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BANACH SPACE PROPERTIES SUFFICIENT FOR THE DOMÍNGUEZ-LORENZO CONDITION

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In this paper, we consider some geometric properties on Banach spaces concerning the García-Falset coefficient and the von Neumann-Jordan type constant, which imply the Domínguez-Lorenzo condition and thus the existence of fixed points for multivalued nonexpansive mappings. The obtained results generalize some previous results in the recent literature. We also show that our results are sharp.

Keywords: Multivalued nonexpansive mapping, Fixed point, (DL)-condition, Normal structure, García-Falset coefficient, von Neumann-Jordan type constant.

1. Introduction

In 1969, Nadler [1] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then, the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. One of the first results in this direction was established by Lim [2] in the framework of a uniformly convex Banach space. Later on, by using Edelstein's method of asymptotic centers, Kirk and Massa [3] proved the existence of a fixed point for a multivalued nonexpansive self-mapping in a Banach space, for which the asymptotic center of any bounded sequence in a closed bounded convex subset is nonempty and compact.

Despite of the above results, some important questions remain still open, for instance, the possibility of extending the celebrated Kirk's theorem [4], i.e., do Banach spaces with weak normal structure (ω-NS) have the fixed point property (FPP) for multivalued nonexpansive mappings?

In 2004, Domínguez Benavides and Lorenzo [5] proved that nearly uniformly convex spaces have the FPP for multivalued nonexpansive mappings with compact and convex values. Dhompongsa et al. [6] noticed that the main tool used in the proof of that result is a relationship concerning the Chebyshev radius of the sequence. Consequently, they introduced the so called Domínguez-Lorenzo condition ((DL)-condition, in short), which implies ω-NS (see [6]) of a Banach space and in turn the FPP for multivalued nonexpansive mappings (see [7]).

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Therefore, a first approach to the problem of extending Kirk's theorem is to study whether properties implying ω-NS also imply the (DL)-condition. Positive results will give only partial answers to the problem, because it is known that uniform normal structure does not imply the (DL)-condition (see [8]).

Recently, many geometric constants for a Banach space have been investigated. Among them, the von Neumann-Jordan constant is one of the most widely studied geometric constants. In connection with the celebrated work of Jordan and von Neumann concerning inner products [9], the von Neumann-Jordan constant of $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [10] as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\| x+y \|^2 + \| x-y \|^2}{2(\| x \|^2 + \| y \|^2)} \leq C$$

holds for all $x, y \in X$ with $(x, y) \neq (0,0)$. If C is the best possible constant on the right-hand side of the above inequality, then so is $\frac{1}{C}$ on the left-hand one. Such a constant is important due to its strong connection with some useful geometric properties.

Throughout this paper, we assume that X be a Banach space with the unit sphere $S_X = \{x \in X \,:\, \| x \| = 1\}$ and the closed unit ball $B_X = \{x \in X \,:\, \| x \| \leq 1\}$.

Recently, Takahashi [11] has introduced the von Neumann-Jordan type constant by

$$C_t(X) = \sup\left\{\frac{J_{X,t}^2(\tau)}{1+\tau^2} \,:\, 0 \leq \tau \leq 1\right\}$$

for $-\infty \leq t < \infty$, where the James type constant $J_{X,t}(\tau)$ is defined as

$$J_{X,t}(\tau) = \begin{cases} \sup\left\{\left(\frac{\| x+\tau y \|^t + \| x-\tau y \|^t}{2}\right)^{\frac{1}{t}} \,:\, x, y \in S_X\right\}, & -\infty < t < \infty, t \neq 0, \\ \sup\left\{\sqrt{\| x+\tau y \|, \| x-\tau y \|} \,:\, x, y \in S_X\right\}, & t = 0, \\ \sup\{\min\{\| x+\tau y \|, \| x-\tau y \|\} \,:\, x, y \in S_X\}, & t = -\infty. \end{cases}$$

Here, we remark that $J(X) = J_{X,-\infty}(1)$. By taking $t = -\infty$ in the definition of $C_t(X)$, we get the constant

$$C_{-\infty}(X) = \sup\left\{\frac{J_{X,-\infty}^2(\tau)}{1+\tau^2} \,:\, 0 \leq \tau \leq 1\right\}.$$

It is obvious that the von Neumann-Jordan type constant includes some known constants, such as the von Neumann-Jordan constant $C_{NJ}(X)$ (see [10]) and the Zbăganu constant $C_Z(X)$ (see [12]). These constants are defined by $C_{NJ}(X) =$

$C_2(X)$ and $C_Z(X) = C_0(X)$. As regards the above constants, the following inequalities do hold (see [11]):

$$\frac{1}{2}(J(X))^2 \leq C_{-\infty}(X) \leq C_Z(X) \leq C_{NJ}(X). \qquad (1.1)$$

Many recent studies have focused on geometric properties concerning some well known moduli and coefficients, which imply normal structure of Banach spaces and the existence of fixed points for multivalued nonexpansive mappings. For more details in this direction, we refer the reader to [6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references mentioned therein.

The purpose of this work is to investigate some geometric conditions on a Banach space X in terms of the García-Falset coefficient and the von Neumann-Jordan type constant, which imply the Domínguez-Lorenzo condition and thus the existence of fixed points for multivalued nonexpansive mappings. Our main results generalize some existing results in the literature on this topic. Moreover, we show that the results are sharp.

2. Preliminaries

In the following lines, we give some notions and definitions which will be needed in the sequel.

Let X be a Banach space and E be a nonempty subset of X. We shall denote by $CB(E)$ the family of all nonempty bounded closed subsets of E and by $KC(E)$ the family of all nonempty compact convex subsets of E.

A multivalued mapping $T: E \longrightarrow CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \| x - y \|, \qquad x, y \in E,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} \| x - y \|, \quad \sup_{y\in B}\inf_{x\in A} \| x - y \|\right\}, \qquad A, B \in CB(X).$$

A point $x \in E$ is a fixed point of T if and only if x is contained in Tx.

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(E, \{x_n\})$ and the asymptotic center $A(E, \{x_n\})$ of $\{x_n\}$ in E are defined by

$$r(E, \{x_n\}) = \inf\left\{\limsup_{n\to\infty} \| x_n - x \| \quad : \quad x \in E\right\}$$

and

$$A(E, \{x_n\}) = \left\{x \in E \quad : \quad \limsup_{n\to\infty} \| x_n - x \| = r(E, \{x_n\})\right\},$$

respectively. It is known that $A(E,\{x_n\})$ is a nonempty weakly compact convex set whenever E is (see [22]).

The sequence $\{x_n\}$ is called regular with respect to E if $r(E,\{x_n\}) = r\left(E,\{x_{n_i}\}\right)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 2.1. (Goebel [23], Lim [2]) *Let $\{x_n\}$ and E be as above. Then there always exists a subsequence of $\{x_n\}$ which is regular with respect to E.*

Let C be a nonempty bounded subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C) \coloneqq \inf\{\sup\{\| x-y \| \ : \ y \in C\} \ : \ x \in E\}.$$

In 2006, Dhompongsa et al. [6] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) as follows.

Definition 2.2. ([6]) A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular with respect to E,

$$r_E\left(A(E,\{x_n\})\right) \le \lambda\, r(E,\{x_n\}).$$

In [6, Theorem 3.2] it was proved that the (DL)-condition implies weak normal structure. We recall that a Banach space X is said to have weak normal structure (ω-NS) if for every weakly compact convex subset K of X with $\operatorname{diam}(K) \coloneqq \sup\{\| x-y \| \ : \ x,y \in K\} > 0$, there exists $x \in K$ such that $\sup\{\| x-y \| \ : \ y \in K\} < \operatorname{diam}(K)$.

The (DL)-condition also implies the existence of fixed points for multivalued nonexpansive mappings.

Theorem 2.3. ([7]) *Let E be a nonempty weakly compact convex subset of a Banach space X and $T: E \longrightarrow KC(E)$ be a nonexpansive mapping. If X satisfies the (DL)-condition, then T has a fixed point.*

In the attempts to find sufficient conditions for the weak fixed point property, many other geometrical properties have been described. Next, we will recall some of these properties involving normal structure.

Definition 2.4. ([24]) A Banach space X has the Opial property if for every weakly null sequence $\{x_n\}$ and every $x \neq 0$ in X,

$$\liminf_{n\to\infty} \| x_n \| < \liminf_{n\to\infty} \| x_n + x \|.$$

We will say that X satisfies the nonstrict Opial property if

$$\liminf_{n\to\infty} \| x_n \| \le \liminf_{n\to\infty} \| x_n + x \|$$

under the same conditions.

The Opial modulus of X [25] is defined for $c \geq 0$ as

$$r_X(c) = \inf \left\{ \liminf_n \| x_n + x \| - 1 \right\},$$

where the infimum is taken over all $x \in X$ with $\| x \| \geq c$ and all weakly null sequences $\{x_n\}$ in X with $\liminf_n \| x_n \| \geq 1$.

This modulus satisfies the following properties (see [26, 27]):

- r_X is nondecreasing and continuous in $[0, \infty)$.
- $c - 1 \leq r_X(c) \leq c$ for all $c \geq 0$. In particular, $r_X(c) > 0$ for all $c > 1$.
- If $r_X(0) < 0$, then r_X is constant in $[0, -r_X(0)]$.
- X has the nonstrict Opial property if and only if $r_X(c) \geq 0$ for all $c \geq 0$.

It is known that a space X with the Opial property has ω-NS. Furthermore, the condition $r_X(1) > 0$ implies weak uniform normal structure (see [25]).

Recall that a Banach space X is called uniformly non-square provided that there exists $\delta > 0$ such that either $\| x + y \| \leq 2 - \delta$ or $\| x - y \| \leq 2 - \delta$ for all $x, y \in B_X$. In [28] it was proved that uniformly non-square Banach spaces are reflexive.

In 1997, García-Falset [29] introduced the following coefficient, the so-called García-Falset coefficient,

$$R(X) = \sup \left\{ \liminf_{n \to \infty} \| x_n + x \| \right\},$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all $x \in S_X$. He proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property (see [22, 30]). Here, we remark that $1 \leq R(X) \leq 2$.

3. Main Results

Throughout this section, let X be a Banach space without the Schur property, that is, there is a weakly convergent sequence which is not norm convergent.

The following result is the heart of this paper.

Theorem 3.1. *Let X be a Banach space and let E be a weakly compact convex subset of X. Suppose that $\{x_n\}$ is a bounded sequence in E which is regular relative to E. Assume that $r_X\left(\frac{1}{2}\right) \geq 0$ and*

$$C_{-\infty}(X) < 1 + \frac{1}{(R(X))^2}.$$

Then there exists $\lambda \in (0,1)$ such that

$$r_C\big(A(E, \{x_n\})\big) < \lambda\, r(E, \{x_n\}).$$

In particular, if X satisfies the nonstrict Opial condition, then

$$\lambda = \frac{C_{-\infty}(X)}{1 + \frac{1}{(R(X))^2}}.$$

Proof. If $r_X\left(\frac{1}{2}\right) > 0$, then $r_X(1) > 0$ and the result follows from Corollary 2 in [8].

Now, suppose that $r_X\left(\frac{1}{2}\right) = 0$. For convenience, we denote $r = r(E, \{x_n\})$ and $A = A(E, \{x_n\})$. We can assume $r > 0$. Since E is a weakly compact set, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in E$.

If $z \in A$ $(z \neq x)$, then $\limsup_n \| x_n - z \| = r$. Since the norm is weak lower semicontinuity, it follows that

$$\|z - x\| \leq \liminf_n \|z - x_n\| \leq \limsup_n \|x_n - z\| = r. \tag{3.1}$$

On the other hand,

$$B \coloneqq \limsup_n \|x_n - x\| \geq r.$$

Fix $\varepsilon > 0$. By passing through a subsequence, if necessary, we can assume that

$$\|x_n - x\| \leq B + \varepsilon, \qquad \text{for all } n \in \mathbb{N} \tag{3.2}$$

By applying (3.1) and (3.2) and taking into account that $x_n - x$ is weakly convergent to 0, we have

$$\begin{aligned}
\liminf_n \| x_n - 2x + z \| &= \liminf_n \left\| \frac{r}{B+\varepsilon}(x_n - x) + \left(1 - \frac{r}{B+\varepsilon}\right)(x_n - x) + \frac{r}{r}(z - x) \right\| \\
&\leq r \liminf_n \left\| \frac{(x_n - x)}{B+\varepsilon} + \frac{z - x}{r} \right\| + \left(1 - \frac{r}{B+\varepsilon}\right) \limsup_n \|x_n - x\| \\
&\leq rR(X) + \left(1 - \frac{r}{B+\varepsilon}\right) \limsup_n \|x_n - x\| \\
&= rR(X) + \left(1 - \frac{r}{B+\varepsilon}\right) B = r\left(R(X) + \frac{B}{r} - \frac{B}{B+\varepsilon}\right).
\end{aligned}$$

Denote $R(\varepsilon) = R(X) + \frac{B}{r} - \frac{B}{B+\varepsilon}$. Since E is convex and $R(\varepsilon) \geq 1$, it follows that $\frac{2}{(R(\varepsilon))^2+1}x + \frac{(R(\varepsilon))^2-1}{(R(\varepsilon))^2+1}z \in C$. Hence, we have

$$\limsup_n \left\| x_n - \left(\frac{2}{(R(\varepsilon))^2 + 1} x + \frac{(R(\varepsilon))^2 - 1}{(R(\varepsilon))^2 + 1} z \right) \right\| \geq r.$$

On the other hand, the weak lower semicontinuity of the norm implies that

$$\liminf_n \|((R(\varepsilon))^2-1)(x_n-x)-((R(\varepsilon))^2-1)(z-x)\| \geq ((R(\varepsilon))^2-1)\|z-x\|.$$

In view of the above inequalities, we can find a natural number N such that

(1) $\|x_N-z\|\leq r+\varepsilon$.

(2) $\|x_N-2x+z\|\leq R(\varepsilon)(r+\varepsilon)$.

(3) $\left\|x_N-\left(\frac{2}{(R(\varepsilon))^2+1}x+\frac{(R(\varepsilon))^2-1}{(R(\varepsilon))^2+1}z\right)\right\|\geq r-\varepsilon$.

(4) $\|((R(\varepsilon))^2-1)(x_n-x)-((R(\varepsilon))^2-1)(z-x)\| \geq ((R(\varepsilon))^2-1)\|z-x\|\left(\frac{r-\varepsilon}{r}\right)$.

We now consider $u=(R(\varepsilon))^2(x_N-z)$ and $v=(x_N-2x+z)$. According to the above estimates, we obtain $\|u\|\leq(R(\varepsilon))^2(r+\varepsilon)$ and $\|v\|\leq R(\varepsilon)(r+\varepsilon)$ and so that

$$\begin{aligned}\|u+v\| &= \left\|(R(X))^2\big((x_N-x)-(z-x)\big)+(x_N-x)+(z-x)\right\|\\ &= ((R(\varepsilon))^2+1)\left\|(x_N-x)-\frac{(R(\varepsilon))^2-1}{(R(\varepsilon))^2+1}(z-x)\right\|\\ &= ((R(\varepsilon))^2+1)\left\|x_N-\left(\frac{2}{(R(\varepsilon))^2+1}x+\frac{(R(\varepsilon))^2-1}{(R(\varepsilon))^2+1}z\right)\right\|\\ &\geq ((R(\varepsilon))^2+1)(r+\varepsilon),\end{aligned}$$

$$\begin{aligned}\|u-v\| &= \left\|(R(X))^2\big((x_N-x)-(z-x)\big)-\big((x_N-x)+(z-x)\big)\right\|\\ &= \|((R(\varepsilon))^2-1)(x_N-x)-((R(\varepsilon))^2+1)(z-x)\|\\ &\geq ((R(\varepsilon))^2-1)\|z-x\|\left(\frac{r-\varepsilon}{r}\right).\end{aligned}$$

By the definition of $C_{-\infty}(X)$, we have

$$C_{-\infty}(X)\geq\frac{\min\{\|u+v\|^2,\|u-v\|^2\}}{\|u\|^2+\|v\|^2}$$

$$\geq\frac{\min\left\{((R(\varepsilon))^2+1)^2(r-\varepsilon)^2,((R(\varepsilon))^2+1)^2\|z-x\|^2\left(\frac{r-\varepsilon}{r}\right)^2\right\}}{(R(\varepsilon))^4(r+\varepsilon)^2+(R(\varepsilon))^2(r+\varepsilon)^2}.$$

Since $\|z-x\|\leq r$, it follows that

$$C_{-\infty}(X)\geq\frac{((R(\varepsilon))^2+1)^2\|z-x\|^2\left(\frac{r-\varepsilon}{r}\right)^2}{(R(\varepsilon))^4(r+\varepsilon)^2+(R(\varepsilon))^2(r+\varepsilon)^2}.$$

On taking the limit as $\varepsilon \to 0$, we get

$$C_{-\infty}(X) \geq \left(1 + \frac{1}{(R(0))^2}\right)\left(\frac{\|z-x\|}{r}\right)^2$$

$$= \left(1 + \frac{1}{\left(R(X) + \frac{B}{r} - 1\right)^2}\right)\left(\frac{\|z-x\|}{r}\right)^2 \qquad (3.3)$$

At this point, we shall distinguish two cases:

Case 1. Suppose that $B = r$, which is the case when X satisfies the nonstrict Opial condition. From (3.3), we get

$$\|z-x\| \leq \left(\sqrt{\frac{C_{-\infty}(X)}{1 + \frac{1}{(R(\varepsilon))^2}}}\right) r. \qquad (3.4)$$

This inequality holds for arbitrary $z \in A$, $z \neq x$. Hence, we have

$$\sup_{z \in A} \|x - z\| \leq \left(\sqrt{\frac{C_{-\infty}(X)}{1 + \frac{1}{(R(\varepsilon))^2}}}\right) r,$$

which implies that

$$r_C(A) \leq \left(\sqrt{\frac{C_{-\infty}(X)}{1 + \frac{1}{(R(\varepsilon))^2}}}\right) r.$$

Case 2. Suppose that $B > r$. In this situation X cannot meet the nonstrict Opial condition. Since $r_X\left(\frac{1}{2}\right) \geq 0$, the continuity of the Opial modulus allows us to find a real number $\lambda \in (0,1)$ such that $r_X\left(\frac{\lambda}{2}\right) < 0$ and for which

$$C_{-\infty}(X) < 1 + \frac{1}{\left(R(X) - \frac{r_X\left(\frac{\lambda}{2}\right)}{1 + r_X\left(\frac{\lambda}{2}\right)}\right)^2} < 1 + \frac{1}{(R(X))^2}.$$

Clearly, we either have

$$\frac{\|z - x\|}{r} \leq k \qquad (3.5)$$

or

$$\frac{\|z - x\|}{r} > k.$$

In the latter case, since $B \leq 2r$ and taking a subsequence of $\{x_n\}$ if necessary, we get

$$r_X\left(\frac{k}{2}\right) \leq r_X\left(\frac{\|z - x\|}{B}\right) \leq \liminf_n \left\|\frac{x_n - x}{B} + \frac{x - z}{B}\right\| - 1$$
$$= \liminf_n \frac{\|x_n - z\|}{B} - 1 \leq \frac{r}{B} - 1 < 0.$$

From the above estimate and since $r_X\left(\frac{k}{2}\right) + 1 > 0$, it is clear that

$$\frac{B}{r} \leq \frac{1}{r_X\left(\frac{k}{2}\right) + 1}.$$

Therefore,

$$R(X) + \frac{B}{r} - 1 \leq R(X) + \frac{1}{r_X\left(\frac{k}{2}\right) + 1} - 1 = R(X) - \frac{r_X\left(\frac{k}{2}\right)}{1 + r_X\left(\frac{k}{2}\right)}.$$

Taking into account (3.3), we deduce

$$C_{-\infty}(X) \geq 1 + \frac{1}{\left(R(X) - \frac{r_X\left(\frac{k}{2}\right)}{1 + r_X\left(\frac{k}{2}\right)}\right)^2}\left(\frac{\|z - x\|}{r}\right)^2,$$

which implies that

$$\|z-x\| \leq \left(\sqrt{\frac{C_{-\infty}(X)}{1+\frac{1}{\left(R(X)-\frac{r_X\left(\frac{k}{2}\right)}{1+r_X\left(\frac{k}{2}\right)}\right)^2}}}\right) r. \qquad (3.6)$$

As a consequence of the inequalities (3.4), (3.5) and (3.6), for any $z \in A$, we obtain

$$\|z-x\| \leq \lambda\, r,$$

where

$$\lambda = \max\left\{\sqrt{\frac{C_{-\infty}(X)}{1+\frac{1}{(R(X))^2}}}, k, \sqrt{\frac{C_{-\infty}(X)}{1+\frac{1}{\left(R(X)-\frac{r_X\left(\frac{k}{2}\right)}{1+r_X\left(\frac{k}{2}\right)}\right)^2}}}\right\} < 1.$$

Consequently,

$$\sup_{z\in A}\|z-x\| \leq \lambda\, r,$$

from which it follows that

$$r_C(A) \leq \lambda\, r.$$

This finishes the proof.

□

Corollary 3.2. *Let X be a Banach space which satisfies $r_X\left(\frac{1}{2}\right) \geq 0$. If*

$$C_{-\infty}(X) < 1 + \frac{1}{(R(X))^2},$$

then X satisfies the (DL)-condition.

Remark 3.3. Corollary 3.2 is sharp in the sense that there is a Banach space X such that $C_{-\infty}(X) = 1 + \frac{1}{(R(X))^2}$ and X does not satisfy the (DL)-condition. Consider the Bynum space $\ell_{2,\infty}$ defined as $\ell_{2,\infty} := \left(\ell_2, \|\cdot\|_{2,\infty}\right)$ where $\| x \|_{2,\infty} := \max\{\| x^+ \|_2, \| x^- \|_2\}$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. We use the computation to conclude that the space $\ell_{2,\infty}$ is a limiting space for Corollary 3.2, i.e., that corollary is sharp. It is known that $C_{NJ}(\ell_{2,\infty}) = \frac{3}{2}$ (see [19]). From the inequality $C_{-\infty}(X) \leq C_{NJ}(X)$ (see [11]), we have $C_{-\infty}(\ell_{2,\infty}) \leq \frac{3}{2}$. Take $x = (-1, 1, 0, \dots) \in \ell_{2,\infty}$ and $y = \left(\frac{1}{2}, \frac{1}{2}, 0, \dots\right) \in \ell_{2,\infty}$. Thus, we obtain $\| x + y \| = \| x - y \| = \frac{3}{2}$, $\| x \| = 1$ and $\| y \| = \frac{1}{\sqrt{2}}$ and so $C_{-\infty}(\ell_{2,\infty}) \geq \frac{3}{2}$. Hence, $C_{-\infty}(\ell_{2,\infty}) = \frac{3}{2}$. It is easy to see that $R(\ell_{2,\infty}) = \sqrt{2}$ (see [19, 29]). Therefore, we have

$$C_{-\infty}(\ell_{2,\infty}) = \frac{3}{2} = 1 + \frac{1}{(R(\ell_{2,\infty}))^2}.$$

However, fails to have weak normal structure and hence does not satisfy the (DL)-condition.

Since the constants $C_Z(X)$ and $C_{NJ}(X)$ are more than or equal to $C_{-\infty}(X)$, we get the following results.

Corollary 3.4. *Let X be a Banach space such that $r_X\left(\frac{1}{2}\right) \geq 0$ and*

$$C_Z(X) < 1 + \frac{1}{(R(X))^2}.$$

Then X satisfies the (DL)-condition.

Corollary 3.5. ([18]) *Let X be a Banach space such that $r_X\left(\frac{1}{2}\right) \geq 0$ and*

$$C_{NJ}(X) < 1 + \frac{1}{(R(X))^2}.$$

Then X satisfies the (DL)-condition.

Recall that for a normed space X, the real number

$$C_P(X) = \sup\left\{\frac{\| x - y \| \| z \|}{\| x - z \| \| y \| + \| z - y \| \| x \|} \ :\ x, y, z \in X\backslash\{0\}, x \neq y \neq z \neq x\right\}$$

is called the Ptolemy constant of X. The notion of the Ptolemy constant of Banach spaces was introduced in [31] and recently it has been studied by Llorens-Fuster et al. in [32].

Because $C_Z(X) \leq C_P(X)$, the next result is a consequence of Corollary 3.4.

Corollary 3.6. *Let X be a Banach space such that $r_X\left(\frac{1}{2}\right) \geq 0$ and*

$$C_P(X) < 1 + \frac{1}{(R(X))^2}.$$

Then X satisfies the (DL)-condition.

Since $r_X\left(\frac{1}{2}\right) \geq 0$ whenever X satisfies the nonstrict Opial condition, we obtain the following result.

Corollary 3.7. *Let X be a Banach space with the nonstrict Opial condition. Suppose that one of the following conditions is satisfied*

(1) $C_{-\infty}(X) < 1 + \frac{1}{(R(X))^2}$,

(2) $C_Z(X) < 1 + \frac{1}{(R(X))^2}$,

(3) $C_{NJ}(X) < 1 + \frac{1}{(R(X))^2}$,

(4) $C_P(X) < 1 + \frac{1}{(R(X))^2}$.

Then X satisfies the (DL)-condition.

Because $R(X) \geq 1$, it follows that each of the conditions $C_{-\infty}(X) < 2$, $C_Z(X) < 2$ and $C_{NJ}(X) < 2$ imply reflexivity of X. Thus, as a consequence of Theorem 2.3 and Corollaries 3.2, 3.4, 3.5 and 3.6, we obtain the following sufficient condition so that a Banach space X has the FPP for multivalued nonexpansive mappings.

Corollary 3.8. *Let E be a nonempty bounded closed convex subset of a Banach space X such that* $r_X\left(\frac{1}{2}\right) \geq 0$ *and* $T: E \longrightarrow KC(E)$ *be a nonexpansive mapping. Suppose that one of the following conditions is satisfied*

(1) $C_{-\infty}(X) < 1 + \frac{1}{(R(X))^2}$,

(2) $C_Z(X) < 1 + \frac{1}{(R(X))^2}$,

(3) $C_{NJ}(X) < 1 + \frac{1}{(R(X))^2}$,

(4) $C_P(X) < 1 + \frac{1}{(R(X))^2}$.

Then T has a fixed point.

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