MULTITIME SCHRÖDINGER SPATIAL SOLITONS

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In this paper, we introduce and analyse a multitime nonlinear Schrödinger PDE, using geometrical ingredients. Then we derive important multitime analytical solutions for the multitime Schrödinger PDE, with a variable nonlinearity coefficient, called multitime Schrödinger spatial solitons. We establish explicit formulas fixing some coefficients. Our results prove that the multitime solitons can be controlled by selecting appropriately the nonlinearity coefficient.

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1. Single-time NLSE and multitime approach

The nonlinear PDEs describe a great range of phenomena in physics, chemistry, biology or telecommunications. The nonlinear Schrödinger PDE

$$i\Psi_t + \Delta\Psi + f(x, |\Psi|^2)\Psi = 0, \quad (1)$$

where $\Psi = \Psi(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is a C^2 unknown complex function and Δ denotes the Laplacian with respect to the space variable $x \in \mathbb{R}^n$, is one of the most important models of modern nonlinear science. The nonlinear Schrödinger equation (NLSE) appears in many branches of physics and applied mathematics, such as hidrodynamics, semiconductor electronics, photonics, dynamics of accelerators, quantum field theory, condensed matter, plasma physics, nonlinear optics and nonlinear acoustics, heat pulses in solids and other various nonlinear instability phenomena (see [1], [2], [4], [10], [11]). The NLSE has important applications in areas like wave propagation in nonlinear media, surface waves on deep waters and signal propagation in optical fibers. In quantum mechanics, the Schrödinger PDE describes how the quantum state of some physical system changes with time.

The best known solutions of the NLSE are those for solitary waves or solitons. Solitons - waves that do not disperse as they travel through a medium, represent the most recent and remarkable development in the revolution of telecommunications technology. Due to their intrinsic stability, optical solitons have been proposed to be used as an information carrier for the long-distance fiber optic communications and the optical signal processing.

A solution of NLSE is a complex valued function $\Psi = \Psi(x, t)$ of two real variables, x-the position and t-the time, called *wave function* and its squared modulus $|\Psi|^2$ is a positive real number, which represents the probability density. In quantum

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mechanics, it describes the quantum state of a system of one or more particles and contains all the information about the physical system. The wave function behaves qualitatively like other waves, like water waves or waves on the string, because the Schrödinger equation is mathematically a type of wave equation, which determines how the wave function evolves over time.

1.1. The NLSE in Differential Geometry

In 1906, the NLSE have appeared in a study of the motion of a isolated vortex filament in an unbounded liquid (see [5]) and later, in 1972, H. Hasimoto have derived it, in a geometric manner, from the motion of a spatial curve, considered inextensible, along the binormal field, which describe the three-dimensional motion of a vortex in a inviscid fluid (see [12]). The time evolution of the curve, given by an extension, satisfies a Cauchy problem, where the PDE models the motion of the curve along the binormal field and, this way, it is generated a soliton surface, called *Hasimoto surface* or *nonlinear Schrödinger surface.*

1.2. Multitime extensions of PDEs

The term of "multitime" or "multi-temporal" for evolutions PDEs appeared to model mathematically certain natural phenomena, in the context of applications in different fields of science.

The ideas to create multitime versions for evolution PDEs are provided by many works of Udrişte and his PhDs or PhD students (see [13]-[17]) and they are based on the ingredients from Differential Geometry (derivation, trace, etc.). Thus, in order to generate multitime PDEs, we will use geometrical objects from the first jet bundle (metric, connection, vector fields, tensor fields, etc.).

2. Multitime NLSE in a given direction

We consider the NLSE (1)

$$i\Psi_t + \Delta\Psi + f(x, |\Psi|^2)\Psi = 0,$$

where $\Psi = \Psi(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is the C^2 unknown complex function and we suppose that the spatial coordinate x is one-dimensional, so that the PDE (1) can be written in the form

$$iu_t + u_{xx} + f(x, |u|^2)u = 0, \tag{2}$$

or, equivalently,

$$u_t = i(u_{xx} + f(x, |u|^2)u), \tag{3}$$

where $u = u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ becomes the C^2 unknown complex function. For this last equation we will introduce the multitime version, supposing that the temporal variable t is m-dimensional, $t = (t^\alpha)$, $\alpha = 1, \dots, m$. Because this multi-temporal evolution parameter $t = (t^1, \dots, t^m)$ is called *multitime*, the new PDE will be called *multi-temporal* or *multitime PDE.*

In order to introduce the new multitime PDE, we will take the multitime $t = (t^1, \dots, t^m)$ in $\mathbb{R}^m$, $\mathbb{R}^m$ being endowed with the *product order* and we will use various objects from the geometry of the jet bundle of order one $J^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{C})$, associated to the C^2 function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{C}$. So, we consider a distinguished vector field $H = (h^\alpha)$, $h^\alpha = h^\alpha(x,t)$, $\alpha = 1,\dots,m$, on $\mathbb{R}^m$ and it will define the *multitime derivation operator (along the direction H)* (see [17])

$$D_H u = h^\alpha\, u_{t^\alpha}. \tag{4}$$

Using this operator, we build the multitime PDE

$$h^\alpha\, u_{t^\alpha} = i(u_{xx} + f(x,|u|^2)u), \tag{5}$$

which will be called *multitime nonlinear Schrödinger equation in the direction of H*, or *multitime NLSE.*

The existence of the solutions to this multitime PDE is ensured by the following theorem, a similar result being provided in a previous paper regarding the subject of the multitime PDEs (see [17]).

Theorem 2.1. *There exists an infinity of distinguished vector fields $H = (h^\alpha)$ on $\mathbb{R}^m$ such that a solution to the nonlinear Schrödinger PDE (3) is also a solution to the multitime nonlinear Schrödinger PDE (5).*

Proof. Let $t^1 = t$ and $u = u(x,t^1)$. Suppose $u = u(x,t^1)$ is a solution to the nonlinear Schrödinger PDE (3). The function $v(x,t^1,...,t^m) = u(x,t^1)$ is a solution to the multitime nonlinear Schrödinger PDE (5) if the family of distinguished vector fields $H = (h^\alpha)$ is fixed by $h^1 = 1$. It is obvious that we have an infinity of distinguished vector fields that satisfies this algebraic equation. □

The foregoing theorem justifies the terminology *multitime geometrical prolongation of the nonlinear Schrödinger PDE.* The reasons and the relations of fixing a vector field $H = (h^\alpha)$ depend on the problem that we want to solve. Many of the core ideas in PDEs theory can be reformulated using an appropriate vector field H.

3. Multitime spatial solitons as solutions to the multitime NLSE in a given direction

We will show that the new multitime PDE (5) admits *multitime soliton solutions*, so this multitime PDE will be called *multitime soliton PDE.* For this reason, we are looking for the solutions of multitime PDE (5) of the form of *multitime spatial solitons*

$$u(x,t) = \phi(x)\, e^{-i\omega_\alpha t^\alpha}, \tag{6}$$

the summation being after α, $\alpha = 1,\dots,m$ and where $\phi : \mathbb{R} \to \mathbb{R}$ is a function so that $|\phi(x)| \to 0$ exponentially as $|x| \to \infty$ (these requirements are an example of a boundary condition) and (ω_α) is a constant index vector. The form (6) of the multitime solutions that we searched was inspired by some directions of the research work in the areas of the soliton theory for NLSE, imposed by the mathematician F. Genoud in many papers (see [6]-[9]).

From (6), we have

$$u_x = \phi'(x)\, e^{-i\omega_\alpha t^\alpha}, \quad u_{xx} = \phi''(x)\, e^{-i\omega_\alpha t^\alpha}, \quad u_{t^\alpha} = -i\,\omega_\alpha\, e^{-i\omega_\alpha t^\alpha},$$

while

$$|u| = |\phi(x)|.$$

Replacing all these from above into equation (5) we get that the function ϕ must satisfy the following second order Riccati ODE

$$\phi''(x) + (h^{\alpha}(x,t)\,\omega_{\alpha} + f(x,\phi^2(x)))\phi(x) = 0, \tag{7}$$

Thus, our problem to find multitime soliton solutions of the multitime NLSE (5) is reducing to the solving of the second order Riccati ODE (7). In order to find some solutions, we make different choices for the combination $h^{\alpha}(x,t)\,\omega_{\alpha}$ and for the function $f(x,\phi^2(x))$ and we will use the direct integration method . All the particular choices we will make are possible because we work on the first order jet bundle $J^1(\mathbb{R}\times\mathbb{R}^m,\mathbb{C})$.

Case 1 If we assume that

$$h^{\alpha}(x,t)\,\omega_{\alpha} = \frac{{\phi'}^2(x)}{\phi^2(x)}, \quad f(x,\phi^2(x)) = -\frac{a}{\phi^2(x)} \quad (a\in\mathbb{R}),$$

the ODE (7) becomes $\phi''(x)\phi(x) + {\phi'}^2(x) - a = 0$, that is $(\phi\phi')' = a$. Integrating twice this last equation, we obtain that $\phi^2 = ax^2 + bx + c$, whence

$$\phi(x) = \pm\sqrt{ax^2+bx+c}, \tag{8}$$

where b, c are real constants, for which $ax^2+bx+c>0,$ for all $x\in\mathbb{R}$.

Case 2 If we impose the conditions

$$h^{\alpha}(x,t)\,\omega_{\alpha} = \frac{\phi'(x)}{x\phi(x)}, \quad f(x,\phi^2(x)) = -\frac{a}{x\phi(x)} \quad (a\in\mathbb{R}),$$

the ODE (7) becomes $x\phi''(x) + \phi'(x) = a$, which leads to $(x\phi'(x))' = a$ and after two successive integrations, we obtain the solution

$$\phi(x) = ax + b\ln|x| + c, \tag{9}$$

where b, c are real constants and $x\in\mathbb{R}-\{0\}$.

Case 3 If we take

$$h^{\alpha}(x,t)\,\omega_{\alpha} + f(x,\phi^2(x)) = -a^2 \quad (a\in\mathbb{R}),$$

the ODE (7) becomes $\phi''(x) - a^2\phi(x) = 0$, which is a linear homogeneous ODE with constant coefficients that admits the general solution

$$\phi(x) = k_1e^{ax} + k_2e^{-ax}, \tag{10}$$

where k_1, k_2 are real constants.

Case 4 If we make the choices

$$h^{\alpha}(x,t)\,\omega_{\alpha} = -a, \quad f(x,\phi^2(x)) = K\phi^2(x) \quad (a\in\mathbb{R}_+,\, K\in\mathbb{R}_+^*),$$

the ODE (7) becomes $\phi''(x) - a\phi(x) + K\phi^3(x) = 0$, which has the solution

$$\phi(x) = \pm\sqrt{\frac{2a}{K}}\,\mathrm{sech}(\sqrt{2a}x), \tag{11}$$

obtained by the cosine method in the paper [3].

Theorem 3.1. *Under the assumptions of the above particular cases, if $\phi(x)$, given by the relations (8)-(11), is a solution to the Riccati ODE (7), then the multitime NLSE (5) admits the following families of multitime spatial solitons, as multitime soliton solutions:*

$$u(x,t) = \pm\sqrt{ax^2 + bx + c}\, e^{-i\omega_\alpha t^\alpha}, \tag{12}$$

where a, b, $c \in \mathbb{R}$ *are fixed by the condition* $ax^2 + bx + c > 0$, *for all* $x \in \mathbb{R}$,

$$u(x,t) = (ax + b\ln|x| + c)\, e^{-i\omega_\alpha t^\alpha}, \tag{13}$$

where $a \in \mathbb{R}$ *is fixed,* b, $c \in \mathbb{R}$ *are arbitrary and* $x \in \mathbb{R} - \{0\}$,

$$u(x,t) = (k_1 e^{ax} + k_2 e^{-ax})\, e^{-i\omega_\alpha t^\alpha}, \tag{14}$$

where $a \in \mathbb{R}$ *is fixed and* k_1, $k_2 \in \mathbb{R}$ *are arbitrary and, respectively,*

$$u(x,t) = \pm\sqrt{\frac{2a}{K}}\,\mathrm{sech}(\sqrt{2a}x)\, e^{-i\omega_\alpha t^\alpha}, \tag{15}$$

where $a \in \mathbb{R}_+$, $K \in \mathbb{R}_+^*$ *are fixed.*

4. **Conclusions**

We introduced a multitime version of the NLSE, using a first order multitime derivation operator, along the direction of a certain distinguished vector field. We proved that the new multitime NLSE admits solutions and we found explicit formulas for the multitime soliton solutions of the form of multitime spatial solitons, by fixing appropriately some coefficients.

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