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FRAMES IN TOPOLOGICAL ALGEBRAS

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We present necessary and sufficient conditions for frames in real or complex locally convex commutative separable topological algebras.

Keywords: frames, topological algebras, locally convex algebra, strongly semi-simple algebra.

1. Introduction and Preliminaries

The theory of topological algebras itself has undergone considerable de-elopement since the appearance of Gelfand's paper [2] on normed algebras. The paper is motivated by Watson's work on bases in topological algebras [7]. First, we collect some basic definitions and notions required in the rest of the paper, which can be found in the book [3]. An interested reader for topological algebras may refer to [8]. Let $\mathcal{A}$ be a linear space over the complex field $\mathbb{C}$ (or the real field $\mathbb{R}$). $\mathcal{A}$ is said to be complex (or real) algebra if for all $x, y \in \mathcal{A}$, the product xy is defined and $xy \in \mathcal{A}$, satisfying the following conditions:

(1) $x(yz) = (xy)z = xyz$,
(2) $x(y + z) = xy + xz$,
(3) $(y + z)x = yx + zx$,
(4) $(\lambda x)(\mu y) = (\lambda\mu)xy$, for all $\lambda, \mu \in \mathbb{C}$.

An algebra $\mathcal{A}$ with a Hausdorff topology is called a semi-topological algebra if the maps: $(x, y) \mapsto x + y$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{C} \times \mathcal{A}$ to $\mathcal{A}$, are continuous and the map: $(x, y) \mapsto xy$ is separately continuous. A semi-topological algebra is said to be a topological algebra if the map: $(x, y) \mapsto xy$ is jointly continuous. Clearly, every topological algebra is a semi-topological algebra but the converse is not true (for instance, see [5]). Each topological vector space E contains a base $\{U\}$ of neighborhoods of 0 such that each U is closed, circled, absorbing and for each $U \in \{U\}$, there is $V \in \{U\}$ such that $V + V \subset U$. If $\mathcal{A}$ is a topological algebra, then there is a base of 0-neighborhoods satisfying these conditions and an additional condition: for each $U \in \{U\}$, there are $V, W \in \{U\}$ such that $VW \subset U$. If each member U of a base $\{U\}$ of 0-neighborhoods in a topological algebra is convex, then it is called a *locally*

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convex algebra (*LC-algebra*, for short). Since each convex, circled (together called absolutely convex) and absorbing set U gives rise to a semi-norm p_u defined by: $p_u(x) = inf\{\lambda > 0 : x \in \lambda\, U\}$, we may alternatively describe an *LC-algebra* $\mathcal{A}$ as a topological algebra whose topology is given by a family $\{p_\alpha\}_{\alpha\in\Gamma}$ of semi-norms satisfying:

(1) $p_\alpha(\lambda x) = |\lambda| p_\alpha(x), \lambda \in \mathbb{C}, x \in \mathcal{A},$

(2) $p_\alpha(x + y) \leq p_\alpha(x) + p_\alpha(y), x, y \in \mathcal{A},$

(3) $p_\alpha(x) = 0$ for all $\alpha \in \Gamma$ if $x = 0,$

(4) for each $p_\alpha \in \{p_\alpha\}$, there is a $p_\beta \in \{p_\alpha\}$ such that

$$p_\beta(xy) \leq p_\alpha(x) p_\alpha(y)\ ,\ x, y \in \mathcal{A}\ .$$

An LC-algebra $\mathcal{A}$ is said to be *locally m-convex* if each $p_\alpha \in \{p_\alpha\}_{\alpha\in\Gamma}$ satisfies: $p_\alpha(xy) \leq p_\alpha(x)\, p_\alpha(y)$ for all $x, y \in \mathcal{A}.$ The last inequality for each p_α is equivalent to that $U_\alpha^2 \subset U_\alpha$ (such an U_α is called idempotent) for each α, where $\{U_\alpha\}$ is a sub-base of absolutely convex 0-neighborhoods in which each U_α determines corresponding p_α. A locally convex topology for a topological algebra $\mathcal{A}$ can always be generated by a directed family of semi-norms. The topology of a locally m-convex algebra $\mathcal{A}$ is given by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of *submultiplicative* (i.e., $p_\alpha(xy) \leq p_\alpha(x)\, p_\alpha(y),\ x, y \in \mathcal{A}$) semi-norms. A linear functional f on a locally convex topological algebra $\mathcal{A}$ whose topology is generated by a directed family of semi-norms $\{p_\alpha : \alpha \in \Lambda\}$, is *continuous* if and only if there exists an $\alpha \in \Lambda$ and $c \geq 0$ such that $|f(x)| \leq c\, p_\alpha(x)$ for all $x \in \mathcal{A}$. A collection of linear functionals Φ on a locally convex topological algebra $\mathcal{A}$ (with a directed family of semi-norms $\{p_\alpha : \alpha \in \Lambda\}$ generating its topology) is *equicontinuous* if and only if there exists $\alpha \in \Lambda$ and $c \geq 0$ such that $|f(x)| \leq c\, p_\alpha(x)$ for all $x \in \mathcal{A}$ and $f \in \Phi$.

A topological algebra $\mathcal{A}$ is called *strongly semi-simple* if for every non zero $x \in \mathcal{A}$, there exists $f \in \mathcal{M}(\mathcal{A})$ with $f(x) \neq 0$, where $\mathcal{M}(\mathcal{A})$ is the maximal ideal space of $\mathcal{A}$ consisting of all non-zero continuous scalar valued homomorphisms of $\mathcal{A}$ with the relative $\sigma(\mathcal{A}, \mathcal{A}')$-topology (the Gelfand topology). There is a bijective correspondence between $\mathcal{M}(\mathcal{A})$ and the closed maximal ideals of codimension one of $\mathcal{A}$ given by $f \mapsto M_f = Ker\, f$.

Lemma 1.1. [4] Let $\{R_\alpha\}$ be a family of linear maps from a topological algebra $\mathcal{A}_1$ into a topological algebra $(\mathcal{A}_2, \tau_2)$, such that $\{R_\alpha(x)\}$ is τ_2 bounded in $\mathcal{A}_2$ for each x in $\mathcal{A}_1$. Then there exists a weakest linear topology τ_1 on $\mathcal{A}_1$, such that $\{R_\alpha\}$ is τ_1-τ_2 equicontinuous. Also, τ_1 is generated by $\overline{D} = \{\, \overline{p} : p \in D\}$, where

$$\overline{p}(x) = \sup_\alpha p\big(R_\alpha(x)\big), x \in \mathcal{A}_1.$$

In this paper, we introduce frames in topological algebras. Necessary and sufficient conditions for frames in a topological algebra are given. Some equivalent conditions for the existence of a frame in topological algebra which satisfies certain conditions are obtained.

2. Main Results

In rest part of the paper, $\mathcal{A}$ denotes a real (or complex) locally convex separable topological algebra, assumed to be commutative, and $\mathcal{A}'$ an algebraic dual of $\mathcal{A}$.

Definition 2.1. A countable sequence $\mathcal{F} \equiv \{x_n\} \subset \mathcal{A}$ is called a τ-frame for $(\mathcal{A}, \tau)$ if there exists a sequence $\{f_n\} \subset \mathcal{A}'$, such that for each $x \in \mathcal{A}$

$$x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) x_i$$

where the sequence $\{\sum_{i=1}^{n} f_i(x) x_i\}$ converges in the topology τ of $\mathcal{A}$.

Remark 2.1. The sequence $\{f_n\} \subset \mathcal{A}'$ is called an associated sequence of functionals, which need not be unique. The associated functionals f_n $(n \in \mathbb{N})$ need not be continuous.

Definition 2.2. A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $(\mathcal{A}, \tau)$ is said to be τ-Schauder frame for $\mathcal{A}$ if all associated functionals are τ-continuous.

Definition 2.3. A τ-frame $\mathcal{F} \equiv \{x_n\}$ for $\mathcal{A}$ is an orthogonal frame if

(1) $x_n \neq 0$ for all $n \in \mathbb{N}$,
(2) each x_n is idempotent, i.e., $x_n^2 = x_n$ for all $n \in \mathbb{N}$, and
(3) $x_n x_m = 0$ $(n \neq m)$ for all $m, n \in \mathbb{N}$.

Or equivalently, a τ-frame $\mathcal{F} \equiv \{x_n\}$ for $\mathcal{A}$ is an orthogonal frame if each $x_n \neq 0$ and $x_n x_m = \delta_{nm} x_n$ for all $m, n \in \mathbb{N}$ (where δ_{nm} is the Kronecker delta).

Example 2.1. Let $\mathcal{A} = \{\{\xi_j\} \subset \mathbb{C}: \sum_{j=1}^{\infty} |\xi_j| < \infty\}$, where $\mathbb{C}$ is the set of all complex numbers. Let τ be the topology induced by the standard metric (on $\mathcal{A}$), that is, $d(x, y) = \sum_{j=1}^{\infty} |\xi_j - \eta_j|$, $x = \{\xi_i\}, y = \{\eta_i\} \in \mathcal{A}$. Then, $\mathcal{A}$ is a locally convex separable topological algebra under pointwise multiplication.
Let $\{\chi_n\} \subset \mathcal{A}$ be sequence of canonical unit vectors, i.e., $\chi_n = \delta_{nm}$, for all $n, m \in \mathbb{N}$.

(1) Choose $x_n = \chi_n$ for all $n, m \in \mathbb{N}$. Then, $\mathcal{F} \equiv \{x_n\}$ is an orthogonal frame for $\mathcal{A}$.

(2) Define $\{y_n\} \subset \mathcal{A}$ by $y_1 = \chi_1$, $y_2 = \chi_1$ and $y_n = \chi_{n-1}, n > 2$. Then, $\mathcal{G} \equiv \{y_n\}$ is a τ-frame (Schauder) for $\mathcal{A}$ which is not orthogonal.

Example 2.2. Let $\mathcal{A} = \{\{\xi_j\} \subset \mathbb{C}: \xi_j = 0$ for all except finitely many $j\}$ with the topology induced by the metric $d(x,y) = \sup_{1\leq j\leq\infty} |\xi_j - \eta_j|$, $x = \{\xi_i\}$, $y = \{\eta_i\} \in \mathcal{A}$. Then, $\mathcal{A}$ is a locally convex separable topological algebra under pointwise multiplication.

Define $\{x_n\} \subset \mathcal{A}$ by $x_1 = \chi_1$, $x_2 = \chi_2$ and $x_n = \chi_2 + \frac{\chi_n}{n}, n \geq 3$, where $\{\chi_n\} \subset \mathcal{A}$ is sequence of canonical unit vectors. Choose $\{f_n\} \subset \mathcal{A}'$ as follows:

$$f_1(x) = \chi_1,\ f_2(x) = \chi_2 - \sum_{j\geq 3} j\xi_j\,,\ f_n(x) = n\xi_n, \text{ where } x = \{\xi_j\} \subset \mathcal{A}.$$

Then, $\mathcal{F} \equiv \{x_n\}$ is a frame for $\mathcal{A}$ which is neither Schauder nor orthogonal.

Remark 2.2. Exact τ-frames are studied in [1] and Paley-Wiener type Perturbation results for τ-frames can be found in [6].

The following theorem provides a necessary condition for τ-frames in $\mathcal{A}$.

Theorem 2.1. Assume that $\mathcal{F} \equiv \{x_n\}$ is a τ-frame for $(\mathcal{A}, \tau)$. For each p in the family $D \equiv D_\tau$ of pseudonorms generating the topology τ, let

$$\overline{p} = \sup \{p(S_n(x)): n \geq 1\}$$

where $S_n(x) = \sum_{i=1}^{n} f_i(x)x_i$, $\{f_i\} \subset \mathcal{A}'$.

Then, $\overline{D} = \{\overline{p} : p \in D\}$ defines a linear Hausdorff topology $\overline{\tau} \supset \tau$, such that $\{S_n\}$ is $\overline{\tau}$-τ equicontinuous and that $\overline{\tau}$ is the coarsest linear topology on $\mathcal{A}$ having these properties. Also, each f_n is continuous on $(\mathcal{A}, \overline{\tau})$.

Proof. Let $p \in D$. Then, there exists a q in D, such that $p(x+y) \leq q(x) + q(y)$, for each x, y in $\mathcal{A}$. Since $\mathcal{F}$ is a τ-frame for $\mathcal{A}$, $x = \tau\text{-}\lim_{n\to\infty} S_n(x)$, for each x in $\mathcal{A}$. Now by hypothesis, $p(x) \leq q(x)$, for each x in $\mathcal{A}$. Therefore, $\tau \subset \overline{\tau}$. By Lemma 1.1, $\{S_n\}$ is $\overline{\tau}$-τ equicontinuous and τ is the coarsest topology on $\mathcal{A}$.

Now, for each $n \in \mathbb{N}$, there exists a p and hence q in D, such that $p(x_n) \neq 0$, and

$$|f_n(x)|p(x_n) \leq q(S_n(x)) + q(S_{n-1}(x)) \leq 2\overline{q}(x) \text{ for all } x \in \mathcal{A}.$$

Hence, each f_n is $\overline{\tau}$-continuous.

Remark 2.3. If $(\mathcal{A}, \tau)$ is a metrizable topological algebra in the above theorem, then τ is generated by a single norm p and the corresponding function $\overline{p}$ is also a norm satisfying $\overline{p}(S_n(x)) < \overline{p}\,(x)$ for all $x \in \mathcal{A}$ $(n \in \mathbb{N})$.

Next we give sufficient conditions for τ-frames in $(\mathcal{A},\tau)$.

Theorem 2.2. Let $\{p_k\}$ be a countable family of seminorms generating the topology τ. Suppose $\{x_n\} \subset (\mathcal{A},\tau)$ is a sequence of non-zero vectors such that for each $x \in \mathcal{A}$, there exists a sequence $\{\alpha_n\} \subset \mathbb{K}$ of scalars such that $x = \tau\text{-}\sum_{n=1}^{\infty} \alpha_n x_n$. Let $\mathcal{X} = \{\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{K}: \sum_{n=1}^{\infty} \alpha_n x_n$ converges in the topology $\tau\}$ be a topological algebra with its topology given by the countable family

$$\left\{q_k: q_k(\{\alpha_n\}) = \sup_m \left\{p_k\left(\sum_{n=1}^{m} \alpha_n x_n\right)\right\}\right\}$$

of seminorms. If $\mathcal{Z} = \{\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{X}: \sum_{n=1}^{\infty} \alpha_n x_n = 0\}$ is a complemented subspace of $\mathcal{X}$, then $\{x_n\}$ is a τ-frame for $\mathcal{A}$.

Proof. Let us write $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$. Define $\Theta: \mathcal{X} \to \mathcal{A}$ by

$$\Theta: \{\alpha_n\} \mapsto \tau\text{-}\sum_{k=1}^{\infty} \alpha_k x_k, \qquad \{\alpha_n\} \in \mathcal{X}.$$

Clearly, Θ is a well-defined operator by the definition of $\mathcal{X}$. The restriction of Θ on $\mathcal{Y}$, $\Theta|_{\mathcal{Y}}$ is an isomorphism of $\mathcal{Y}$ onto $\mathcal{A}$.

Define $\{f_k\} \subset \mathcal{A}'$ by

$$\{f_k(x)\} = (\Theta|_{\mathcal{Y}})^{-1}(x), x \in \mathcal{A}$$

Then, each f_k is linear and $x = \Theta\{f_k(x)\} = \tau - \sum_{k=1}^{\infty} f_k(x) x_k$ for all $x \in \mathcal{A}$. Hence $\{f_k\} \subset \mathcal{A}'$ is such that $\{x_n\}$ is a τ -frame for $\mathcal{A}$.

The following theorem provides equivalent conditions for the existence of an orthogonal frame for $\mathcal{A}$

Theorem 2.3. In $(\mathcal{A}, \tau)$, the following are equivalent.

(1) $\mathcal{A}$ has a τ-frame $\mathcal{F} \equiv \{x_n\}$ with $x_n x_m = 0$ for $n \neq m$ $(n, m \in \mathbb{N})$, $x_n \neq 0$ and $x_n^2 \neq 0$, for all $n \in \mathbb{N}$.

(2) $\mathcal{A}$ has a τ-frame $\mathcal{G} \equiv \{y_n\}$ with $y_n y_m = 0$ for $n \neq m$ $(n, m \in \mathbb{N}), y_n \neq 0$, and $y_n^2 = c_n y_n$ $(c_n \neq 0)$, for all $n \in \mathbb{N}$.

(3) $\mathcal{A}$ has an orthogonal frame $\mathcal{H} \equiv \{z_n\}$.

Proof. $(1) \Longrightarrow (2)$: Fix $n \in \mathbb{N}$. Assume that $\mathcal{F} \equiv \{x_n\}$ is a τ-frame for $\mathcal{A}$. Then,

$$x_n^2 = \tau\text{ - }\lim_{k\to\infty} \sum_{i=1}^{k} f_i(x_n^2) x_i, \text{ for some } \{f_i\} \subset \mathcal{A}'.$$

Therefore, by (i), we have

$$0 = x_m x_n^2 = x_m(\tau\text{-}\lim_{k\to\infty} \sum_{i=1}^{k} f_i(x_n^2) x_i) = f_m(x_n^2) x_m^2, \text{ for } m \neq n.$$

By hypothesis, $x_m^2 \neq 0$. Therefore, $f_m(x_n^2) = 0$, for all $m \neq n$ $(m, n \in \mathbb{N})$. Hence $x_n^2 = f_n(x_n^2)x_n$ with $f_n(x_n^2) \neq 0$, for all $n \in \mathbb{N}$. Choose $f_n(x_n^2) = c_n, n \in \mathbb{N}$. Then, $x_n^2 = c_n x_n$ with $c_n \neq 0$, for all $n \in \mathbb{N}$. Let $y_n = x_n$, for all $n \in \mathbb{N}$. Then, $\mathcal{G} \equiv \{y_n\}$ is a τ-frame for $\mathcal{A}$ with desired properties.

(2) $\Longrightarrow$ (3): Suppose that $\mathcal{G} \equiv \{y_n\}$ is a τ-frame satisfying (2). Choose $z_n = \frac{y_n}{c_n}, n \in \mathbb{N}$. First we claim that, $\mathcal{H} \equiv \{z_n\}$ is a frame for $\mathcal{A}$. Let $x \in \mathcal{A}$ be arbitrary. Then, $x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)y_i$ for some $\{f_n\} \subset \mathcal{A}'$. Define $\{g_i\} \subset \mathcal{A}'$ by $g_i(x) = c_i f_i(x)$, for all $i \in \mathbb{N}$. Then

$$x = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} c_i f_i(x)\frac{y_i}{c_i} = \tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} g_i(x) z_i.$$

Hence $\mathcal{H} \equiv \{z_n\}$ is a τ -frame for $\mathcal{A}$.
Now for each $n \in \mathbb{N}$, we have

$$z_n^2 = \frac{y_n^2}{c_n^2} = \frac{c_n y_n}{c_n^2} = \frac{y_n}{c_n} = z_n.$$

Also, for $n \neq m$ $(n, m \in \mathbb{N})$, $z_n z_m = \frac{y_n}{c_n}\frac{y_m}{c_m} = 0$. Therefore, $z_n z_m = \delta_{nm} z_n$, for all $n, m \in \mathbb{N}$. Hence (3) is proved.

(3) $\Longrightarrow$ (1) Obvious. Indeed, let $x_n = z_n$, for all $n \in \mathbb{N}$. Then, by hypothesis $\mathcal{F} \equiv \{x_n\}$ is a τ-frame for $\mathcal{A}$ with desired properties.

Corollary 2.1. If $\mathcal{A}$ has an identity, then each of (1), (2), (3) in Theorem 2.3, is equivalent to the following condition.

(4) $\mathcal{A}$ has a τ-frame $\mathcal{F} \equiv \{x_n\}$ with each $x_n \neq 0$ and $x_n x_m = 0$ $(n \neq m)$, for all $n, m \in \mathbb{N}$.

Proof. It is sufficient to show that (4) is equivalent to (1).
(1)$\Longrightarrow$ (4) : Obvious.
(4)$\Longrightarrow$ (1): Suppose that $e \in \mathcal{A}$. Then, $e = \tau\text{-}\mathit{lim}_{n\to\infty} \sum_{i=1}^{n} f_i(e)x_i$ for some $\{f_n\} \subset A'$. For a fixed j $\in \mathbb{N}$, by hypothesis we have

$$x_j = x_j e = x_j\left(\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(e)x_i\right) = f_j(e)x_j^2\ . \tag{1}$$

Now $x_j \neq 0$, so by using equation (1), $x_j^2 \neq 0$. Hence $x_n^2 \neq 0$, for all $n \in N$.

Remark 2.4. The result given in Theorem 2.3 is a generalization of Theorem 1.10 in [7].

The following proposition gives a necessary condition for $\mathcal{A}$ to be strongly semi-simple in terms of τ-frames for $\mathcal{A}$.

Proposition 2.1. Let $\mathcal{F} \equiv \{x_n\}$ be a τ -frame for $\mathcal{A}$. Then, for all non-zero $x_n, x_n^2 \neq 0$, provided $\mathcal{A}$ is strongly semi-simple.

Proof. Assume that $x_n \neq 0$ and $x_n^2 = 0$ for some n. Then, for each $f \in \mathcal{M}(\mathcal{A})$, we have

$$\left(f(x_n)\right)^2 = f(x_n^2) = f(0) = 0.$$

Therefore, $f(x_n) = 0$, for all $f \in \mathcal{M}(\mathcal{A})$. Thus, $\mathcal{A}$ is not strongly semi-simple, a contradiction. The proposition is proved.

The following theorem shows that every element of the maximal ideal space $\mathcal{M}(\mathcal{A})$ is some coefficient functional associated with an orthogonal frame for $\mathcal{F}$. Let us denote the system of coefficient functional associated with $\mathcal{F}$ by $\mathcal{A}'|_{\mathcal{F}}$.

Theorem 2.4. Let $\mathcal{F} \equiv \{x_n\}$ be an orthogonal frame for $(\mathcal{A}, \tau)$. Then, $\mathcal{M}(\mathcal{A}) \subset \mathcal{A}'|_{\mathcal{F}}$.

Proof. Let $f \in \mathcal{M}(\mathcal{A})$ be arbitrary. Then, by the definition of $\mathcal{M}(\mathcal{A})$, $f \neq 0$. So, there exists an $x_n \in \mathcal{F}$ such that $f(x_n) \neq 0$. Since $\mathcal{F}$ is orthogonal, $x_m x_n = 0$ $(m \neq n)$.Therefore, $f(x_m x_n) = 0$ $(m \neq n)$. Also, since $f \in \mathcal{M}(\mathcal{A})$, $f(x_m x_n) = f(x_m) f(x_n)$. So, $f(x_m) f(x_n) = 0$. But $f(x_n) \neq 0$. Thus, $f(x_m) = 0$ for all $m \neq n$ $(m, n \in \mathbb{N})$. Let $x \in \mathcal{A}$ be arbitrary. Then, there exists $\{f_n\} \subset \mathcal{A}'$ such that

$$f(x) = f\left(\tau\text{-}\lim_{l\to\infty} \sum_{i=1}^{l} f_i(x) x_i\right) = \tau\text{-}\lim_{l\to\infty} \sum_{i=1}^{l} f_i(x) f(x_i) = f_n(x) f(x_n) \qquad (2)$$

Now $f \in \mathcal{M}(\mathcal{A})$, f is multiplicative. Therefore, $f(x_n) = f(x_n^2) = f\,(x_n)^2$. Thus, by using the fact that $f(x_n) \neq 0$, we have

$$f(x_n) = 1. \qquad (3)$$

By using (2) and (3), we have $f(x) = f_n(x)$, for all $x \in \mathcal{A}$. Hence $\mathcal{M}(\mathcal{A}) \subset \mathcal{A}'|_{\mathcal{F}}$.

To conclude the paper, we show that the finite product of locally convex commutative separable topological algebras has a frame provided each component space has a frame.

Theorem 2.5. Let $(\mathcal{A}, \tau_1)$ and $(\mathcal{B}, \tau_2)$ be locally convex commutative separable topological algebras with τ-frames $\mathcal{F} \equiv \{x_n\}$ and $\mathcal{G} \equiv \{y_n\}$, respectively. Then, $(\mathcal{A}\times\mathcal{B}, \tau_0)$, where τ_0 is the product topology on $\mathcal{A}\times\mathcal{B}$, has a τ_0-frame. Furthermore, if $\mathcal{F}$ and $\mathcal{G}$ are orthogonal, then the respective τ_0-frame for $(\mathcal{A}\times\mathcal{B}, \tau_0)$ is also orthogonal.

Proof. Let $(x, y) \in \mathcal{A}\times\mathcal{B}$ be arbitrary. Then, $x = \tau_1\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i$ and $y = \tau_2\text{-}\lim_{m\to\infty} \sum_{j=1}^{m} g_j(y)y_j$, for some $\{f_i\} \subset \mathcal{A}'$ and $\{g_j\} \subset \mathcal{B}'$. Define a sequence $\mathcal{H} \equiv \{z_n\} \subset \mathcal{A}\times\mathcal{B}$ by

$$z_{2n-1} = (x_n, 0), n \in \mathbb{N} \quad \text{and} \quad z_{2n} = (0, y_n), n \in \mathbb{N}.$$

Choose $\{h_n\} \subset (\mathcal{A}\times\mathcal{B})'$ as follows:

$$h_{2j-1}(x, y) = f_j(x), j \in \mathbb{N}, (x, y) \in \mathcal{A}\times\mathcal{B},$$
$$h_{2j}(x, y) = g_j(y), j \in \mathbb{N}, (x, y) \in \mathcal{A}\times\mathcal{B}.$$

Then

$$(x, y) = \tau_0 - \lim_{n\to\infty} \sum_{j=1}^{n} h_j(x, y)z_j.$$

Hence $\mathcal{H} \equiv \{z_n\}$ is a τ -frame for $(\mathcal{A}\times\mathcal{B}, \tau_0)$.

To show $\mathcal{H} \equiv \{z_n\}$ is orthogonal whenever $\mathcal{F}$ and $\mathcal{G}$ are orthogonal. Note that the τ -frame $\mathcal{H}$ has the property that $z_n^2 = z_n$, for all $n \in \mathbb{N}$. Indeed

$$z_{2n}^2 = (0, y_n)^2 = (0, y_n^2) = (0, y_n) = z_{2n}, n \in \mathbb{N},$$

and

$$z_{2n-1}^2 = (x_n, 0)^2 = (x_n^2, 0) = (x_n, 0) = z_{2n-1}, n \in \mathbb{N}$$

Now for $m \neq n$ $(m, n \in \mathbb{N})$, the following cases arise.

(1) m is even, n is odd. Then, $z_m z_n = (0,\ y_m)(x_n, 0) = (0, 0)$.

(2) m is odd, n is even. Then, $z_m z_n = (x_m, 0)(0,\ y_n) = (0, 0)$.

(3) m, n both are even. Then, $z_m z_n = (0, y_m)(0, y_n) = (0,\ y_m y_n) = (0, 0)$.

(4) m, n both are odd. Then, $z_m z_n = (x_m, 0)(x_n, 0) = (x_m x_n,\ 0) = (0, 0)$.

Thus, $z_m z_n = 0$, for all n with $m \neq n$ $(m, n \in \mathbb{N})$. Hence $\mathcal{H} \equiv \{z_n\}$ is an orthogonal frame for $\mathcal{A}\times\mathcal{B}$.

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