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FIXED POINT RESULTS FOR POINTWISE CHATTERJEA TYPE MAPPINGS WITH RESPECT TO A c-DISTANCE IN CONE METRIC SPACES ENDOWED WITH A GRAPH

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In this paper we study the existence of the fixed points for pointwise Chatterjea type mappings with respect to a c-distance in cone metric spaces endowed with a graph. Our results are generalizations of some fixed point theorems given in terms of a c-distance from cone metric spaces equipped with a partial order to cone metric spaces endowed with a graph. The main improvements of our results refer to the fact that we do not need to suppose the continuity of the mapping nor the normality of the cone.

Keywords: c-distance; Cone metric space; Fixed point; Orbitally G-continuous mapping; Pointwise Chatterjea type mapping.

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1. Introduction and preliminaries

Ordered normed spaces and cones have many applications in applied mathematics. Hence, fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century. In 2007, Huang and Zhang [6], using the concept of cone metric space (i.e., by replacing, in the definition of the metric, the set of real numbers by a cone in an ordered Banach space, see also [14]) proved some fixed point theorems for contraction type mappings on complete cone metric spaces (see also [8] and the references cited therein).

Let E be a real Banach space with the zero element θ. A proper nonempty and closed subset P of E is called a cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{\theta\}$. Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \text{ if and only if } y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Moreover, we denote $x \ll y$ if and only if $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P.

If $\operatorname{int} P \neq \emptyset$, then the cone P is called solid. The cone P is named normal if there is a number $k > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq k\|y\|$. The least positive number satisfying the above is called the normal constant of P.

Let $P \subset E$ be a cone, $\preceq$ a partial ordering with respect to P and X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

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(d3) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric [6] or K-metric [14] on X and (X,d) is called a cone metric space [6] or K-metric space [14].

For notions such as convergent and Cauchy sequences, completeness, continuity and etc. in cone metric spaces, we refer to [6, 8]. We shall also make use of the following properties for all $u, v, w, c \in E$ when the cone P may be non-normal.

(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(p_2) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.

(p_3) If $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.

(p_4) Let $x_n \to \theta$ in E, $\theta \preceq x_n$ and $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

In 1996, Kada et al. [9] defined the concept of w-distance in metric spaces. Further, Cho et al. [5] defined the concept of c-distance in cone metric spaces and obtained some fixed point results (see also [12, 13] and the references cited therein).

Definition 1.1 ([5, 13]). *Let (X,d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:*

(q_1) *$\theta \preceq q(x,y)$ for all $x, y \in X$;*

(q_2) *$q(x,z) \preceq q(x,y) + q(y,z)$ for all $x, y, z \in X$;*

(q_3) *for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;*

(q_4) *for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.*

Each w-distance in a metric space (in the sense of Kada et al. [9]) is a c-distance in the cone metric space (X,d) (in the sense of Cho et al. [5]) with $E = \mathbb{R}$ and $P = [0, \infty)$. Moreover, we have three important results

- Each cone metric d on X with a normal cone is a c-distance q on X.
- For a c-distance q, $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.
- For a c-distance q, $q(x,y) = q(y,x)$ does not necessarily hold for all $x, y \in X$.

Lemma 1.1 ([5, 13]). *Let (X,d) be a cone metric space, q be a c-distance on X, let $\{x_n\}$ be a sequence in X and let $\{u_n\}$ be a sequence in P converging to θ. If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

The most important graph theory approach to metric fixed point theory introduced so far is attributed to Jachymski [7]. In this approach, the underlying metric space is equipped with a directed graph and the Banach contraction is formulated in a graph language. Subsequently, Beg et al. [1] and Nicolae et al. [11] extended some results in [7] for the case of set-valued mappings. In 2012, Bojor [2] followed Jachymski's idea for Kannan contractions using a new assumption called the weak T-connectedness of the graph.

We next review some basic notions of graph theory in relation to a cone metric space that we need in the sequel. For more details on the theory of graphs, see [3, 7].

Consider a directed graph G with $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops (that is, $\Delta(X) \subseteq E(G)$, where $\Delta(X) := \{(x,x) \in X \times X : x \in X\}$) and suppose that G has no parallel edges. Then G can be represented by the ordered pair $(V(G), E(G))$. If a cone metric space (X,d) is endowed with the graph G, then we denote it by (X,d,G). Notice also that cone metric space may be also endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$.

If x and y are two vertices in a graph G, then a path in G from x to y is a finite sequence $(x_i)_{i=0}^{N}$ consisting of N + 1 vertices of G such that $x_0 = x$, $x_N = y$, and (x_{i-1}, x_i) is an edge of G for $i = 1, \cdots, N$ and $N \in \mathbb{N}$. A graph G is said to be connected if there exists a path in G between every two vertices of G.

The main purpose of this paper is to give fixed point results for a new type of generalized contraction of Chatterjea type with respect to a c-distance in cone metric spaces endowed with a graph. Our results are generalizations of some fixed point theorems given in terms of a c-distance from cone metric spaces equipped with a partial order to cone metric spaces endowed with a graph. The main improvements of our results refer to the fact that we do not need to suppose the continuity of the mapping nor the normality of the cone. For related results see [2], [4], [7], [10], etc.

2. **Main results**

Following Jachymski [7, Definition 2.4], we define the concept of orbitally G-continuous for self-map T on cone metric spaces.

Definition 2.1. *Let (X, d, G) be a cone metric space endowed with a graph G. A mapping $T : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{b_n\}$ of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$, the convergence $T^{b_n}x \to y$ implies $T(T^{b_n}x) \to Ty$.*

Trivially, a continuous mapping on a cone metric space is orbitally G-continuous for all graphs G but the converse is not generally true. The next example shows that a graph plays an effective role to imply a weaker type of continuity.

Example 2.1. *Let $Y = \mathbb{R}$ and $P = \{x \in Y : x \geq 0\}$. Let $X = [0, +\infty)$ and define a mapping $d : X \times X \to Y$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define the mapping $T : X \to X$ by $T(0) = 1$ and $T(x) = \frac{x}{2}$ for all $x \in X$ with $x \neq 0$. Obviously, T is not continuous at $x = 0$, and in particular, on the whole X. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, x) : x \in X\}$; that is, $E(G)$ contains nothing but all loops. If $x, y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{b_n}x \to y$, then $\{T^{b_n}x\}$ is necessarily a constant sequence. Thus, $T^{b_n}x = y$ for all $n \geq 1$ and so $T(T^{b_n}x) \to Ty$. Hence T is orbitally G-continuous on X.*

In this section, let (X, d) be a cone metric space associated with a c-distance q and endowed with a directed graph G with $V(G) = X$ and $\Delta(X) \subseteq E(G)$. Throughout this section, we denote

$$X_T := \{x \in X : \quad (x, Tx) \in E(G)\}.$$

Our main result is the following theorem for mappings satisfying Chatterjea type conditions with respect to a given c-distance in a complete cone metric space.

Theorem 2.1. *Let (X, d) be a complete cone metric space associated with a c-distance q and endowed with a graph G and $T : X \to X$ be a orbitally G-continuous mapping on X. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- **t1)** *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$, $\gamma(Tx) \leq \gamma(x)$ and $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;*
- **t2)** *T preserves the edges of G; that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*

t3) *for all $x, y \in X$ with $(x, y) \in E(G)$,*

$$q(Tx, Ty) \preceq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(y, Tx),$$
$$q(Ty, Tx) \preceq \alpha(x)q(y, x) + \beta(x)q(Ty, x) + \gamma(x)q(Tx, y).$$

If $X_T \neq \emptyset$, then T has a fixed point on X. Moreover, if $Tv = v$, then $q(v, v) = \theta$.

Proof. Let $x_0 \in X_T$. If $Tx_0 = x_0$, then x_0 is a fixed point of T and the proof is finished. Now, suppose that $Tx_0 \neq x_0$. Since T preserves the edges of G and $(x_0, Tx_0) \in E(G)$, then it follows that by induction $(x_n, x_{n+1}) \in E(G)$, where $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Now, set $x = x_n$ and $y = x_{n-1}$ in (**t3**). Since $(x_{n-1}, x_n) \in E(G)$, we have

$$\begin{aligned} q(x_{n+1}, x_n) &= q(Tx_n, Tx_{n-1}) \\ &\preceq \alpha(x_n)q(x_n, x_{n-1}) + \beta(x_n)q(x_n, x_n) + \gamma(x_n)q(x_{n-1}, x_{n+1}) \\ &\preceq \alpha(Tx_{n-1})q(x_n, x_{n-1}) + \beta(Tx_{n-1})[q(x_n, x_{n+1}) + q(x_{n+1}, x_n)] \\ &\quad + \gamma(Tx_{n-1})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\ &\preceq \alpha(x_{n-1})q(x_n, x_{n-1}) + (\beta + \gamma)(x_{n-1})q(x_n, x_{n+1}) \\ &\quad + \beta(x_{n-1})q(x_{n+1}, x_n) + \gamma(x_{n-1})q(x_{n-1}, x_n) \\ &\vdots \\ &\preceq \alpha(x_0)q(x_n, x_{n-1}) + (\beta + \gamma)(x_0)q(x_n, x_{n+1}) + \beta(x_0)q(x_{n+1}, x_n) \\ &\quad + \gamma(x_0)q(x_{n-1}, x_n). \end{aligned} \tag{1}$$

Similarly, set $x = x_n$ and $y = x_{n-1}$ in (**t3**). Since $(x_{n-1}, x_n) \in E(G)$, we have

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \alpha(x_0)q(x_{n-1}, x_n) + \beta(x_0)q(x_n, x_{n+1}) \\ &\quad + (\beta + \gamma)(x_0)q(x_{n+1}, x_n) + \gamma(x_0)q(x_n, x_{n-1}). \end{aligned} \tag{2}$$

Adding up (1) and (2). Then

$$\begin{aligned} q(x_{n+1}, x_n) + q(x_n, x_{n+1}) &\preceq (\alpha + \gamma)(x_0)[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] \\ &\quad + (2\beta + \gamma)(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n+1})]. \end{aligned}$$

Set $u_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. We get that

$$u_n \preceq (\alpha + \gamma)(x_0)u_{n-1} + (2\beta + \gamma)(x_0)u_n.$$

Thus, we have $u_n \preceq \lambda u_{n-1}$, where $\lambda = \dfrac{(\alpha + \gamma)(x_0)}{1 - (2\beta + \gamma)(x_0)} < 1$ by (**t2**). By repeating the procedure, we get $u_n \preceq \lambda^n u_0$ for all $n \in \mathbb{N}$. Hence,

$$q(x_n, x_{n+1}) \preceq u_n \preceq \lambda^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{3}$$

Now, let $m > n$. It follows from (3) and $\lambda \in [0, 1)$ that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq (\lambda^n + \cdots + \lambda^{m-1})[q(x_1, x_0) + q(x_0, x_1)] \\ &\preceq \frac{\lambda^n}{1 - \lambda}[q(x_1, x_0) + q(x_0, x_1)]. \end{aligned}$$

Since $\frac{\lambda^n}{1-\lambda}[q(x_1, x_0) + q(x_0, x_1)]$ converges to θ, Lemma 1.1 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $\widehat{x} \in X$ such that $x_n = T^n x_0 \to \widehat{x}$ as $n \to \infty$.

We next show that $\widehat{x}$ is a fixed point for T. To this end, note first that since T preserves the edges of G, it follows by induction that $T^n x_0 \in X_T$ for all $n \geq 0$. Thus,

$(T^n x_0, T^{n+1}x_0) \in E(G)$ for all $n \geq 0$. Now, since T is orbitally G-continuous on X, thus $T^{n+1}x_0 = T(T^n x_0) \to T\widehat{x}$ as $n \to \infty$. Because the limit of a convergent sequence is unique, we get $T\widehat{x} = \widehat{x}$. Now, suppose that $Tv = v$. Then, from (**t3**), we have

$$\begin{aligned} q(v,v) &= q(Tv,Tv) \\ &\preceq \alpha(v)q(v,v) + \beta(v)q(v,Tv) + \gamma(v)q(v,Tv) \\ &= (\alpha+\beta+\gamma)(v)q(v,v). \end{aligned}$$

which implies that $q(v,v) = \theta$ by (**t2**) and (p_3). This completes the proof. □

In particular, if the Chatterjea type assumptions are no longer pointwise (i.e., the mappings α, β, γ are constant), we get the following consequence of Theorem 2.1.

Corollary 2.1. *Let (X,d) be a complete cone metric space associated with a c-distance q and endowed with a graph G and $T : X \to X$ be a orbitally G-continuous mapping on X. Suppose that there exist $\alpha, \beta, \gamma \in [0,1)$ such that the following conditions hold:*

- **t1)** $\alpha + 2\beta + 2\gamma < 1$;
- **t2)** *T preserves the edges of G; that is, $(x,y) \in E(G)$ implies $(Tx,Ty) \in E(G)$ for all $x, y \in X$;*
- **t3)** *for all $x, y \in X$ with $(x,y) \in E(G)$,*

$$\begin{aligned} q(Tx,Ty) &\preceq \alpha q(x,y) + \beta q(x,Ty) + \gamma q(y,Tx), \\ q(Ty,Tx) &\preceq \alpha q(y,x) + \beta q(Ty,x) + \gamma q(Tx,y). \end{aligned}$$

If $X_T \neq \emptyset$, then T has a fixed point on X. Moreover, if $Tv = v$, then $q(v,v) = \theta$.

Several consequences of our main result follow now for particular choices of the graph.

For example, if we consider (X,d) endowed with the complete graph G_0 whose vertex set coincides with X (that is, $V(G_0) = X$ and $E(G_0) = X \times X$), then we get the following corollary.

Corollary 2.2. *Let (X,d) be a complete cone metric space associated with a c-distance q and $T : X \to X$ be a orbitally G_0-continuous mapping on X. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0,1)$ such that the following conditions hold:*

- **t1)** $\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$, $\gamma(Tx) \leq \gamma(x)$ *and* $(\alpha + 2\beta + 2\gamma)(x) < 1$ *for all* $x \in X$;
- **t2)** *for all $x, y \in X$,*

$$\begin{aligned} q(Tx,Ty) &\preceq \alpha(x)q(x,y) + \beta(x)q(x,Ty) + \gamma(x)q(y,Tx), \\ q(Ty,Tx) &\preceq \alpha(x)q(y,x) + \beta(x)q(Ty,x) + \gamma(x)q(Tx,y). \end{aligned}$$

Then T has a fixed point on X. Moreover, if $Tv = v$, then $q(v,v) = \theta$.

Suppose now that $(X, \sqsubseteq)$is a poset. Consider on the poset X the graph G_1 given by $V(G_1) = X$ and $E(G_1) = \{(x,y) \in X \times X : x \sqsubseteq y\}$. Since $\sqsubseteq$ is reflexive, it follows that both $E(G_1)$ contain all loops. If we set $G = G_1$ in Theorem 2.1, then the following version of our fixed point theorem in complete cone metric spaces associated with a c-distance q and endowed with a partial order is obtained.

Corollary 2.3. $(X, \sqsubseteq)$ *be a poset and d be cone metric on X such that (X,d) is a complete cone metric space associated with a c-distance q and $T : X \to X$ be a nondecreasing and orbitally G_1-continuous mapping on X. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0,1)$ such that the following conditions hold:*

- **t1)** $\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$, $\gamma(Tx) \leq \gamma(x)$ *and* $(\alpha + 2\beta + 2\gamma)(x) < 1$ *for all* $x \in X$;

t2) *for all* $x, y \in X$ *with* $x \sqsubseteq y$,

$$q(Tx, Ty) \preceq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(y, Tx)$$
$$q(Ty, Tx) \preceq \alpha(x)q(y, x) + \beta(x)q(Ty, x) + \gamma(x)q(Tx, y).$$

Then T *has a fixed point on* X *if there exists* $x_0 \in X$ *such that* $x_0 \sqsubseteq Tx_0$. *Moreover, if* $Tv = v$, *then* $q(v, v) = \theta$.

For our next consequence, suppose again that $(X, \sqsubseteq)$ is a poset. Consider on the poset X the graph G_2 defined by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \sqsubseteq y \ \vee \ y \sqsubseteq x\}$. Then, an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \sqsubseteq)$. If we set $G = G_2$ in Theorem 2.1, then we obtain another fixed point theorem in complete cone metric spaces associated with a c-distance q and endowed with a partial order.

Corollary 2.4. $(X, \sqsubseteq)$ *be a poset and* d *be cone metric on* X *such that* (X, d) *is a complete cone metric space associated with a c-distance* q *and* $T : X \to X$ *be a mapping which maps comparable elements of* X *onto comparable elements. Also let* T *be orbitally* G_2*-continuous on* X. *Suppose that there exist mappings* $\alpha, \beta, \gamma : X \to [0, 1)$ *such that the following conditions hold:*

t1) $\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$, $\gamma(Tx) \leq \gamma(x)$ *and* $(\alpha + 2\beta + 2\gamma)(x) < 1$ *for all* $x \in X$;

t2) *for all* $x, y \in X$ *such that* x *and* y *are comparable,*

$$q(Tx, Ty) \preceq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(y, Tx),$$
$$q(Ty, Tx) \preceq \alpha(x)q(y, x) + \beta(x)q(Ty, x) + \gamma(x)q(Tx, y).$$

Then T *has a fixed point on* X *if there exists* $x_0 \in X$ *such that* x_0 *and* Tx_0 *are comparable. Moreover, if* $Tv = v$, *then* $q(v, v) = \theta$.

Let $e \in int\ P$ is a fixed. Recall that two elements $x, y \in X$ are said to be e-close if $d(x, y) \preceq e$. Define the e-graph G_3 by $V(G_3) = X$ and $E(G_3) = \{(x, y) \in X \times X : \ d(x, y) \preceq e\}$. We see that $E(G_3)$ contains all loops. Finally, if we set $G = G_3$ in Theorem 2.1, then we get the following consequence of our fixed point theorem in complete cone metric spaces associated with a c-distance q.

Corollary 2.5. *Let* (X, d) *be a complete cone metric space associated with a c-distance q,* $e \in int\ P$ *and* $T : X \to X$ *be a mapping which maps e-close elements of* X *onto e-close elements. Also let* T *be orbitally* G_3*-continuous on* X. *Suppose that there exist mappings* $\alpha, \beta, \gamma : X \to [0, 1)$ *such that the following conditions hold:*

t1) $\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$, $\gamma(Tx) \leq \gamma(x)$ *and* $(\alpha + 2\beta + 2\gamma)(x) < 1$ *for all* $x \in X$;

t2) *for all* $x, y \in X$ *such that* x *and* y *are e-close elements,*

$$q(Tx, Ty) \preceq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(y, Tx),$$
$$q(Ty, Tx) \preceq \alpha(x)q(y, x) + \beta(x)q(Ty, x) + \gamma(x)q(Tx, y).$$

Then T *has a fixed point on* X *if there exists* $x_0 \in X$ *such that* x_0 *and* Tx_0 *are e-close elements. Moreover, if* $Tv = v$, *then* $q(v, v) = \theta$.

The following example shows the usefulness of our main results.

Example 2.2. *Let* $E = C^1_{\mathbb{R}}[0, 1]$ *with the norm* $\|\varphi\| = \|\varphi\|_\infty + \|\varphi'\|_\infty$, $X = [0, 1]$ *and consider the non-normal cone* $P = \{\varphi \in E : \varphi(t) \geq 0$ *on* $[0, 1]\}$. *Also, let a mapping* $d : X \times X \to Y$ *introduced by* $d(x, y) = |x - y| \cdot \varphi(t)$ *for all* $x, y \in X$, *where* $\varphi(t) = 2^t \in P \subset E$ *with* $t \in [0, 1]$. *Then* (X, d) *is a cone metric space with non-normal solid cone. Take mapping*

$q : X \times X \to E$ defined by $q(x,y)(t) = y \cdot 2^t$ for all $x,y \in X$, where $t \in [0,1]$. Then q is a c-distance. Consider the mapping $T : X \to X$ by $T(\frac{1}{2}) = \frac{1}{16}$ and $T(x) = \frac{x^3}{4}$ for all $x \in X$ with $x \neq \frac{1}{2}$. Obviously, T is not continuous at $x = \frac{1}{2}$, and in particular, on the whole X. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x,x) : x \in X\}$; that is, $E(G)$ contains nothing but all loops. Observe that for all $x,y \in X$ such that $(x,y) \in E(G)$, we get $x = y$. If $x,y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{b_n}x \to y$, then $\{T^{b_n}x\}$ is necessarily a constant sequence. Thus, $T^{b_n}x = y$ for all $n \geq 1$ and so $T(T^{b_n}x) \to Ty$. Hence, T is orbitally G-continuous on X. Take mappings $\alpha(x) = \frac{x^2}{2}$, $\beta(x) = \frac{x}{3}$ and $\gamma(x) = 0$ for all $x \in X$. Observe that:

1) *if $x \neq \frac{1}{2}$, then $\alpha(Tx) = \alpha(\frac{x^3}{4}) = \frac{x^6}{32} \leq \frac{x^2}{2} = \alpha(x)$ and if $x = \frac{1}{2}$, then $\alpha(T\frac{1}{2}) = \frac{1}{512} \leq \frac{1}{8} = \alpha(\frac{1}{2})$;*
2) *if $x \neq \frac{1}{2}$, then $\beta(Tx) = \beta(\frac{x^3}{4}) = \frac{x^3}{12} \leq \frac{x}{3} = \beta(x)$ and if $x = \frac{1}{2}$, we have $\beta(T\frac{1}{2}) = \frac{1}{48} \leq \frac{1}{6} = \beta(\frac{1}{2})$;*
3) *$\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
4) *$\alpha(x) + \beta(x) + \gamma(x) = \frac{x^2}{2} + \frac{x}{3} < 1$ for all $x \in X$;*
5) *let $x \in X$ with $(x,x) \in E(G)$. If $x \neq \frac{1}{2}$, then*
$$q(Tx,Tx)(t) = \frac{x^3}{4} \cdot 2^t \leq \alpha(x)q(x,x)(t) + \beta(x)q(x,Tx)(t) + \gamma(x)q(x,Tx)(t)$$
and if $x = \frac{1}{2}$, then
$$q(T\frac{1}{2}, T\frac{1}{2})(t) = \frac{1}{16} \cdot 2^t \leq \alpha(\frac{1}{2})q(\frac{1}{2},\frac{1}{2})(t) + \beta(\frac{1}{2})q(\frac{1}{2},T\frac{1}{2})(t) + \gamma(\frac{1}{2})q(\frac{1}{2},T\frac{1}{2})(t).$$
Similarly, for other relation, one can apply above approach with substitute first component with second component.
6) *Since $(0, T0) = (0,0) \in E(G)$, so $X_T \neq \emptyset$.*

Thus, all the conditions of Theorem 2.1 are satisfied. Clearly, T has a fixed point $x = 0 \in [0,1]$ and $q(0,0) = 0$.

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