, 2)$. Then the result can be deduced by a combination with Lemma 3.

Based on Lemma 4, for $u,\ v \in V_h^0$, we can define the following inner product

$$(u, v)_{\mathcal{A}^{\alpha}} = h\sum_{i=-\infty}^{\infty}(\mathcal{A}^{\alpha}u_i)v_i$$

and the corresponding discrete norm $\|u\|_{\mathcal{A}^{\alpha}} = (\mathcal{A}^{\alpha}u, u)$. Furthermore, $\|u\|_{\mathcal{A}^{\alpha}}$ is equivalent to $\|u\|$.

3. The compact Crank-Nicolson difference scheme

Now we derive the compact Crank-Nicolson difference scheme for the problem (1). Considering ${}_{-\infty}D_x^\alpha u(x,t) ={}_0 D_x^\alpha u(x,t)$, ${}_xD_\infty^\alpha u(x,t) ={}_x D_{L_1}^\alpha u(x,t)$, by use of Lemma 2 one can obtain that at the point (i,n)

$$\begin{aligned}
&\mathcal{A}^\alpha[{}_0D_x^\alpha u(x,t) -{}_x D_L^\alpha u(x,t)]_{(i,n)}\\
&= \frac{1}{h^\alpha}\sum_{k=0}^{\infty}\lambda_k^{(\alpha)}u_{i-k+2}^n - \frac{1}{h^\alpha}\sum_{k=0}^{\infty}\lambda_k^{(\alpha)}u_{i+k-2}^n + O(h^6)\\
&= \frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^n + O(h^6). \qquad (7)
\end{aligned}$$

where

$$\begin{cases}
\omega_0^{(\alpha)} = \lambda_2^{(\alpha)} - \lambda_2^{(\alpha)} = 0,\\
\omega_1^{(\alpha)} = \lambda_3^{(\alpha)} - \lambda_1^{(\alpha)},\\
\omega_2^{(\alpha)} = \lambda_4^{(\alpha)} - \lambda_0^{(\alpha)},\\
\omega_k^{(\alpha)} = \lambda_{k+2}^{(\alpha)},\ k = 3,4,...,\\
\omega_{-k}^{(\alpha)} = -\omega_k^{(\alpha)},\ k = 1,2,....
\end{cases}$$

After applying the operator $\mathcal{A}^\alpha$ on both sides of the first equation of (1), by use of the center difference approximation formula for $u_t(x,t)$, together with the average of the approximation formula (7) at the point (i,n) and $(i,n-1)$ one can deduce that

$$\mathcal{A}^\alpha(\delta_t U_i^{n-\frac{1}{2}}) = \frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}U_{i-k}^{n-\frac{1}{2}} + \mathcal{A}^\alpha f_i^{n-\frac{1}{2}} + O(\tau^2 + h^6). \qquad (8)$$

Then the compact Crank-Nicolson difference scheme approximating the problem (1) can be denoted as follows:

$$\begin{cases}
\mathcal{A}^\alpha(\delta_t u_i^{n-\frac{1}{2}}) = \frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^{n-\frac{1}{2}} + \mathcal{A}^\alpha f_i^{n-\frac{1}{2}}, 1\le i\le M-1,\ 1\le n\le N,\\
u_i^0 = \varphi(x_i),\ 1\le i\le M-1,\\
u_i^n = 0,\ i\le 0,\ or\ i\ge M.
\end{cases} \qquad (9)$$

4. Unique solvability, stability and convergence analysis

In this section, we research the unique solvability, stability and convergence of the present Crank-Nicolson difference scheme (9). For further use, the definition domain of the function $f(x,t)$ is extended to $\mathbb{R}\times[0,T]$ such that f is smooth enough and $f\in U_h^0$.

Lemma 5. For $u\in U_h^0$, it holds that $\sum_{i=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}u_i] = 0$.

Proof. We have the following observations

$$\begin{aligned}
\sum_{i=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}u_i] &= \sum_{k=-\infty}^{\infty}[\sum_{i=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}u_i]\\
&= \sum_{k=-\infty}^{\infty}[\sum_{i=-\infty}^{\infty}\omega_k^{(\alpha)}u_i u_{i+k}] = \sum_{k=-\infty}^{\infty}[\sum_{i=-\infty}^{\infty}\omega_{-k}^{(\alpha)}u_i u_{i-k}]
\end{aligned}$$

$$= -\sum_{i=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}u_i],$$

which implies $\sum_{i=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}u_i] = 0$. The proof is complete.

Theorem 1. The Crank-Nicolson difference scheme denoted by (9) has unique solution.

Proof. The corresponding homogeneous difference equation of the first equation of (9) is denoted by

$$A^\alpha u_i^n = \frac{\tau}{2h^\alpha}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^n. \tag{10}$$

Taking the inner product of (10) with u^n, by use of Lemma 5 we have $\|u^n\|^2_{\mathcal{A}^\alpha} = 0$, and then $\|u^n\| = 0$. So there is only zero solution for (10), which implies the Crank-Nicolson difference scheme denoted by (9) has unique solution. The proof is complete.

Theorem 2. The Crank-Nicolson difference scheme (9) is unconditionally stable on the initial value and the the right source term f.

Proof. Setting $r = \frac{\tau}{h^\alpha}$, the first equation of (9) can be rewritten as follows

$$\mathcal{A}^\alpha(u_i^n - u_i^{n-1}) = r\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^{n-\frac{1}{2}} + \tau\mathcal{A}^\alpha f_i^{n-\frac{1}{2}}. \tag{11}$$

Multiplying $hu_i^{n-\frac{1}{2}}$ on both sides of Eq. (11) and a summation with respect to i from $-\infty$ to ∞ yields that

$$h\sum_{i=-\infty}^{\infty}[\mathcal{A}^\alpha(u_i^n - u_i^{n-1})](\frac{u_i^n + u_i^{n-1}}{2})$$
$$= rh\sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^{n-\frac{1}{2}}u_i^{n-\frac{1}{2}} + \tau h\sum_{i=-\infty}^{\infty}(\mathcal{A}^\alpha f_i^{n-\frac{1}{2}})u_i^{n-\frac{1}{2}}.$$

Since $\sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\omega_k^{(\alpha)}u_{i-k}^{n-\frac{1}{2}}u_i^{n-\frac{1}{2}} = 0$ according to Lemma 5, one has

$$h\sum_{i=-\infty}^{\infty}[\mathcal{A}^\alpha(u_i^n - u_i^{n-1})](\frac{u_i^n + u_i^{n-1}}{2}) = \tau h\sum_{i=-\infty}^{\infty}(\mathcal{A}^\alpha f_i^{n-\frac{1}{2}})u_i^{n-\frac{1}{2}},$$

that is,

$$\|u^n\|^2_{\mathcal{A}^\alpha} - \|u^{n-1}\|^2_{\mathcal{A}^\alpha} = 2\tau(f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}})_{\mathcal{A}^\alpha}.$$

Furthermore, we have

$$\|u^n\|^2_{\mathcal{A}^\alpha} - \|u^{n-1}\|^2_{\mathcal{A}^\alpha} \le 2\tau[\frac{1}{2+\tau}\|u^{n-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha} + \frac{2+\tau}{4}\|f^{n-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}]$$
$$= \frac{\tau}{2(2+\tau)}(\|u^n + u^{n-1}\|^2_{\mathcal{A}^\alpha}) + \frac{\tau(2+\tau)}{2}\|f^{n-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}$$
$$\le \frac{\tau}{(2+\tau)}(\|u^n\|^2_{\mathcal{A}^\alpha} + \|u^{n-1}\|^2_{\mathcal{A}^\alpha}) + \frac{\tau(2+\tau)}{2}\|f^{n-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha},$$

which implies

$$\|u^n\|^2_{\mathcal{A}^\alpha} \le (1+\tau)\|u^{n-1}\|^2_{\mathcal{A}^\alpha} + \tau(1+\frac{\tau}{2})^2\|f^{n-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}.$$

Moreover,

$$\|u^n\|^2_{\mathcal{A}^\alpha} \le (1+\tau)^n\|u^0\|^2_{\mathcal{A}^\alpha} + \sum_{m=0}^{n-1}(1+\tau)^m\tau(1+\frac{\tau}{2})^2\|f^{n-m-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}$$
$$\le (1+\tau)^n\|u^0\|^2_{\mathcal{A}^\alpha} + [\sum_{m=0}^{n-1}(1+\tau)^m]\tau(1+\frac{\tau}{2})^2 \max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}$$
$$\le (1+\tau)^n[\|u^0\|^2_{\mathcal{A}^\alpha} + (1+\frac{\tau}{2})^2 \max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}]$$
$$\le \exp^{n\tau}\|u^0\|^2_{\mathcal{A}^\alpha} + \exp^{(n+1)\tau} \max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}$$
$$\le \exp^{T}\|u^0\|^2_{\mathcal{A}^\alpha} + \exp^{2T} \max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2_{\mathcal{A}^\alpha}.$$

From the inequality above one can see that the solution u^n of the Crank-Nicolson difference scheme (9) depends continuously on the initial value u^0 and the right term f, which shows that the difference scheme (9) is unconditionally stable. The proof is complete.

Now we prove the convergence of the difference scheme (9).

Let $\epsilon^n = U^n - u^n,\ n = 0, 1, ..., N$ denote the errors between the exact solutions and the numerical solutions, and $\epsilon^n = (..., \epsilon^n_{-2},\ \epsilon^n_{-1},\ \epsilon^n_0,\ \epsilon^n_1,\ \epsilon^n_2, ...)^T$. Then from (8), (9), (11) we have

$$\begin{cases} \mathcal{A}^\alpha(\epsilon^n_i - \epsilon^{n-1}_i) = r\sum_{k=-\infty}^{\infty}\omega^{(\alpha)}_k\epsilon^{n-\frac{1}{2}}_{i-k} + \tau\mathcal{A}^\alpha R(\tau,h),\ 1\le n\le N,\ i = 0, \pm1, \pm2, ..., \\ \epsilon^0_i = 0,\ i = 0, \pm1, \pm2, ..., \end{cases} \tag{12}$$

where $\mathcal{A}^\alpha R(\tau, h) = O(\tau^2 + h^6)$.

Similar to the proof process of Theorem 2 one can deduce that

$$\|\epsilon^n\|^2_{\mathcal{A}^\alpha} \le \exp^T\|\epsilon^0\|^2_{\mathcal{A}^\alpha} + \exp^{2T}\|R(\tau,h)\|^2_{\mathcal{A}^\alpha} = \exp^{2T}\|R(\tau,h)\|^2,$$

which implies

$$\|\epsilon^n\|_{\mathcal{A}^\alpha} \le \exp^T\|R(\tau,h)\|.$$

Furthermore, according to lemma 4 we have $\|\epsilon^n\| \le C_1\tau^2 + C_2h^6$, where $C_1,\ C_2$ are two positive constants. So we have the following theorem.

Theorem 3. The Crank-Nicolson difference scheme denoted by (9) is convergent with the accuracy $O(\tau^2 + h^6)$.

5. Numerical experiments

In this section, we present two numerical examples for testing the theoretical analysis results above. In the first example, approximation accuracy for the Riemann-Liouville derivative by use of (6) is checked, while in the second example, the efficiency of the Crank-Nicolson difference scheme (9) is tested.

Example 1. Let $u(x) = x^7,\ x \in [0, 1]$, and consider ${}_0D^\alpha_x u(x) = \dfrac{7!x^{7-\alpha}}{\Gamma(8-\alpha)}$.

In Table 1, we list the errors in L_2 norm and the convergence rates generated by use of the approximation formula (6), where the error is denoted by $\|R(h)\|$, and the convergence rate is defined by $Rate = \dfrac{\ln(\|R(\tau,h_1)\|/\|R(\tau,h_2)\|)}{\ln(h_1/h_2)}$.

Table 1: The L_2 errors and convergence rates for (6)

	$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$	
h	$\Vert R(h)\Vert$	*Rate*	$\Vert R(h)\Vert$	*Rate*	$\Vert R(h)\Vert$	*Rate*
$\frac{1}{8}$	4.7724×10^{-5}		5.3243×10^{-5}		4.8612×10^{-5}	
$\frac{1}{10}$	1.2699×10^{-5}	5.9331	1.4390×10^{-5}	5.8632	1.3465×10^{-5}	5.7532
$\frac{1}{12}$	4.3004×10^{-6}	5.9389	4.9381×10^{-6}	5.8663	4.7128×10^{-6}	5.7580
$\frac{1}{14}$	1.7215×10^{-6}	5.9390	1.9986×10^{-6}	5.8678	1.9348×10^{-6}	5.7755
$\frac{1}{16}$	7.7942×10^{-7}	5.9342	9.1325×10^{-7}	5.8654	8.9897×10^{-7}	5.7402
$\frac{1}{18}$	3.8600×10^{-7}	5.9662	4.6010×10^{-7}	5.8205	4.5676×10^{-7}	5.7486
$\frac{1}{20}$	2.0669×10^{-7}	5.9284	2.5056×10^{-7}	5.7682	2.4616×10^{-7}	5.8675

From the results in Table 1 one can see that the errors are about $O(h^6)$, and the convergence rates are about sixth order, which coincide with the conclusion of (6).

Example 2. Consider the problem (1) with an exact analytical solution

$$u(x,t) = \begin{cases} (t^2+1)x^3(1-x)^3, \ x \in [0,1], \\ 0, \ x \in (-\infty, 0) \bigcup (1, \infty), \end{cases}$$

and satisfies

$$\begin{cases} k = L = 1, \\ f(x,t) = 2tx^3(1-x)^3 - \sum\limits_{n=3}^{6} [\frac{c_n n! x^{-\alpha+n}}{\Gamma(1-\alpha+n)} - \frac{c_n n!(1-x)^{-\alpha+n}}{\Gamma(1-\alpha+n)}], \\ u(x,0) = \varphi(x) = x^3(1-x)^3, \end{cases}$$

where $x^3(1-x)^3 = \sum\limits_{n=3}^{6} c_n x^n$.

Let $\Vert e\Vert_\infty = \max\limits_i |U_i^n - u_i^n|$ denotes the maximum absolute error between the exact solutions and the numerical solutions. By use of the Crank-Nicolson difference scheme (9) we obtain corresponding numerical results, which are shown in Fig. 2 and Table 2 respectively under certain conditions.

Table 2: The maximum absolute errors at $\tau = 10^{-5}$, $h = \frac{1}{10}$

	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
time steps	$\Vert e\Vert_\infty$	$\Vert e\Vert_\infty$	$\Vert e\Vert_\infty$	$\Vert e\Vert_\infty$
10	1.0633×10^{-6}	1.9728×10^{-6}	3.3266×10^{-6}	5.0501×10^{-6}
20	1.0631×10^{-6}	1.9724×10^{-6}	3.3262×10^{-6}	5.0497×10^{-6}
30	1.0626×10^{-6}	1.9717×10^{-6}	3.3255×10^{-6}	5.0491×10^{-6}
40	1.0620×10^{-6}	1.9708×10^{-6}	3.3246×10^{-6}	5.0481×10^{-6}
50	1.0611×10^{-6}	1.9695×10^{-6}	3.3233×10^{-6}	5.0469×10^{-6}
60	1.0601×10^{-6}	1.9680×10^{-6}	3.3218×10^{-6}	5.0454×10^{-6}

From Fig. 2 one can see that the maximum absolute errors between the numerical solutions and the exact solutions lie in a low level with about $O(h^6)$, and the results in both Fig. 2 and Table 2 show that the maximum

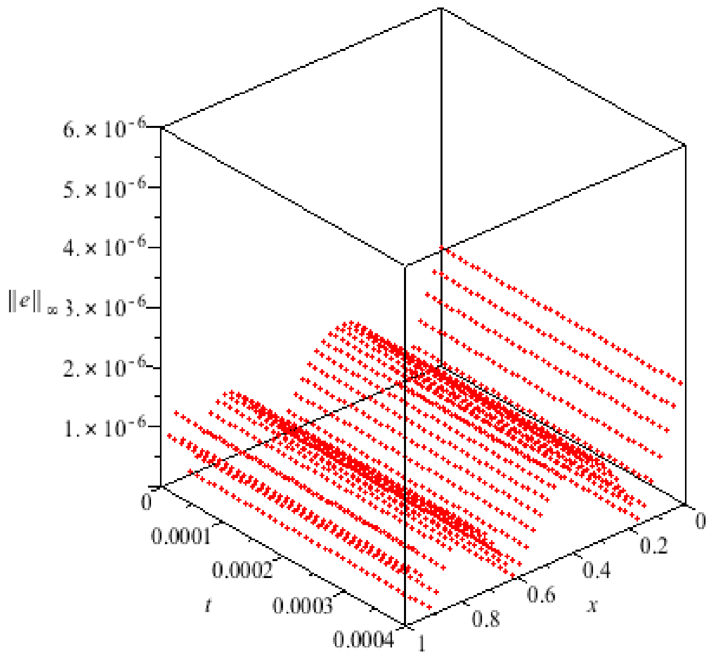

Fig 2. The maximum absolute errors with h=1/40;$\tau=10^{-5}$;α=1.5

absolute errors are stable with the time steps increasing, which coincides with the stability analysis in Section 4 for the Crank-Nicolson difference scheme (9).

6. Conclusions

In this paper, by use of the weighted and shifted Grünwald-Letnikov approximation technique, we have derived an approximation formula with sixth order accuracy for the Riemann-Liouville derivative, and based on this formula constructed a compact Crank-Nicolson difference scheme for a class of space fractional differential equations. The present difference scheme is proved to be unconditionally stable and convergent with accuracy $O(\tau^2 + h^6)$. Numerical experiments are carried out to test the approximation formula and the Crank-Nicolson difference scheme, and the numerical results show their good agreement with the theoretical analytical results.

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