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BIFLATNESS, BIPROJECTIVITY, $\varphi-$AMENABILITY AND $\varphi-$CONTRACTIBILITY OF A CERTAIN CLASS OF BANACH ALGEBRAS.

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Given a Banach algebra A and $\varepsilon \in \overline{B_1^{(0)}}$ (the closed unit ball of A), the biflatness, biprojectivity, $\varphi-$amenability and $\varphi-$contractibility of a new Banach algebra A_ε are investigated.

Keywords: Biflatness, Biprojectivity, Character module homomorphism, $\varphi-$amenability, $\varphi-$contractibility.

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1. INTRODUCTION AND PRELIMINARIES

Let A be a Banach algebra. In [4] R. A. Kamyabi-Gol and M. Janfada defined a new product " $\odot$ " on A by $a \odot c = a\varepsilon c$ for all $a, c \in A$, where ε is a fixed element of the closed unit ball $\overline{B_1^{(0)}}$ of A. $(A, \odot)$ is an associative Banach algebra which is denoted by A_ε. Some miscellaneous algebraic properties of A_ε such as when A_ε has a unit element, when an element of A_ε is invertible and the necessary and sufficient conditions for the existence of involution on A_ε are investigated in [4]. The Arens regularity and amenability of A_ε and also derivations on A_ε and when is A_ε a C^*-algebra are studied in [4].

For a Banach algebra A let $\triangle_A : A\hat{\otimes}A \longrightarrow A$ be the multiplication map, where $A\hat{\otimes}A$ is the projective tensor product. $\triangle_A$ is an $A-$bimodule map that is a bounded linear map such that $\triangle_A(a\cdot u) = a\cdot\triangle_A(u)$ and $\triangle_A(u\cdot a) = \triangle_A(u)\cdot a$ for all $a \in A$ and $u \in A\hat{\otimes}A$. It is well known that the $A-$module actions on $A\hat{\otimes}A$ is defined by

$$a\cdot(c\otimes d) = ac\otimes d, \quad (c\otimes d)\cdot a = c\otimes da, \quad a, c, d \in A.$$

A Banach algebra A is said to be biprojective if $\triangle_A : A\hat{\otimes}A \longrightarrow A$ has a bounded right inverse which is an $A-$bimodule map. It means that there exists a bounded linear map $\lambda_A : A \longrightarrow A\hat{\otimes}A$ such that $\triangle_A \circ \lambda_A = I_A$ and

$$\lambda_A(ac) = a\cdot\lambda_A(c) = \lambda_A(a)\cdot c,$$

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for all $a, c \in A$.
A Banach algebra A is said to be biflat if the adjoint ${\triangle_A}^* : A^* \longrightarrow (A\hat{\otimes}A)^*$ of $\triangle_A$ has a bounded left inverse which is an $A-$bimodule map. Recall that every biprojective Banach algebra is biflat. Indeed, if A is biprojective then there exists an $A-$bimodule map $\lambda_A : A \longrightarrow A\hat{\otimes}A$ such that $\triangle_A \circ \lambda_A = I_A$. So ${\lambda_A}^* \circ {\triangle_A}^* = I_{A^*}$. It follows that ${\lambda_A}^*$ is a left inverse of ${\triangle_A}^*$ that is an $A-$bimodule map. The basic properties of biprojectivity and biflatness are investigated in [3] and also [1, 9].
Also biflatness and biprojectivity of Lau product of Banach algebras are investigated in [5].
Let A be a Banach algebra and $\triangle(A)$ be the set of all homomorphisms from A onto $\mathbb{C}$. The character space of A is denoted by $\triangle(A)\bigcup\{0\}$.
A new version of amenability which is related to characters was introduced and investigated by E. Kaniuth and A. T.-M. Lau and J. Pym in [7]. Also M. S. Monfared independently studied this concept in [8].
Let A be a Banach algebra and $\varphi \in \triangle(A)$. Then A is said to be $\varphi-$ amenable if there exists an $m \in A^{**}$ such that $m(\varphi) = 1$ and for all $a \in A$ and $f \in A^*$, $m(f \cdot a) = \varphi(a)m(f)$. Such an m is called a $\varphi-$mean.
A Banach algebra A is said to be $\varphi-$ contractible if there exists an $u \in A$ such that $\varphi(u) = 1$ and $au = \varphi(a)u$ for all $a \in A$. The notion of $\varphi-$contractibility of Banach algebras was introduced by Z. Hu, M. S. Monfared and T. Traynor in [2]. Recall that each $\varphi-$contractible Banach algebra is $\varphi-$amenable.
Let A and B be two normed algebras and let $\triangle(B) \neq \emptyset$. Then we say that a bounded linear map $T : A \longrightarrow B$ is character module homomorphism if there exists a $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $g \in B^*$ and $b \in B$. The set of all non-zero character module homomorphisms from A into B is denoted by $CMH(A, B)$. In particular in the case where $A = B$, $CMH(A, B)$ is denoted by $CMH(A)$. Some basic and hereditary properties of character module homomorphisms are investigated in [6].

2. Main Results

In this section let A be a Banach algebra and $\overline{B_1^{(0)}}$ be the closed unit ball of A. Also let $\varepsilon \in \overline{B_1^{(0)}}$ and A_ε be the Banach space A equipped with the new multiplication " $\odot$ ".
The aim of this section is to study the relation between biflatness, biprojectivity, $\varphi-$amenability and also $\varphi-$contractibility of A and A_ε. Also we present the relation between $CMH(A)$ and $CMH(A_\varepsilon)$.
In this section we use the following results repeatedly.

Proposition 2.1 ([4, Proposition 2.3]). *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then A_ε is unital if and only if A is unital and ε is invertible.*

The relation between $\triangle(A)$ and $\triangle(A_\varepsilon)$ are given by,

Proposition 2.2 ([4, proposition 2.4]). *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *If φ is a multiplicative linear functional on A, then $\psi = \varphi(\varepsilon)\varphi$ is a multiplicative linear functional on A_ε.*
(2) *If A_ε is unital and ψ is a multiplicative linear functional on A_ε, then $\varphi(a) = \psi(\varepsilon^{-1}a)$ is a multiplicative linear functional on A.*

We give the following proposition that we use it repeatedly.

Proposition 2.3. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. If A_ε is unital then $(A_\varepsilon)_{\varepsilon^{-2}} = A$, (isometrically isomorphism).*

Proof. Let " $\cdot$ ", " $\odot$ " and " $\circledcirc$ " be the products on A, A_ε and $(A_\varepsilon)_{\varepsilon^{-2}}$ respectively. Let $I : (A, \|\cdot\|, \cdot\ ) \longrightarrow ((A_\varepsilon)_{\varepsilon^{-2}}, \|\cdot\|, \circledcirc)$ be the identity map. We shall show that I is an algebraic homomorphism.

$$\begin{aligned} I(a) \circledcirc I(c) &= a \circledcirc c \\ &= a \odot \varepsilon^{-2} \odot c \\ &= a \cdot \varepsilon \cdot \varepsilon^{-2} \cdot \varepsilon \cdot c \\ &= a \cdot c \\ &= I(a \cdot c). \end{aligned}$$

This shows that I is an isometric isomorphism. Note that if $\|\varepsilon\| \leq 1$ then $\|\varepsilon^{-1}\| \geq 1$. But for each $a, c \in (A_\varepsilon)_{\varepsilon^{-2}}$ we have,

$$\begin{aligned} \|a \circledcirc c\| &= \|a \odot \varepsilon^{-2} \odot c\| \\ &= \|a \cdot \varepsilon \cdot \varepsilon^{-2} \cdot \varepsilon \cdot c\| \\ &= \|a \cdot c\| \\ &\leq \|a\|\|c\|. \end{aligned}$$

□

The following proposition reveal some equalities concerning $A_\varepsilon-$module actions on ${A_\varepsilon}^*, A_\varepsilon \hat{\otimes} A_\varepsilon$ and $(A_\varepsilon \hat{\otimes} A_\varepsilon)^*$ that we apply them in the sequel.

Proposition 2.4. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *$f \odot a = f \cdot a\varepsilon$ and $a \odot f = \varepsilon a \cdot f$, for all $f \in {A_\varepsilon}^*$ and $a \in A_\varepsilon$.*
(2) *$a \odot (c \otimes d) = a\varepsilon c \otimes d$ and $(c \otimes d) \odot a = c \otimes d\varepsilon a$ for all $a, c, d \in A_\varepsilon$. In particular, $a \odot u = a\varepsilon \cdot u$ and $u \odot a = u \cdot \varepsilon a$ for all $a \in A_\varepsilon$ and $u \in A_\varepsilon \hat{\otimes} A_\varepsilon$.*
(3) *$h \odot a = h \cdot a\varepsilon$ and $a \odot h = \varepsilon a \cdot h$ for all $a \in A_\varepsilon$ and $h \in (A_\varepsilon \hat{\otimes} A_\varepsilon)^*$.*

Proof. (1) : Let $a, c \in A_\varepsilon$ and $f \in {A_\varepsilon}^*$. Then,

$$\begin{aligned} \langle f \odot a, c\rangle &= \langle f, a \odot c\rangle = \langle f, a\varepsilon c\rangle \\ &= \langle f \cdot a\varepsilon, c\rangle. \end{aligned}$$

It follows that $f \odot a = f \cdot a\varepsilon$. Similarly $a \odot f = \varepsilon a \cdot f$.
(2) : Let $a, c, d \in A_\varepsilon$. Then

$$\begin{aligned} a \odot (c \otimes d) &= a \odot c \otimes d \\ &= a\varepsilon c \otimes d. \end{aligned}$$

Similarly $(c \otimes d) \odot a = c \otimes d\varepsilon a$. Now let $u = \sum_{i=1}^{\infty} c_i \otimes d_i \in A_\varepsilon \hat{\otimes} A_\varepsilon$ and $a \in A_\varepsilon$. Then,

$$\begin{aligned} a \odot u = a \odot \sum_{i=1}^{\infty} c_i \otimes d_i &= \lim_{n \longrightarrow \infty} a \odot \sum_{i=1}^{n} c_i \otimes d_i \\ &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} a \odot c_i \otimes d_i \\ &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} a\varepsilon c_i \otimes d_i \\ &= a\varepsilon \cdot \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} c_i \otimes d_i \\ &= a\varepsilon \cdot u. \end{aligned}$$

Similarly $u \odot a = u \cdot \varepsilon a$.
(3) : Let $a \in A_\varepsilon$ and $h \in (A_\varepsilon \hat{\otimes} A_\varepsilon)^*$. Then for all $c, d \in A_\varepsilon$ we have,

$$\begin{aligned} \langle h \odot a, c \otimes d \rangle = \langle h, a \odot (c \otimes d) \rangle &= \langle h, a\varepsilon c \otimes d \rangle \\ &= \langle h \cdot a\varepsilon, c \otimes d \rangle. \end{aligned}$$

Now let $u = \sum_{i=1}^{\infty} c_i \otimes d_i \in A_\varepsilon \hat{\otimes} A_\varepsilon$. Then,

$$\begin{aligned} \langle h \odot a, u \rangle &= \langle h \odot a, \sum_{i=1}^{\infty} c_i \otimes d_i \rangle \\ &= \lim_{n \longrightarrow \infty} \langle h \odot a, \sum_{i=1}^{n} c_i \otimes d_i \rangle \\ &= \lim_{n \longrightarrow \infty} \langle h, a \odot \sum_{i=1}^{n} c_i \otimes d_i \rangle \\ &= \lim_{n \longrightarrow \infty} \langle h, a\varepsilon \cdot \sum_{i=1}^{n} c_i \otimes d_i \rangle \\ &= \lim_{n \longrightarrow \infty} \langle h \cdot a\varepsilon, \sum_{i=1}^{n} c_i \otimes d_i \rangle \\ &= \langle h \cdot a\varepsilon, u \rangle. \end{aligned}$$

It follows that $h \odot a = h \cdot a\varepsilon$. Similarly $a \odot h = \varepsilon a \cdot h$. □

In the following results we characterize the relation between $\varphi-$contractibility of A and A_ε.

Theorem 2.1. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *If A is $\varphi-$contractible and $\varphi(\varepsilon) \neq 0$ then A_ε is $\psi-$contractible, where $\psi = \varphi(\varepsilon)\varphi$.*
(2) *If A_ε is unital and $\psi-$contractible then A is $\varphi-$contractible, where $\varphi(a) = \psi(\varepsilon^{-1}a), a \in A$.*

Proof. (1) : As A is $\varphi-$contractible then there exists an $u \in A$ such that $\varphi(u) = 1$ and $au = \varphi(a)u$ for all $a \in A$. Let $V = \frac{u}{\varphi(\varepsilon)}$. Then

$$\begin{aligned} a \odot V = a\varepsilon V &= a\varepsilon \frac{u}{\varphi(\varepsilon)} \\ &= \frac{1}{\varphi(\varepsilon)} a\varepsilon u = \frac{1}{\varphi(\varepsilon)}\varphi(a\varepsilon)u \\ &= \varphi(a)u = \varphi(\varepsilon)\varphi(a)\frac{u}{\varphi(\varepsilon)} \\ &= \psi(a)V. \end{aligned}$$

Also

$$\begin{aligned} \psi(V) &= \psi(\frac{u}{\varphi(\varepsilon)}) = \varphi(\varepsilon)\varphi(\frac{u}{\varphi(\varepsilon)}) = \varphi(u) \\ &= 1. \end{aligned}$$

So A_ε is $\psi-$contractible.
(2) : Let $\varphi(a) = \psi(\varepsilon^{-1}a)$ and let A_ε be unital and $\psi-$contractible. So $\psi(a) = \varphi(\varepsilon a)$. Also there exists an $u \in A_\varepsilon$ such that $\psi(u) = 1$ and $a \odot u = \psi(a)u$ for all $a \in A_\varepsilon$. It follows that

$$\begin{aligned} a\varepsilon u &= \psi(a)u = \varphi(\varepsilon a)u = \varphi(\varepsilon)\varphi(a)u \\ &= \varphi(a\varepsilon)u, \quad a \in A_\varepsilon. \end{aligned}$$

So

$$a\varepsilon u = \varphi(a\varepsilon)u. \tag{1}$$

Upon substituting $a = c\varepsilon^{-1}$ in (1) we can conclude that

$$cu = \varphi(c)u, \quad c \in A. \tag{2}$$

On the other hand the equality $1 = \psi(u) = \varphi(\varepsilon u) = \varphi(\varepsilon)\varphi(u)$ implies that $\varphi(u) \neq 0$.
Choose $V = \frac{u}{\varphi(u)}$. So $\varphi(V) = 1$ and for all $c \in A$,

$$cV = \frac{cu}{\varphi(u)} = \varphi(c)\frac{u}{\varphi(u)} = \varphi(c)V.$$

This shows that A is $\varphi-$contractible. □

The following Theorem reveals the relation between $\varphi-$amenability of A and A_ε.

Theorem 2.2. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *If A is $\varphi-$amenable and $\varphi(\varepsilon) \neq 0$ then A_ε is $\psi-$amenable, where $\psi = \varphi(\varepsilon)\varphi$.*
(2) *If A_ε is unital and $\psi-$amenable then A is $\varphi-$amenable, where $\varphi(a) = \psi(\varepsilon^{-1}a)$.*

Proof. (1) : Let $\psi = \varphi(\varepsilon)\varphi$ and $\varphi(\varepsilon) \neq 0$. Also let A be $\varphi-$amenable. So there exists an $m \in A^{**}$ such that $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A^*$. Hence $\frac{m}{\varphi(\varepsilon)}(\psi) = m(\varphi) = 1$.
Also

$$\begin{aligned}\frac{m}{\varphi(\varepsilon)}(f \odot a) &= \frac{m}{\varphi(\varepsilon)}(f \cdot a\varepsilon)\\ &= \frac{1}{\varphi(\varepsilon)}\varphi(a\varepsilon)m(f) = \varphi(a)m(f)\\ &= \varphi(\varepsilon)\varphi(a)\frac{m}{\varphi(\varepsilon)}(f)\\ &= \psi(a)\frac{m}{\varphi(\varepsilon)}(f).\end{aligned}$$

So $\frac{m}{\varphi(\varepsilon)}$ is a $\psi-$mean and A_ε is $\psi-$amenable.
(2) : Let $\varphi(a) = \psi(\varepsilon^{-1}a)$ and let A_ε be unital and $\psi-$amenable. So by part (1) $(A_\varepsilon)_{\varepsilon^{-2}}$ is $\phi-$amenable, where

$$\begin{aligned}\phi(a) &= \psi(\varepsilon^{-2})\psi(a) = \varphi(\varepsilon\varepsilon^{-2})\varphi(\varepsilon a)\\ &= \varphi(\varepsilon^{-1})\varphi(\varepsilon)\varphi(a)\\ &= \varphi(a), \quad a \in A.\end{aligned}$$

But it is obvious that $(A_\varepsilon)_{\varepsilon^{-2}} = A$. Hence A is $\varphi-$amenable. □

In the sequel we investigate the relations between biprojectivity and biflatness of A and A_ε.

Theorem 2.3. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *If A is biprojective and A_ε is unital then so is A_ε.*
(2) *If A_ε is biprojective and unital then so is A.*

Proof. (1) : Let A_ε be unital and let A be biprojective. Then there exists an $A-$bimodule map $\lambda_A : A \longrightarrow A\hat{\otimes}A$ such that $\triangle_A \circ \lambda_A = I_A$. Clearly λ_A is an $A_\varepsilon-$bimodule map. Indeed,

$$\begin{aligned}\lambda_A(a \odot c) &= \lambda_A(a\varepsilon c) = a\varepsilon \cdot \lambda_A(c)\\ &= a \odot \lambda_A(c), \quad a, c \in A_\varepsilon.\end{aligned}$$

Similarly

$$\begin{aligned}\lambda_A(a \odot c) &= \lambda_A(a\varepsilon c) = \lambda_A(a) \cdot \varepsilon c\\ &= \lambda_A(a) \odot c, \quad a, c \in A_\varepsilon.\end{aligned}$$

Let $\kappa : A_\varepsilon \hat{\otimes} A_\varepsilon \longrightarrow A_\varepsilon \hat{\otimes} A_\varepsilon$ be the bounded linear map such that
$\kappa(a \otimes c) = a\varepsilon^{-1} \otimes c, \quad a, c \in A_\varepsilon$. κ is an $A_\varepsilon-$bimodule map. Indeed,

$$\begin{aligned}\kappa(a \odot (c \otimes d)) &= \kappa(a \odot c \otimes d) = \kappa(a\varepsilon c \otimes d)\\ &= a\varepsilon c\varepsilon^{-1} \otimes d\\ &= a \odot c\varepsilon^{-1} \otimes d\\ &= a \odot (c\varepsilon^{-1} \otimes d)\\ &= a \odot \kappa(c \otimes d), \quad a, c, d \in A_\varepsilon.\end{aligned}$$

Similarly

$$\begin{aligned}\kappa((c \otimes d) \odot a) &= \kappa(c \otimes d \odot a)\\ &= c\varepsilon^{-1} \otimes d \odot a\\ &= (c\varepsilon^{-1} \otimes d) \odot a\\ &= \kappa(c \otimes d) \odot a, \quad a, c, d \in A_\varepsilon.\end{aligned}$$

Set $\lambda_{A_\varepsilon} = \kappa \circ \lambda_A$. As λ_{A_ε} is the composition of two $A_\varepsilon-$bimodule maps so it is an $A_\varepsilon-$bimodule map. Let $\lambda_A(a) = \sum_{j=1}^{\infty} f_j(a) \otimes g_j(a)$. So,

$$\begin{aligned}\triangle_{A_\varepsilon} \circ \lambda_{A_\varepsilon}(a) &= \triangle_{A_\varepsilon} \circ \kappa \circ \lambda_A(a) = \triangle_{A_\varepsilon} \circ \kappa(\sum_{j=1}^{\infty} f_j(a) \otimes g_j(a))\\ &= \triangle_{A_\varepsilon}(\sum_{j=1}^{\infty} f_j(a)\varepsilon^{-1} \otimes g_j(a)) = \sum_{j=1}^{\infty} f_j(a)\varepsilon^{-1} \odot g_j(a)\\ &= \sum_{j=1}^{\infty} f_j(a)\varepsilon^{-1}\varepsilon g_j(a) = \sum_{j=1}^{\infty} f_j(a)g_j(a)\\ &= \triangle_A(\sum_{j=1}^{\infty} f_j(a) \otimes g_j(a)) = \triangle_A(\lambda_A(a))\\ &= \triangle_A \circ \lambda_A(a)\\ &= a, \quad a \in A_\varepsilon.\end{aligned}$$

Hence A_ε is biprojective.
(2) : As $A = (A_\varepsilon)_{\varepsilon^{-2}}$ so the proof is an immediate consequence of part (1). $\square$

Theorem 2.4. *Let A be a Banach algebra and $\varepsilon \in \overline{B_1^{(0)}}$. Then,*

(1) *If A is biflat and A_ε is unital then so is A_ε.*
(2) *If A_ε is biflat and unital then so is A.*

Proof. Let A be biflat and A_ε be unital. Then there exists an $A-$bimodule map $\rho_A : (A\hat{\otimes}A)^* \longrightarrow A^*$ such that $\rho_A \circ {\triangle_A}^* = I_{A^*}$. Clearly ρ_A is an $A_\varepsilon-$bimodule map. Indeed

$$\begin{aligned}\rho_A(h\odot a) &= \rho_A(h\cdot a\varepsilon)\\ &= \rho_A(h)\cdot a\varepsilon\\ &= \rho_A(h)\odot a, \quad a\in A_\varepsilon, h\in (A\hat{\otimes}A)^*.\end{aligned}$$

Similarly

$$\begin{aligned}\rho_A(a\odot h) &= \rho_A(\varepsilon a\cdot h)\\ &= \varepsilon a\cdot \rho_A(h)\\ &= a\odot \rho_A(h), \quad a\in A_\varepsilon, h\in (A\hat{\otimes}A)^*.\end{aligned}$$

Suppose that $l : A_\varepsilon\hat{\otimes}A_\varepsilon \longrightarrow A_\varepsilon\hat{\otimes}A_\varepsilon$ is the bounded linear map such that $l(a\otimes c) = a\varepsilon\otimes c, \quad a, c\in A_\varepsilon$. We shall show that l is an $A_\varepsilon-$bimodule map.

$$\begin{aligned}l(a\odot(c\otimes d)) &= l(a\odot c\otimes d) = l(a\varepsilon c\otimes d)\\ &= a\varepsilon c\varepsilon\otimes d = a\odot(c\varepsilon)\otimes d\\ &= a\odot(c\varepsilon\otimes d)\\ &= a\odot l(c\otimes d), \quad a, c, d\in A_\varepsilon.\end{aligned}$$

Similarly

$$\begin{aligned}l((c\otimes d)\odot a) &= l(c\otimes d\odot a) = c\varepsilon\otimes d\odot a\\ &= (c\varepsilon\otimes d)\odot a\\ &= l(c\otimes d)\odot a, \quad a, c, d\in A_\varepsilon.\end{aligned}$$

One can easily check that

$$\triangle_{A_\varepsilon} = \triangle_A\circ l.$$

It follows that $l^*\circ {\triangle_A}^* = {\triangle_{A_\varepsilon}}^*$.
Let $\sigma : A_\varepsilon\hat{\otimes}A_\varepsilon \longrightarrow A_\varepsilon\hat{\otimes}A_\varepsilon$ be the bounded linear map such that $\sigma(a\otimes c) = a\varepsilon^{-1}\otimes c, \quad a, c\in A_\varepsilon$. Obviously σ is an $A_\varepsilon-$bimodule map. Define

$$\rho_{A_\varepsilon} : (A_\varepsilon\hat{\otimes}A_\varepsilon)^* \longrightarrow {A_\varepsilon}^*$$

by $\rho_{A_\varepsilon}(h) = \rho_A\circ\sigma^*(h), h\in (A_\varepsilon\hat{\otimes}A_\varepsilon)^*$.
As ρ_{A_ε} is the composition of two $A_\varepsilon-$bimodule maps, so it is an $A_\varepsilon-$bimodule map. Also,

$$\begin{aligned}(\rho_{A_\varepsilon}\circ {\triangle_{A_\varepsilon}}^*)(g) &= \rho_{A_\varepsilon}({\triangle_{A_\varepsilon}}^*(g))\\ &= \rho_{A_\varepsilon}(l^*\circ {\triangle_A}^*(g)) = \rho_{A_\varepsilon}(l^*({\triangle_A}^*(g)))\\ &= \rho_A(\sigma^*(l^*({\triangle_A}^*(g))))\\ &= \rho_A(l^*({\triangle_A}^*(g))\circ\sigma) = \rho_A({\triangle_A}^*(g)) = I_{A^*}(g)\\ &= g, \quad g\in (A_\varepsilon)^*.\end{aligned}$$

Note that

$$l^*({\triangle_A}^*(g))\circ\sigma = {\triangle_A}^*(g).$$

Indeed,

$$\begin{aligned}\langle l^*(\triangle_A{}^*(g))\circ\sigma, c\otimes d\rangle &= \langle l^*(\triangle_A{}^*(g)), \sigma(c\otimes d)\rangle\\ &= \langle l^*(\triangle_A{}^*(g)), c\varepsilon^{-1}\otimes d\rangle\\ &= \langle \triangle_A{}^*(g), l(c\varepsilon^{-1}\otimes d)\rangle\\ &= \langle \triangle_A{}^*(g), c\varepsilon^{-1}\varepsilon\otimes d\rangle\\ &= \langle \triangle_A{}^*(g), c\otimes d\rangle, c, d\in A_\varepsilon.\end{aligned}$$

It follows that

$$l^*(\triangle_A{}^*(g))\circ\sigma = \triangle_A{}^*(g).$$

(2) : As $A = (A_\varepsilon)_{\varepsilon^{-2}}$ and A_ε is biflat and unital so the proof is an immediate consequence of part (1). □

In the following results we characterize the relation between $CMH(A)$ and $CMH(A_\varepsilon)$.

Proposition 2.5. *Let A be a Banach algebra and $\varepsilon\in\overline{B_1^{(0)}}$. If A_ε is unital then $CMH(A) = CMH(A_\varepsilon)$.*

Proof. Let $T\in CMH(A)$. Then there exists a $\varphi\in\triangle(A)$ such that $T^*(g\cdot a) = \varphi(a)T^*(g), a\in A, g\in A^*$. So,

$$\begin{aligned}T^*(g\odot a) &= T^*(g\cdot a\varepsilon) = \varphi(a\varepsilon)T^*(g)\\ &= \varphi(\varepsilon)\varphi(a)T^*(g)\\ &= \psi(a)T^*(g), \quad a\in A_\varepsilon, g\in {A_\varepsilon}^*.\end{aligned}$$

It follows that $CMH(A)\subseteq CMH(A_\varepsilon)$.
We shall show that $CMH(A_\varepsilon)\subseteq CMH(A)$. Let $T\in CMH(A_\varepsilon)$. So there exists a $\psi\in\triangle(A_\varepsilon)$ such that $T^*(g\odot a) = \psi(a)T^*(g), a\in A_\varepsilon, g\in {A_\varepsilon}^*$. Set

$$\varphi(a) = \psi(\varepsilon^{-1}a), a\in A. \tag{3}$$

So by substituting $a = \varepsilon c\varepsilon^{-1}$ in (3) we can conclude that

$$\begin{aligned}\psi(c\varepsilon^{-1}) &= \varphi(\varepsilon c\varepsilon^{-1})\\ &= \varphi(c), c\in A.\end{aligned}$$

Hence

$$\begin{aligned}T^*(g\cdot a) &= T^*(g\odot a\varepsilon^{-1}) = \psi(a\varepsilon^{-1})T^*(g)\\ &= \varphi(a)T^*(g), a\in A, g\in A^*.\end{aligned}$$

It follows that $T\in CMH(A)$. So $CMH(A_\varepsilon) = CMH(A)$. □

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